# PRODUCTION FUNCTION ANALYSIS OF A FERTILIZER TRIAL ON BARLEY *

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## I. *Introduction*

Production functions of one type or another have been well explored and widely discussed, no less in the literature of agricultural economics than elsewhere. A particularly fruitful area of application in farm economics is the quantification of the relationship between fertilizer inputs and crop output. In this context currently in Australia and New Zealand, more than passing interest is being devoted to problems of experimental design and subsequent interpretation of results.[1] Another problem area relates to the choice of the most appropriate type of function to use. This note examines possible relevant functions for crop fertilization experiments. Subsequently, a set of empirical data are fitted and the derivation of the more elementary economic optima are demonstrated.

## II. *Selection of a Function*

What agreement there is in the literature on fertilizer production functions[2] suggests that the form of function should be some compromise between what is biologically compatible and what is statistically sound. Less emphasis has been placed on the ease and reliability with which economically meaningful quantities can be derived from the function. A practical consideration which must be taken into account is that time and research funds are not limitless. This is true for researchers working in a non-academic environment and for some university personnel. It may be desirable for work to be pressed forward in seeking an "ideal" function characterizing response of plants to fertilizer. The fact remains that guidance must be given now to extension workers and others.

Fortunately, the choice of a suitable form of function is not as

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[1] See, for example, P. J. Skerman and A. G. Lloyd, "Agricultural Experiments and their Economic Significance," *Aust. Jour. Ag. Econ.*, 2 : 24-44, 1958.

[2] Heady, E. O., "Organization Activities and Criteria in Obtaining and Fitting Technical Production Functions," *Jour. Farm Econ.*, 39 : 360-369, 1957.

Mason, D. D., "Functional Models and Experimental Designs for Characterizing Response Curves and Surfaces." In Baum, E. L., et al., *Methodological Procedures in the Economic Analysis of Fertilizer Use Data*, pp. 76-98, Iowa State College Press, 1956.

difficult as it might appear. The more generally acceptable[3] types of functions fall into three groups: (1) exponentials; (2) the power function (Cobb-Douglas); and (3) polynomials.

(1) *Exponentials*

The most widely known exponential is the Mitscherlich.[4]

$$Y = A\,[1 - 10^{-c(x+b)}] \qquad (1)$$

where Y is product output, A is the theoretical maximum yield, x is fertilizer input, c represents the efficiency of the fertilizer, and b measures the amount of fertilizer already present in the soil in control plots. Spillman[5] expressed the exponential relationship in another form—

$$Y = M - aR^{x} \qquad (2)$$

where M represents the limiting or maximum yield, a is the theoretical maximum increase in yield and R is the ratio by which successive increments are added to total production. For both functions, the total product curve is regarded as asymptotic to some maximum yield. This assumption is not reconcilable with observed phenomena of negative marginal products found in some fertilizer experiments. Ratios of successive increments to total yield, per unit increase in fertilizer input, are assumed equal. Again, this may be true only within a narrow range of the total product curve. The assumption that the elasticity of response is less than 1 over all ranges of inputs may not be realistic at lower rates of fertilization, especially in the case of an impoverished soil.

There are also statistical problems associated with the use of exponential forms. Non-linear parameters are involved and hence transformations must be made before simplified least squares techniques can be used. While Hartley[6] and Pimentel-Gomes[7] have outlined methods for reducing the computational burden involved, adequate goodness of fit tests are still lacking.

Therefore, it appears that for multifactorial fertilizer trials the use of exponential forms involves not only questionable assumptions concerning plant response, but also places limitations on statistical tests of significance.

(2) *The Power Function* (Cobb-Douglas)[8]

$$Y = aX_1^{b}X_2^{c} \ldots\ldots X_n^{n} \qquad (3)$$

Y equals product output, $X_1 \ldots\ldots X_n$ are factor inputs and a, b……n are parameters.

[3] Ruling out such special cases as the Bray modification of the Mitscherlich, Janisch's complex exponential on Briggs' hyperbolic form.

