Linear programming

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1978

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WORKING PAPER SERIES

University of California, Davis
Department of Agricultural Economics

CONSISTENT AGGREGATION OF LINEAR COMPLEMENTARITY PROBLEMS

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Working Paper No. 78-1

ABSTRACT

Consistent Aggregation of Linear Complementarity Problems

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Aggregating linear complementarity problems under a general definition of constrained consistency leads to the possibility of consistent aggregation of linear and quadratic programming models. Under this formulation, consistent aggregation of dual variables is also achieved. Furthermore, the existence of multiple sets of aggregation operators is illustrated with a numerical example. Such multiple operators allow considerable flexibility of the microstructures admitting consistent aggregation.

Consistent Aggregation of Linear Complementarity Problems

Constrained consistency--rather than total consistency--was suggested in this journal by Guccione and Oguchi [2] as the proper framework to analyze the aggregation of linear programs. They dealt, however, with a very special case of constrained consistency and, furthermore, they did not consider the aggregation of dual variables.

In this note we discuss the aggregation of linear complementarity (LC) problems under a definition of constrained consistency which is more general than that suggested in [2]. Yet, it is empirically applicable, as demonstrated by a numerical example. We achieve the following results: (a) aggregation conditions are extended to cover any LP problem, symmetric and asymmetric quadratic programs of any structure, and two-person-non-zero-sum games; (b) perfect aggregation under constrained consistency includes that of dual variables in all the models admitted by the LC problem; (c) further restrictions on the structure of the LC problem allow the interpretation of the aggregation conditions within a variety of empirical contexts; (d) a numerical example illustrates the feasibility (in principle) of the approach.

Specification of the Aggregation Problem

The micro linear complementarity problem to be aggregated is defined as follows: find vectors z and w such that

$$-q = -w + Mz, \qquad w \geq 0, \qquad z \geq 0, \tag{1}$$

and

$$z'w = 0, \tag{2}$$

where q is a $[(m+n)x1]$ vector of known coefficients and M is a given $[(m+n)x(m+n)]$ positive semidefinite matrix.

Instead of solving problem (1) and (2), it is desired to consider a problem of the same form but of smaller dimensions such as: find vectors z_a and w_a which solve

$$-q_a = -w_a + M_a z_a, \qquad w_a \geq 0, \qquad z_a \geq 0, \tag{3}$$

and

$$z_a' w_a = 0, \tag{4}$$

where q_a is a [(h+k)x1] arbitrary vector of known coefficients, M_a is a given [(h+k)x(h+k)] positive semidefinite matrix and, of course (h+k) < (m+n).

The aggregation problem relating (1) and (2) to (3) and (4)--as indicated by Guccione and Oguchi [2]--must be considered under some condition of constrained consistency. This notion was originally introduced by Ijiri [4] in his masterful survey of aggregation theory. It should be clear that an appropriate defintion of constrained consistency will allow to achieve the solution of the aggregation problem under empirically flexible structures. Thus, our choice is the following: the domain of the microvariables q, w and z is restricted by linear rules such that

$$q = Uq_a \tag{5}$$

$$w^* = Uw_a^* \tag{6}$$

$$z^* = Tz_a^* \tag{7}$$

where U and T are nonnegative linear operators of full rank and of dimensions [(m+n)x(h+k)]; the starred elements are solutions of the respective LC problems. Under this specification, any arbitrary choice of the vector q_a is admitted. This, in turn, will generate a wide spectrum of solutions (z_a^*, w_a^*) of the aggregate LC problem (3) and (4). The stringency of the restrictions on the domains of themicrovariables q, w and z, imposed by the linear rules specified above depend upon the structure of operators T and U. Guccione and Oguchi chose to work with very simplified operators requiring a proportionality relation between the micro and the macroelements of their LP problems. In this paper, this proportionality relation is unnecessary.

It is now possible to state the perfect aggregation problem under constrained consistency: for any set of microvariables (q, w*, z*) satisfying (5), (6) and (7), the following relations must hold

$$q_a = T'q \tag{8}$$

$$w^*_a = T'w^* \tag{9}$$

$$z^*_a = U'z^*. \tag{10}$$

Therefore, the conditions imposed upon the structure of problems [(1), (2)] and [(3), (4)] for achieving exact aggregation as defined above are given in the following

Theorem. Perfect aggregation of LC problems under constrained consistency is obtained if and only if

$$M'T = UM'_a \tag{11}$$

$$T'U = I \tag{12}$$

where I is an identity matrix.

Proof. (Necessity.) Premultiplying (1) by T' and substituting (10) into (3) we get

$$T'(-q) + T'w = T'Mz \tag{13}$$

$$-q_a + w_a = M_a U'z. \tag{14}$$

Hence, in view of (8) and (9),

$$z'M'T = z'UM'_a \tag{15}$$

and (11) follows because (15) must hold for any z satisfying (1) and (2) under parametric variation of q as defined by (5) (recall that q_a is arbitrary). Condition (12) is easily obtained by premultiplying (5) and (7) by T' and comparing the result to (8) and (10), respectively.

