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AN ALTERNATIVE SOLUTION TO LINEAR PROGRAMMING PROBLEMS WITH STOCHASTIC INPUT-OUTPUT COEFFICIENTS

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Deriving acceptable farm plans where input-output coefficients are stochastic is a complex problem. Previous formulations have required many simplifying assumptions about the stochastic variables in the analysis. This paper presents an alternative approach based on the mean absolute deviation, which permits solution by a conventional linear programming algorithm whilst avoiding some of those assumptions previously required. The formulation also incorporates a stochastic objective function. Examples are provided using the situation of stochastic feed supply with reference to representative sheep-grain farms on the Northern Tablelands of New South Wales. Results from these suggest that this alternative approach is a distinct improvement on earlier stochastic formulations which utilize linear programming algorithms.

Introduction

An essential, but often limiting, assumption of conventional linear programming models is that all objective function, resource constraint and input-output coefficients are known with certainty. Relaxation of the assumption for just one of these groups of coefficients greatly increases the complexity of determining an optimal solution. For the simplest of these cases—risk in the elements of the objective function—quadratic programming provides one possible method of solution [4, 21]. However, practical application of the technique has been restricted by the limited availability of suitable algorithms, and by doubts about its adequacy as a means of taking account of risk [28]. Consequently, alternative formulations which are capable of handling this form of risk within the conventional linear programming framework have evolved. These include Hazell's original MOTAD formulation [11], Hazell and Scandizzo's modification of MOTAD for the estimation of (E, V) or (E, σ) efficient production frontiers [13], marginal risk constraint linear programming [6], focus-loss constrained programming [3] and separable programming [35].

By contrast, solutions for problems where the risk is contained in one of the other sets of coefficients (i.e., in the resource supply or input-output coefficients) are less well developed, and have been considered in a relatively sparse literature. Sengupta and Tintner [32] have

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suggested that problems with risky resource constraint coefficients can, in most cases, be solved through the dual formulation of the problem. However, they also indicate that 'there are some additional problems in this case due to the inequalities in the constraint space', and therefore outline some alternative characterizations of the problem.

Solution of problems with risk in the input-output coefficients presents a far more formidable problem [1]. In reviewing the various methods of solution proposed in the literature, it is useful to classify the different types of problems. Following Hadley [10], stochastic programming problems may be conveniently divided into sequential and non-sequential categories. Although this paper is concerned with non-sequential problems, several of the sequential stochastic programming approaches are referenced, since these also have treated the problem of stochastic input-output coefficients.

Further classification of non-sequential problems, on the basis of the number of constraints involving stochastic input-output coefficients, isolates the category which includes only one constraint having stochastic coefficients. These problems are solved relatively easily since in such cases the objective function and the constraint can be interchanged, and the problem reduced to one of parametric quadratic programming [5, 23]. As soon as stochasticity is extended to coefficients of more than one constraint, this approach is no longer possible.

One approach to the more general situation is to assume that the multivariate joint probability distribution of the elements of the stochastic constraints is discrete. Solutions can then be obtained either through a game theoretic approach [22, 26] or by using the active discrete stochastic programming approach developed by Cocks [7] and extended by Rae [28, 29]. The game theoretic approach has the advantage of being solved relatively easily, but also has two distinct disadvantages. First, a large number of solutions are required to obtain an acceptable answer to any problem. This results not only in the need for considerable computational effort but also in the presentation of numerous alternative strategies to the decision maker. Second, there is an implicit assumption that the optimal solution associated with one of the observations on the multivariate distribution is optimal for the decision maker. Conceptually, the grounds for such an assumption are highly tenuous.

By contrast, the discrete stochastic programming approach does not require a large number of solutions but, because of the number of discrete states of nature which must be modelled for any realistic analysis of risk, it does give rise to a very large programming matrix. In the discrete procedure used by Cocks [7] and Rae [28, 29], all the variability of the input-output coefficients is transferred directly into the objective function of the model, whereas in the present approach, as in the earlier work of Merrill [23], chance constraints and continuous distributions are used to represent the stochasticity of the input-output coefficients. Although Merrill formulated the general problem with m chance constraints, he in fact only attempted to solve the simpler problem with a single chance constraint.