[4] Mitscherlich, E. A., *Die Bestimmung des Dungerbedurfnisses des Bodens.* P. Parey, Berlin, 1930.

[5] Spillman, W. J., "Exponential Yield Curves in Fertilizer Experiments," *U.S.D.A. Tech. Bull* 348, 1933.

[6] Hartley, H. O., "The Estimation of Non-Linear Parameters by 'Internal Least Squares'," *Biometrika,* 35 : 32-45, 1948.

[7] Pimentel-Gomes, F., "The Use of Mitscherlich's Regression Law in the Analysis of Experiments with Fertilizers," *Biometrics,* 9 : 498-516, 1953.

[8] P. H. Douglas and C. W. Cobb, "A Theory of Production," *Am. Econ. Rev.,* 18, No. 1 : 139-165 Proceedings.

The power function presents no computational problems and is amenable to statistical tests. However, the assumption of constant elasticity of production, i.e. the percentage increase in yield is constant for all increments of fertilizer, does not permit negative marginal products (except when the exponents are negative, in which case positive marginal products are ruled out altogether). Total yield increases continuously so that the production surface does not form a peak. Unlike the Mitscherlich or Spillman functions, no maximum or limiting yield is envisaged. Allowance is not made for possible yield depression at high fertilization rates.

Isoquants for the power function are asymptotic to the axes implying complementarity of inputs, i.e. product is not forthcoming from one input alone. Furthermore, the isoclines are linear and pass through the origin. This means that, for a given fertilizer price ratio, the minimum cost combination of fertilizers for one level of production is also the least cost combination for all levels of output.

It is suggested that response of plant growth to fertilizer does not conform to the above patterns.

Relaxation of the assumptions of constant elasticity and symmetry gives a more flexible predicting function[9] as follows:

$$Y = cx_1^{a_1}e^{b_1x_1}x_2^{a_2}e^{b_2x_2} \ldots\ldots\ldots x_n^{a_n}e^{b_nx_n} \ldots\ldots\ldots (4)$$

Y represents product output, $x_1$ ........... $x_n$ are factor inputs, e is the base of natural logarithms, and c, $a_1$ ........... $a_n$, $b_1$ ........... $b_n$ are parameters. As far as agricultural economists are concerned, it is unfortunate that the computations necessary for deriving economic optima become extensive.

Heady[10] has suggested the addition of constants to the input quantities. Thus, for two inputs, equation (3) becomes—

$$Y = a(p + X_1)^b (q + X_2)^c \ldots\ldots\ldots (5)$$

p and q being arbitrary constants.

The isocline equation (the ratio of the marginal products equated to a fertilizer price ratio r) may be solved for $X_1$ to give—

$$X_1 = \left(\frac{rbq}{c} - p\right) + \left(\frac{rb}{c}\right) X_2 \ldots\ldots\ldots (6)$$

The isoclines[11] no longer pass through the origin but through the $X_1$ or $X_2$ axis (depending on the sign of the $\left(\frac{rbq}{c} - p\right)$ term); therefore the same combination of fertilizer inputs is no longer required to be a least cost one for all levels of output.

(3) *Polynomial Functions*

Polynomial equations for one input X, usually include a linear term,

[9] Halter, A. N., Carter, H. O., and Hocking, J. G., "A Note on the Transcendental Production Function," *Jour. Farm Econ.*, 39 : 966-974, 1957.

[10] See Footnote 2.