(Sufficiency.) If (11) holds, any solution (z, w) of the expanded LC problem (1) and (2) which also satisfies (5), (6) and (7), produces a solution (z_a, w_a) for the aggregate problem (3) and (4). In fact, assume that (z*, w*) represents a solution to (1) and (2). Then, $z^*_a \geq 0$ and $w^*_a \geq 0$, since T and U are nonnegative operators, and

$$z^{*\prime}_a M'_a = z^{*\prime} U M'_a = z^{*\prime} M' T = (-q' + w^{*\prime})T = -q'_a + w^{*\prime}_a \tag{16}$$

which establishes the feasibility of z^*_a and w^*_a. To show that they constitute also a complementary solution satisfying (4) it is sufficient to recall that (z*, w*) is a complementary solution (by assumption) and

$$z^{*\prime}_a w^*_a = z^{*\prime} U T' w^* = z^{*\prime} w^* = 0 \tag{17}$$

because $UT'w^* = Uw^*_a = w^*$, according to (6) and (9). Q.E.D.

Implications

It is interesting to note that the essential role of constrained consistency is required only for proving the complementarity of the aggregate solution in the sufficiency part of the theorem. For problems such as Leontief input-output analysis, where complementarity is trivially satisfied, it is not necessary (Hatanaka [3]) to invoke any form of constrained consistency to achieve the solution of the aggregation problem.

Another implication, especially important for empirical applications, concerns the multiplicity of the aggregation operators T and U. Hence, any other pair of matrices (T*, U*) such that (6), (7), (11) and (12) are satisfied, constitutes an alternative pair of aggregation operators.

The LC problem encompasses several mathematical programming structures. It is, thus, of interest to analyze the aggregation conditions more explicitly. To begin, let us consider the dual pair of symmetric quadratic programming problems formulated by Cottle [1]:

$$\max \{c'x - x'Qx - y'Ey\} \tag{18}$$

$$\text{subject to} \quad Ax - 2Ey \le b \qquad \underline{\text{Primal}}$$

$$\text{and} \qquad x \ge 0,\ y \ge 0$$

$$\min \{b'y + x'Qx + y'Ey\} \tag{19}$$

$$\text{subject to} \quad A'y + 2Qx \ge c \qquad \underline{\text{Dual}}$$

$$\text{and} \qquad x \ge 0,\ y \ge 0$$

where Q and E are known positive semidefinite symmetric matrices of order n and m, respectively; c and b are given vectors of coefficients which admit parametric variations. It is easy to show that this specification may be stated as a LC problem when the following correspondence is established

$$q = \begin{bmatrix} b \\ -c \end{bmatrix},\ z = \begin{bmatrix} y \\ x \end{bmatrix},\ M = \begin{bmatrix} 2E & -A \\ A' & 2Q \end{bmatrix},\ w = \begin{bmatrix} v \\ u \end{bmatrix}$$

where v and u are vectors of slack variables associated with the primal and dual constraints, respectively. For this symmetric quadratic programming structure the aggregation operators T and U are specified as block-diagonal matrices

$$T = \begin{bmatrix} W & 0 \\ 0 & D \end{bmatrix}, \qquad U = \begin{bmatrix} P & 0 \\ 0 & G \end{bmatrix}. \tag{20}$$

The submatrices W and P are of dimensions (mxh), while D and G are of dimensions (nxk), h < m, k < n. The aggregation condition (11) now becomes

$$\begin{bmatrix} 2E & A \\ -A' & 2Q \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} 2E_a & A_a \\ -A' & 2Q_a \end{bmatrix} \tag{21}$$

or, equivalently

$$EW = PE_a \tag{22}$$

$$A'W = GA_a' \tag{23}$$

$$AD = PA_a \tag{24}$$

$$QD = GQ_a. \tag{25}$$

The matrices E_a and Q_a are symmetric and positive semidefinite of order h and k, respectively, and belong to the aggregate symmetric quadratic programming problem. The matrix A_a is of dimensions (hxk). The aggregation condition (12) corresponds to

$$W'P = I_h \tag{26}$$

$$D'G = I_k \tag{27}$$

where I_h and I_k are identity matrices of order h and k, respectively. Notice that if $E = 0$ and $E_a = 0$, condition (22) vanishes and the remaining relations constitute the conditions for aggregating the traditional asymmetric quadratic programming problem. Furthermore, if also $Q = 0$ and $Q_a = 0$, relation (25) vanishes and the residual relations express the requirements for aggregating linear programs.

Further insight into the structure of conditions (22) through (25) is achieved, if by using (26) and (27) as needed, we restate the aggregation conditions as

$$W'EW = E_a \tag{28}$$

$$D'A'W = A_a' \tag{29}$$

$$W'AD = A_a \tag{30}$$

$$D'QD = Q_a. \tag{31}$$

This set of relations provides the fundamental guidelines for constructing consistent aggregate structures. In order to compare the generality of these conditions with those suggested by Guccione and Oguchi notice that, for them, $E = 0$, $E_a = 0$, $Q = 0$, $Q_a = 0$; the operators D and W are composed of identity submatrices stacked atop one another, and the matrix A is block diagonal with identical submatrices along the diagonal. None of these restrictions are implied by the above development. Finally, notice that exact aggregation of dual variables in quadratic programs is also achieved.