In this paper an alternative method for solving non-sequential stochastic programming problems of the type originally outlined by Merrill

[23] is developed. The problem is rewritten in the chance-constrained form[1] that the most restrictive constraint must not be violated with more than some pre-specified level of probability.[2] Solution is then achieved by applying Hazell's MOTAD method for estimating the mean absolute deviation to a sample set of input-output coefficients for each constraint and converting this into a standard deviation. The value of the standard deviation can then be incorporated into the chance constraint. Solution is thus achieved by a standard linear programming routine. Risk may be simultaneously introduced into the objective function using the procedure outlined by Hazell and Scandizzo [13].

The objective of the proposed methodology is to provide an alternative computationally feasible procedure for specifying optimal equilibrium activity levels on farms, subject to the usual resource constraints and to a broader perception of risk by individual farmers than can be accommodated in the usual quadratic programming or MOTAD procedures. Although both yield and price variability are reflected in the sample gross margins of cropping activities producing final products, in both these formulations of risk programming, only price (cost) variability typically appears in the sample gross margins in the case of pasture and feed crop activities producing intermediate products.

Further, because animal activities are typically specified on a per animal basis, only price variability tends to be reflected in sample gross margins for livestock activities producing final products. The important yield variability in pasture and feed crop activities, and the intimately linked yield variability of animal activities per unit of land area, are normally excluded from consideration. The methodology developed here provides a procedure for taking such yield variability into account in farm planning. It also facilitates the direct consideration of other aspects of risk associated with the input-output coefficients.

Theoretical aspects of the model are developed in the following section. An example using three representative sheep-grain farms[3] on the Northern Tablelands of New South Wales is given in the third section. Additional data requirements are defined and solutions obtained for various levels of risk aversion. This enables the derived plans to be

[1] Following Anderson, Dillon and Hardaker [1], two different formulations of the chance constraints may be constructed. The first,

$$P(A_{hj}x_j \leqslant b_h) \geqslant p_h, \quad h = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n,$$

implies that the most restrictive constraint must not be violated with more than some pre-specified level of probability. The second,

$$P(A_{1j}x_j \leqslant b_1, A_{2j}x_j \leqslant b_2, \ldots, A_{mj}x_j \leqslant b_m) \geqslant p, \quad j = 1, 2, \ldots, n,$$

implies that none of the constraints must be violated with more than some pre-specified level of joint probability. This paper provides a method of solution for the former formulation.

[2] Choice of the pre-specified level of probability is a decision problem in itself. No attempt will be made in this paper to solve the problem, but rather a range of probabilities will be examined.

[3] The three representative sheep-grain farms used in the example have been constructed to represent all such farms on the Northern Tablelands of New South Wales. The population on which these farms are modelled is defined on the basis of the Commonwealth Bureau of Census and Statistics [8] definition of holdings as sheep-grain 'if the combined receipts from these activities accounted for at least 75 per cent of the total receipts of the holdings, and if neither contributed more than four times the other'.

compared with the actual 1970/71 situation[4], providing an approximate validation of the model [14]. Finally some conclusions are drawn with respect both to the theoretical aspects of the model and to its practical applicability to stochastic linear programming problems.

A MOTAD Formulation for Chance Constraints

Consider the general deterministic linear programming problem:

Maximize $z = c'x$ subject to $Ax \leqq b$, $x \geqq 0$,
where x is an $n \times 1$ vector of activities,
c is an $n \times 1$ vector of gross margins,
b is an $m \times 1$ vector of resource availabilities, and
A is an $m \times n$ matrix of input-output coefficients.

Riskiness in activity returns may be incorporated into this model, following Hazell [11] and Hazell and Scandizzo [13], if the available information concerning the vector c, instead of being exact, comprises a $T \times n$ matrix of observations, C, of actual values of activity gross margins per unit of the respective activities.