[11] The slope of the isocline is equal to $\left(\frac{rb}{c}\right)$

positive in sign, which "explains" yield increases caused by the variable, and a higher order term, negative in sign, which accounts for diminishing returns. Signs are reversed when increasing returns are present. Thus, if X is used as a linear term, $X^{6/4}$, $X^{7/4}$, or $X^{8/4}$ might be selected to represent diminishing returns. Conversely, when X represents diminishing returns $X^{\frac{1}{2}}$ or $X^{2/3}$ might be chosen for a linear term. For a two variable input case, the most widely used forms[12] are either—

$$Y = a + bX_1^{\frac{1}{2}} + cX_2^{\frac{1}{2}} - dX_1 - eX_2 + fX_1^{\frac{1}{2}} X_2^{\frac{1}{2}} \quad \text{.........} \quad (7)$$

or

$$Y = a + bX_1 + cX_2 - dX_1^2 - eX_2^2 + fX_1X_2 \quad \text{..............} \quad (8)$$

where Y is total yield, a represents the yield intercept term, $X_1$ and $X_2$ are fertilizer inputs and b, c, d, e and f are parameters. Use of the square root function is usually restricted to cases where the soil is initially low in plant nutrients or where marginal products are large at first but decrease rapidly. In such situations, isoclines would be expected to pass through the origin and this in fact is the pattern followed by square root function isoclines. Isoclines for the quadratic function intersect the axes. However, for polynomial forms in general, isoclines converge to a point on the input plane which represents the fertilizer combination required to maximize physical product. Consequently, production surfaces achieve a definite peak thus conforming to some input-output theories.

By contrast with some of the functions discussed, polynomial models are easy to fit and permit adequate statistical tests. They are flexible in the sense that terms can be added or dropped without complete reworking of the regression problem. Furthermore (and probably most important) no assumptions are made about the elasticity of response. In the quadratic form, for example, negative marginal products are allowed for by the inclusion of a squared term with a (theoretically) negative sign.[13]

The chief criticism of such functions is that only linear and interaction terms can be justified as far as plant growth is concerned. The same cannot be said for squared or cubed terms, or terms raised to some power, e.g. square root transformations.

Thus, inconsistencies of one sort or another can be found in all these functions seeking to quantitfy input-ouput relationships. Eventually a choice, not based entirely on objective grounds, must be made. As far as the present study is concerned, a polynomial function with an interaction term was used. Specifically, its form is as follows:

$$Y = a + bP + cN - dP^2 - eN^2 + fPN \quad \text{....................} \quad (8a)$$

This is function (8) with P and N (representing phosphorus and nitrogen) substituting for $X_1$ and $X_2$.

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[12] Heady, E. O., et al., "Crop Response Surfaces and Economic Optima in Fertilizer Use," *Iowa Ag. Expt. Stn. Bull.* 424, 1955.

[13] When the regression problem has been worked through, the signs do not always work out to be negative, in which case the problem for adjusting for diminishing returns remains unsolved.

## III. *Empirical Analysis*

### 1. *Source of Experimental Data*

The experiment from which the data were obtained was a barley fertilization trial carried out at Rangiora, N.Z., by the N.Z. Department of Agriculture in the 1957-58 season. The design was a 4 x 4 factorial, replicated 3 times giving a total of 48 observations. Three of these were checks. Phosphorus and nitrogen in the form of "triple" superphosphate (P), and ammonium sulphate (N) respectively were each applied at 4 levels (0, 112, 224, 336 lbs. per acre). The soil type was Temuka silt loam which had been unfertilized for the previous three years. Kenia Barley was sown at the rate of 100 lbs. per acre. Table 1 includes the yields for each replicate of the experiment.

*Table 1*
*TEMUKA SILT LOAM—YIELDS OF BARLEY, 1958*
*(bushels per acre)*

| Level of Fertilization* P | N | Replicate I | Replicate II | Replicate III |
|---|---|---|---|---|
| 0 | 0 | 82.4 | 77.6 | 69.5 |
| 1 | 0 | 88.1 | 80.0 | 77.6 |
| 2 | 0 | 84.0 | 77.6 | 81.6 |
| 3 | 0 | 80.8 | 85.7 | 82.4 |
| 0 | 1 | 80.8 | 82.4 | 73.5 |
| 1 | 1 | 88.1 | 81.6 | 77.6 |
| 2 | 1 | 90.5 | 105.0 | 87.3 |
| 3 | 1 | 83.2 | 87.3 | 84.0 |
| 0 | 2 | 84.0 | 80.8 | 74.3 |
| 1 | 2 | 87.3 | 85.7 | 81.6 |
| 2 | 2 | 93.7 | 93.7 | 90.5 |
| 3 | 2 | 94.5 | 85.7 | 87.3 |
| 0 | 3 | 88.1 | 77.6 | 69.5 |
| 1 | 3 | 85.7 | 89.7 | 80.8 |
| 2 | 3 | 88.9 | 93.7 | 92.1 |
| 3 | 3 | 93.7 | 95.4 | 90.5 |