An Example of Exact Aggregation Under Constrained Consistency

Tables 1 and 2 present a numerical illustration of the consistent aggregation outlined in this paper. The example hypothesizes a block diagonal matrix of the microtechnology A with unequal submatrices along the diagonal. We assume that the two submatrices A_1 and A_2 represent the technologies of two firms (regions, sectors) of different dimensions. The first activity of each firm is assumed to correspond to a homogeneous commodity whose quantities are, therefore, aggregated in terms of their original units (the weights of the G matrix are unitary). The first resources (land) of each firm are imperfectly homogeneous but it is desirable to aggregate them into a single measure. Hence, it is necessary to transform the original measures of the resources into efficiency units. For this reason, the nonzero weights of the W matrix are not unitary. The programming problems considered in this example are asymmetric quadratic programs with different matrices Q_1 and Q_2 corresponding to the quadratic forms in the objective functions of the two firms.

It is also of interest to show the existence of multiple sets of aggregation operators, as stated above. The structure of such alternative matrices can be judiciously chosen for the purpose of allowing greater

TABLE 1: Aggregation of Quadratic Programs Under Constrained Consistency

Set Number		Aggregate		Firm 1	Firm 2	
	A_k', k=a,1,2	2	3/2	49/36	49/10	12/5
		3	4	---	21/5	16/5
	Q_k, k=a,1,2	$\frac{337}{2800}$	$\frac{3}{40}$	359/3600	5/16	21/160
		$\frac{3}{40}$	$\frac{1}{10}$	---	21/160	1/10
1	c_k, k=a,1,2	2	31/12	1/24	111/32	31/12
	b_k, k=a,1,2	5	9	13/12	254/35	36/5
	x_k^*, k=a,1,2	.4875	1.8844	.2089	.2787	1.8842
	y_k^*, k=a,1,2	0.0000	.5333	.0000	.0000	.6667
	$z_k^*=r_k^*$, k=a,1,2	5.3215		.0044	5.3171	
2	c_k, k=a,1,2	31/9	31/12	217/216	1519/288	31/12
	b_k, k=a,1,2	7	12	5/3	352/35	48/5
	x_k^*, k=a,1,2	2.8571	.8571	1.2245	1.6326	.8571
	y_k^*, k=a,1,2	1.3036	.0070	.5587	.8147	.0087
	$z_k^*=r_k^*$, k=a,1,2	10.6322		1.0806	9.5516	
3	c_k, k=a,1,2	2	31/12	1/24	111/32	31/12
	b_k, k=a,1,2	607/28	55	59/84	1676/49	44
	x_k^*, k=a,1,2	.4866	12.5530	.2089	.2783	12.5519
	y_k^*, k=a,1,2	.0000	.0000	.0000	.0000	.0000
	$z_k^*=r_k^*$, k=a,1,2	16.6993		.0044	16.6949	

TABLE 2: Consistent Aggregation Operators

Aggregation Operators	Firm 1		Firm 2	
W_k, k=1,2	3/7	0	5/8	0
	---	---	0	5/4
D_k, k=1,2	3/7	0	4/7	0
	---	---	0	1
R_k, k=1,2	2/3	-1/2	5/4	3/8
	---	---	0	1
S_k, k=1,2	2/3	-1/4	8/7	6/35
	---	---	0	4/5
G_k, k=1,2	1	0	1	0
	---	---	0	1
P_k, k=1,2	1344/1801	0	1960/1801	0
	---	---	0	4/5

flexibility of the microtechnologies. For example, we hypothesize that the matrix $U^* = \begin{bmatrix} S & 0 \\ 0 & R \end{bmatrix}$ is an alternative aggregation operator fulfilling the same role as the matrix U previously defined. To be such an operator, the submatrices S and R must satisfy conditions (21) or, more explicitly

$$EW = SE_a \tag{32}$$

$$A'W = RA'_a \tag{33}$$

$$AD = SA_a \tag{34}$$

$$QD = RQ_a \tag{35}$$

and condition (12), or

$$W'S = I_h \tag{36}$$

$$D'R = I_k . \tag{37}$$

Notice that if relations (32) and (34) are premultiplied by W' and relations (33) and (35) by D, the same conditions (28) through (31) are obtained. Table 2 presents a set of aggregation operators which satisfy conditions (32) - (37). Table 1 exhibits the structural elements and optimal primal and dual solutions of the micro and macroproblems relative to three different sets of b and c vectors. These values were selected to represent the following cases: (a) boundary solution for firm 2 and interior solution for firm 1; (b) boundary solution for both firms; (c) interior solution for both firms. It can be verified that the aggregation is exact in all three cases for both primal and dual variables.

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