If the matrix C can then be assumed to be a sample drawn from an underlying multivariate normal distribution, and riskiness is determined on the basis of mean absolute values of negative deviations about activity expected values, the standard MOTAD model may be represented as follows:

Problem 1

$$\begin{aligned} &\text{Maximize } z = e'Cx \\ &\text{subject to} \quad Ax \leqq b \\ &\qquad -MCx - I_c y_c \leqq 0 \\ &\qquad ge'y_c \leqq \hat{\sigma} \\ &\text{and} \quad x \geqq 0, y_c \geqq 0, \end{aligned}$$

where the estimated standard deviation of total gross margin, $\hat{\sigma}$, is parametized from zero to a finite upper bound at which risk becomes irrelevant.

The additional symbols incorporated in this problem comprise:

e a $T \times 1$ vector of elements each taking the value $(1/T)$ where T is the size of the sample of observations on gross margins[5], it being assumed that $T > (n + 1)$,

I_c an identity matrix of order T,

$M = (I_c - Tee')$ and creates deviations from sample mean values with the matrix C,

[4] The Agricultural Census data from which the model is constructed related to 1970/71, and all ten-year averages used in the model are based on the ten years prior to this date. Even so, a comparison between linear programming results (these being equilibrium results) and an actual situation can never be regarded as anything better than an approximate validation.

[5] Here $e'C = \bar{c}'$, where $\bar{c}$ is the column vector of sample means of the activity gross margins. If τ $(< n)$ of the activities have known returns then these can be partitioned off from those with stochastic returns and treated as in the usual LP formulation. In such a case the sample size for stochastic returns need only be such that $T > (n - \tau) + 1$.

y_c is a $T \times 1$ vector containing elements representing sums of absolute values of weighted (by the x vector) negative deviations from sample mean gross margins, and

g is a scalar for transforming mean deviations into standard deviations.[6]

Solutions to this model should trace out approximately the risk efficient frontier as defined in the usual expectation-variance framework and provide an appropriate ranking of farm plans according to their associated levels of risk [36]. When the normality assumption holds [11], estimates of $\hat{\sigma}$ based on mean deviations are somewhat less efficient than estimates based on the usual sum of squares approach.[7] Nevertheless, there is a very significant tradeoff in computational simplicity when $\hat{\sigma}$ is estimated from mean deviations since a linear, rather than a quadratic, programming algorithm suffices for obtaining problem solutions. All that is required is the addition of $T + 1$ rows and T columns to the programming tableau. Alternatively, if a specific value for a linear risk aversion coefficient (ϕ) is known then, rather than parametizing the problem, one may add a further column, transferring $\hat{\sigma}$ to the left hand side of the inequality where it appears and adding the term $-\phi\hat{\sigma}$ to the objective function.[8]

In extending the model to incorporate a set of risky input-output coefficients the case where all resource constraints include such risky coefficients is developed.[9] Obviously where some constraints are non-

[6] If the expectations and variances of gross margins are unknown (as is typically the case) but the normality assumption may be validly introduced, then $\hat{\sigma} = \Delta d$ where d is the estimated mean absolute deviation and

$$\Delta = [(T + 1 - n)(T + 1)\pi/(2T(T - 1 - n))]^{\frac{1}{2}}$$

where π is the mathematical constant. See Barry [2]. Since the MOTAD formulation [11] estimates $0{\cdot}5Td$ and the vector e in our formulation provides for division by T, the scalar g is given by 2Δ. In problems where τ of the gross margins are non-stochastic $(n - \tau)$ is substituted for n in the above formula.

[7] If the normality assumption does not hold, the problem of relative efficiencies is open to question. Tukey [37, p. 474] states that 'Nearly imperceptible nonnormalities may make conventional relative efficiencies of estimates of scale and location entirely useless', and 'In small samples, the use of the mean deviation may be a frequently useful compromise'.