* Coded.

| *Code* | *Lbs. Applied Per Acre* |
|---|---|
| 0 | 0 |
| 1 | 112 |
| 2 | 224 |
| 3 | 336 |

The yields range from 69.5 to 105.0 bushels per acre with an average of 84.8 bushels and are thus considerably higher than average New Zealand figures.

### 2. *Regression Analysis*

Equation (9) is the least squares regression function derived from the data in Table 1.

$$\hat{Y} = 76.094437 + .067381P + .031533N - .000152P^2 - .000079N^2 + .000082PN \quad \text{.................... (9)}$$

$\hat{Y}$ is the expected yield and P and N are the fertilizer inputs. Both squared terms are negative—hence negative marginal products are allowed for. The value of $R^2$ is .855, indicating that the fit of the regression plane is good. The overall significance of the regression was tested by means of the F ratio (the null hypothesis is $b'y_1 = b'y_2$ etc. $= 0$).

Table 2 includes the analysis of variance for the yield data corresponding to the regression equation:

*Table 2*

*TEMUKA SILT LOAM, 1958. ANALYSIS OF VARIANCE FOR BARLEY YIELDS*

| *Source of Variation* | *Degrees of Freedom* | *Sum of Squares* | *Mean Sq.* | *F* |
|---|---|---|---|---|
| Total | 47 | 2288.48 | 48.69 | |
| Replicates | 2 | 318.50 | | |
| Treatments | 15 | 1466.00 | 97.73 | 5.82** |
| Due to regression | 5 | 1253.44 | 250.69 | |
| Lack of fit | 10 | 212.56 | 21.26 | |
| Error | 30 | 503.98 | | |

** Significant at the 1% level.

The F value in Table 2 is significant at less than the 1% level. A further criterion of goodness of fit of a function[14] is that the lack of fit term should be of the same order of magnitude (or less) than the experimental error. In Table 2 the lack of fit term is considerably less than the experimental error. On the basis of these tests it is assumed that the quadratic function characterizes the data adequately.

Table 3 includes t values for individual terms in equation (9):

*Table 3*

*t VALUES AND PROBABILITY LEVELS FOR TERMS IN EQUATION (9)*

| | *t Value* | *Approximate Probability Level** |
|---|---|---|
| P | 4.19 | .001 |
| N | 1.96 | .05 |
| $P^2$ | 3.55 | .001 |
| $N^2$ | 1.84 | .05 |
| PN | 2.39 | .025 |

* Probability of drawing a t value as large or larger by chance, gives the null hypothesis.

All terms are significant at the 5% level or less and on this basis are retained.

3. *The Nature of the Production Surface*

Regression equation (9) was used to derive expected barley yields for various phosphorus and nitrogen levels.

[14] Refer footnote 2 — (Mason).

These are shown in Table 4:

*Table 4*

*EXPECTED YIELDS OF BARLEY (BUSHELS PER ACRE) FOR VARIOUS P AND N LEVELS (lbs./acre)*

| P↓ N→ | 0 | 84 | 168 | 252 | 336 |
|---|---|---|---|---|---|
| 0 | 76.09 | 78.18 | 79.16 | 79.03 | 77.76 |
| 84 | 80.68 | 83.35 | 84.91 | 85.35 | 84.66 |
| 168 | 83.12 | 86.37 | 88.50 | 89.53 | 89.42 |
| 252 | 83.42 | 87.24 | 89.96 | 91.57 | 92.03 |
| 336 | 81.57 | 85.97 | 89.27 | 91.45 | 92.50 |

The data from this table have been used to construct the production surface of Figure 1. With P held at zero level, application of increasing amounts of N raises barley yield to a maximum of 79.16 bushels. Further top-dressing with N fertilizer then actually decreases total yield. However, the higher the level that P is held constant with increases of N levels, the greater the total barley yield. Thus, with P held at 168 lbs. per acre, an increase of N from 84 to 336 lbs. per acre increases output by 3.05 bushels. On the other hand, with P at 336 lbs. per acre, an increase of N from 84 to 336 lbs. per acre increases yield by 6.53 bushels.

If N is held at zero level and P increased, barley yield is maximized with an application of 252 lbs. per acre of P. Thereafter, increasing amounts of P depress total yield. As the level at which N is held constant is raised, response to P also increases. Thus, with zero N, an increase of P level from zero to 336 lbs. increases yield by 5.48 bushels, while if N is held at 336 lbs., an extra 14.74 bushels results from increasing P from zero to 336 lbs. per acre.

4. *Yield Isoquants*

Isoquants were derived from production function (9) by assuming values for one fertilizer and yield ($\hat{Y}$) and solving for the other nutrient. The isoquant equation for P is given in equation (10).

$$P = \frac{(.067381 + .000082N) \pm \sqrt{.050805 - .000608\hat{Y} + .000008N}}{.000304} \quad \text{.................. (10)}$$

This equation was used to derive the isoquants shown in Figure 2 which predict various combinations of P and N required to produce barley yields of 78, 80, 82 and 84 bushels per acre. The isoquants are convex to the origin confirming diminishing marginal rates of substitution. The change in slope from left to right becomes more gradual with increasing yield, indicating that N and P are better substitutes at higher than at lower levels.

A returns to scale test line would show decreasing returns. This is particularly noticeable at low N rates when successively larger increments of P have to be applied to maintain the two bushel increment in yield.