[8] When specified values of ϕ (the risk aversion coefficient) are introduced, it is vital to include the scalar g in the determination of $\hat{\sigma}$. This is because when the population means and variances of the gross margins are unknown $\hat{\sigma}$ must incorporate measures of both the inherent variability of returns and the additional variability arising from lack of knowledge of the true parameter values; see Klein and Bawa [18] and Barry [2]. If $\hat{\sigma}$ is parametized then the same efficient frontier of solutions is obtained whether or not the scalar g is included, although the risk associated with particular solutions is liable to be assessed incorrectly if g is omitted, or incorporated using $\Delta = [T\pi/2(T - 1)]^{\frac{1}{2}}$ as suggested by Hazell [11] and Hazell and Scandizzo [13].

[9] At no stage do we consider the situation where elements of the vector b may be stochastic. The reasons for this are first, that we consider resource availabilities to be a minor part of the uncertainties encountered by a farmer when he is planning. For example, many resources such as land areas, initial working capital and building capacities may be known exactly. Second, stochastic resource availabilities may often have been assumed in the past on the grounds that they

stochastic these may be treated in the usual LP manner. Since even resource constraints containing stochastic input-output coefficients include many exact coefficients, for example, the zero elements, the minimum sample sizes required for estimating variability of input-output coefficients will depend on the number of stochastic coefficients in a particular constraint. Consequently, in what follows we allow sample size T_i $(i = 1, \ldots, m)$ to differ for each of the basic resource constraints and define $k_i (i = 1, \ldots, m)$ to be the (variable) number of stochastic input-output coefficients in each constraint.

The extended model with *r*isky *in*put-*o*utput *co*efficients (MOTAD with RINOCO) may be represented as follows:

Problem 2

$$\begin{array}{llllll}
\text{Maximize } z = & e'Cx & -\phi\hat{\sigma} & & & \\
\text{subject to} & EA^*x & & & +Hw & \leqq b \\
 & -MCx & -I_c y_c & & & \leqq 0 \\
 & & ge'y_c - \hat{\sigma} & & & \leqq 0 \\
 & M^*A^*x & I_c y_c & & & \leqq 0 \\
 & & & GEy_a & -I_m w & \leqq 0
\end{array}$$

and $x \geqq 0,\ \hat{\sigma} \geqq 0,\ y_c \geqq 0,\ y_a \geqq 0,\ w \geqq 0.$

Additional notation incorporated in this problem comprises:

- E an $m \times \Sigma T_i$ matrix having T_i non-zero elements in each row, each taking the value $1/T_i$, these being placed block diagonally,
- A^* a $\Sigma T_i \times n$ matrix containing in the first T_1 rows the sample observations on the elements of the first row of the matrix[10] and so on for each of the m sets of coefficients defined in A,
- M^* a $\Sigma T_i \times \Sigma T_i$ matrix with submatrices M_{ii} down the diagonal and zeros elsewhere, where each submatrix M_{ii} has the same structure as M (defined previously for Problem 1) but is of order T_i,
- I^* the identity matrix of order ΣT_i,
- I_m the identity matrix of order m,
- G an $m \times m$ diagonal matrix with elements g_{ii} which are estimated by $g_{ii} = 2\,[(T_i + 1 - k_i)(T_i + 1)\pi/(2T_i(T_i - 1 - k_i))]^{\frac{1}{2}}$. The g_{ii} transform mean deviations into standard deviations,
- y_a a $\Sigma T_i \times 1$ vector containing elements representing absolute sums of weighted (by the x vector) positive deviations of input-output coefficients from their sample means,

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were analytically more tractable, when in fact stochastic input-output coefficients were appropriate. This we consider, and hope to show, should be unnecessary in future. Third, where uncertainty may be significant, as in the case of labour availability, the simplest approach is to define a level of the resource known to be available with high probability and then add a resource hiring activity to the set of intermediate activities to meet any deficit. A parametric run on the level of the resource available, after each basis change in the parametization of the standard deviation of total gross margin, should allow a thorough analysis of the tradeoffs involved in reducing the level of probability with which the resource constraint can be satisfied.