5. *Yield Isoclines*

Yield isoclines (least cost expansion paths) were derived by equating

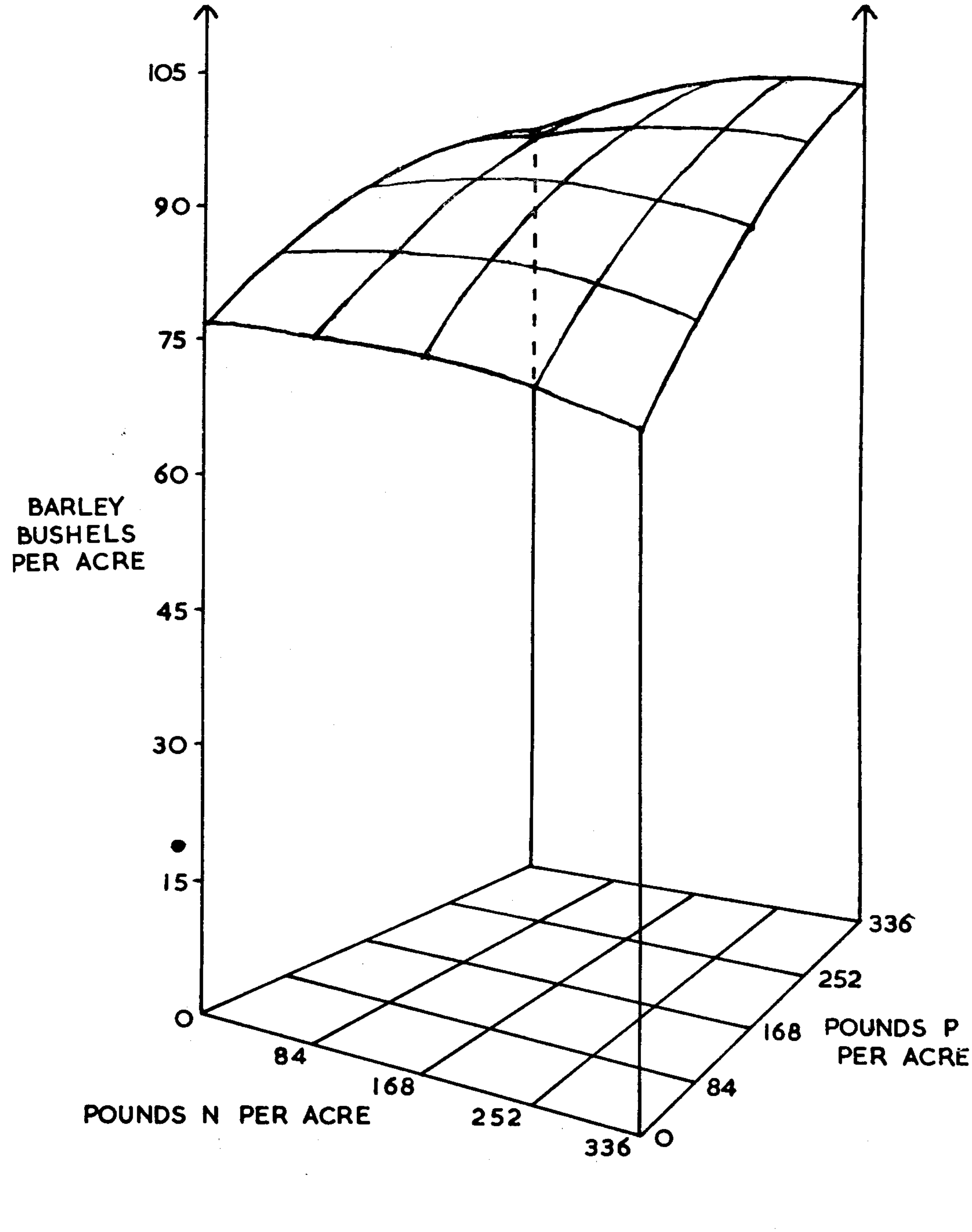


FIGURE 1. PRODUCTION SURFACE FOR BARLEY TRIAL.

the marginal products of basic function (9) to a given nutrient price ratio and solving for one nutrient. Thus:

$$\frac{d\hat{Y}}{dP} = .067381 - .000304P + .000082N \quad \ldots\ldots\ldots\ldots\ldots \quad (11)$$

$$\frac{d\hat{Y}}{dN} = .031533 - .000158N + .000082P \quad \ldots\ldots\ldots\ldots\ldots \quad (12)$$

The isocline equation then becomes:

$$\frac{.067381 - .000304P + .000082N}{.031533 - .000158N + .000082P} = a \quad \ldots\ldots\ldots\ldots\ldots \quad (13)$$

"a" represents a given nutrient price ratio.

The equation was then solved for either P or N. A series of isoclines are shown in Figure 2.

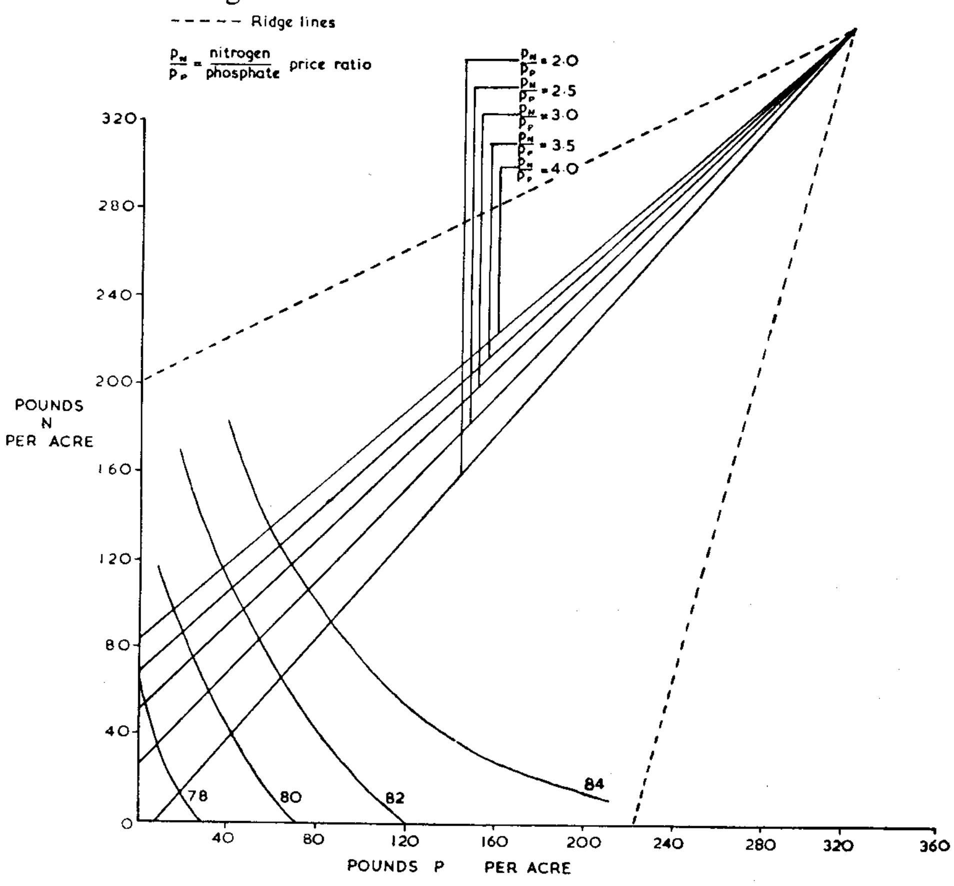


FIGURE 2. YIELD ISOQUANTS, ISOCLINES & RIDGE LINES FOR BARLEY TRIAL.

The intersection of an isocline and an isoquant gives the lowest cost combination of fertilizers required for a particular yield. For example, when the ratio of the price of P and the price of N is 3 : 1 (on current prices the actual ratio is approximately 2.5) the lowest cost combination of P and N to give an 82 bushel yield of barley is 46.5 lbs. of P and 95 lbs. of N per acre.

On any production surface where the yield attains a maximum, the family of isoclines converge to a point. The point is where the partial derivatives of both inputs are zero. In the case under consideration the maximum yield of 92.66 bushels of barley is obtained with application of 320 lbs. of P and 364 lbs. of N per acre. Thereafter, increasingly large amounts of fertilizer depress total yield. The dotted lines are ridge lines—along these lines the marginal rate of substitution between P and N equals infinity (top ridge line) or zero (lower ridge line). The ridge lines define the technical limits of replacing one fertilizer with the other to attain a specified output. They connect the points on the isoquants where the two fertilizers become technical complements. Isoquants become vertical along the upper ridge line and horizontal along the lower ridge line.

6. *Economic Optima*

Consider the situation in which a farmer concerned with maximizing profits from a barley crop wishes to apply the optimum amounts of fertilizer. Possibly an extension worker requires similar information for use in advisory work. The optimum combination of fertilizers will depend not only on the price of barley but also on the prices of P and N. Profit maximizing quantities of fertilizer are obtained by setting the partial derivatives of production function (9) with respect to P and N equal to the corresponding nutrient/barley price ratio and solving simultaneously. The general equations are as follows:—

$$.067381 - .000304P + .000082N = \frac{p_P}{p_B} \quad \text{..........} \quad (14)$$

$$.031533 - .000158N + .000082P = \frac{p_N}{p_B} \quad \text{..........} \quad (15)$$

$p_P$ and $p_N$ represent the prices per lb. of P and N fertilizer respectively and $p_B$ is the price per bushel of barley.

Profit maximizing rates of fertilization are shown in Table 5.

*Table 5*

*PROFIT MAXIMIZING RATES OF P AND N FERTILIZATION FOR VARIOUS BARLEY AND FERTILIZER PRICES*

| *Price barley/ bushel* | *Price fertilizer £ per ton* | | *Profit max. quantity of fertilizer* | | *Barley yield bushel/acre* | *Net profit* shillings/ acre* |
|---|---|---|---|---|---|---|
| | P | N | P (lb.) | N (lb.) | | |
| 6/- | 9 | 20 | 210 | 120 | 88.2 | 491 |
| 6/- | 12 | 25 | 178 | 56 | 85.6 | 483 |
| 6/- | 15 | 30 | 146 | 0 | 82.7 | 477 |
| 6/- | 15 | 20 | 176 | 102 | 87.1 | 481 |
| 6/- | 9 | 30 | 180 | 10 | 83.7 | 485 |
| 8/- | 9 | 20 | 237 | 181 | 90.2 | 670 |
| 8/- | 12 | 25 | 213 | 133 | 88.7 | 657 |
| 8/- | 15 | 30 | 190 | 86 | 86.9 | 647 |
| 8/- | 15 | 20 | 212 | 168 | 89.5 | 658 |
| 8/- | 9 | 30 | 215 | 99 | 87.6 | 657 |
| 10/- | 9 | 20 | 254 | 218 | 91.1 | 852 |
| 10/- | 12 | 25 | 235 | 180 | 90.1 | 836 |
| 10/- | 15 | 30 | 216 | 142 | 88.9 | 822 |
| 10/- | 15 | 20 | 233 | 207 | 90.6 | 838 |
| 10/- | 9 | 30 | 236 | 152 | 89.4 | 834 |

* Gross return per acre minus fertilizer cost per acre.

The data in the table confirm elementary input-output theory. As the price of barley increases with the price of N and P remaining constant, profits are maximized by increasingly heavier fertilizer application.

Thus, when the price of barley is 6/- per bushel and P and N are £9 and £20 per ton respectively, 210 lbs. of P and 120 lbs. of N per acre maximize profits. However, if the price of barley rises to 8/- per bushel, a further 27 lbs. per acre of P and 61 lbs. of N increase profits by 179/- per acre. When barley is selling at 10/- per bushel, an additional 17 lbs. of P and 37 lbs. of N per acre again raise profits by 182/- per acre.

On the other hand, if the price of barley remains constant but the prices of both fertilizers rise, profits are maximized by cutting down fertilizer use. With barley at 8/- per bushel, a rise in price of P from £9 to £12 per ton requires a reduction of the rate of application from 237 to 213 lbs. per acre if net profits are to be maximized.

The table illustrates a fundamental principle in the economics of fertilizer use which is sometimes overlooked by extension workers. This is that the combination of fertilizers which maximizes yield may not necessarily be that which maximizes net profits. Whether profits are increased by applying the yield maximizing amount of fertilizers will depend entirely on the prices of the crop and the fertilizers. Reference to Figure 1 indicates that yield is maximized with the application of 320 lbs. of P and 364 lbs. of N per acre (to give a return of 92.66 bushels of barley). With barley at 10/- per bushel and N and P at £20 and £9 per ton respectively, net returns are 836/- per acre. This represents a decrease from the possible maximum net returns of 852/- obtainable by applying 254 lbs. of P and 218 lbs. of N, as indicated in Table 5.

## IV. *Conclusion*

Through the use of production function techniques, greater precision than currently evident can be incorporated into fertilizer use advice.

There are dangers in using production function data for predictive purposes. (These dangers are inherent in whatever type of inferences are drawn from fertilizer trials.) The fundamental weakness is that *ex post* experimental knowledge is used as a basis for *ex ante* advice to farmers and others. Also, recommendations are sometimes made on the basis of one experiment carried out under particular environmental conditions in a single year (unlikely as this may seem). But, unless the circumstances are similar in future years, rates of fertilization suggested may not maximize profit. If data are built up for various soil types under changeable environmental conditions, greater predictive accuracy can be achieved.