[10] Non-stochastic elements of the matrix A are treated as if they were repeated T_i times in A^*. Consequently $EA^* = \overline{A}$ where $\overline{A}$ is the matrix of sample mean values of the input-output coefficients.

w an $m \times 1$ vector of estimated standard deviations for requirements of each resource, and

H an $m \times m$ diagonal matrix of risk aversion coefficients (η_{ii}) associated with the individual resource constraints, these reflecting the permissible probability of each particular constraint being violated.[11]

Given the definition of the matrices G and H as diagonal, it is obvious that two basic statistical assumptions of the model are (a) that each of the m sets of T_i rows in the matrix A^* may be regarded as a sample drawn from an underlying multivariate normal distribution, and (b) that each such multivariate normal distribution is distributed independently of all other such distributions, thus making each subset of aggregate random variables in the vector y_a independent of each other subset. The validity of these assumptions is open to empirical examination in any particular context. While instances of their non-applicability may be proposed involving jointly distributed chance constraints possibly of non-normal form, these assumptions seemed appropriate to the empirical situation we were considering, and were also analytically tractable.[12]

A simple example of the matrix structure associated with the estimation of mean absolute deviations of input-output coefficients is set out in Table 1. This involves a sample of four observations on each of two stochastic coefficients in each of two resource constraints. Naturally one enters only the numerical result of evaluating $(a_{ij}(t) - \bar{a}_{ij})$ in the computing tableau. With $T_i = 4$ and $k_i = 2$ the non-zero elements of the matrix product GE are all equal to 1·21. The η_{ii} values have both been assumed to be 1·17 in this example. A more comprehensive example which utilizes this matrix structure is presented in the following section.

An Application Involving Northern Tableland Representative Sheep-Grain Farms

The representative farms used in this example were selected from those included in the revised version of APMAA[13] described by Walker and Dillon [38]. To permit sufficient possibilities for diversification in the farm plans, whilst being able to capitalize fully on available data, the three representative Northern Tablelands sheep-grain farms were selected for analysis. Classification of these was on the basis of total area, and they will be referred to here as small, medium and large.

For this application of the model only the non-zero input-output

[11] Given the underlying assumption of multivariate normality, the estimated standard deviation for constraint i must be associated with a standardized t-variate. If α_i represents the permissible probability of constraint i being violated then the diagonal elements in the matrix H are $\eta_{ii} = F^{-1}(1 - \alpha_i)$ where $F(.)$ is the cumulative t distribution with $(T_i - k_i)$ degrees of freedom. An alternative approach suggested to us by Hazell involves the use of Herrey's H-statistic [15] in direct conjunction with estimated mean deviations. The two approaches should give identical results. We preferred to retain the estimated standard deviation with t-variate approach since it is more widely understood.

[12] Methods of handling chance constraints which are jointly distributed and may involve non-normal distributions are discussed by Sengupta [31].

[13] APMAA is an acronym for the *A*ggregative *P*rogramming *M*odel of *A*ustralian *A*griculture developed at the University of New England.

TABLE 1

Example Matrix Showing Part of the Structure of a Problem Formulated as MOTAD with RINOCO

	Pasture crop 1	Pasture crop 2	Feed transfer acti-vity	Live-stock acti-vity	Transfer activities for first set of deviations				Transfer activities for second set of deviations				Standard deviation transfer activities		Sign	RHS
First feedpool constraint	—11·2	—6·3	1	12·1									1·17		⩽	0
Second feedpool constraint	—7·1	—2·5	—0·7	10·9										1·17	⩽	0
Sample of feed output deviations* for first feedpool constraint	2·3	0·5			—1										⩽	0
	—4·1	—0·7				—1									⩽	0
	—0·6	0·9					—1								⩽	0
	2·7	—1·3						—1							⩽	0
Sample of feed output deviations* for second feedpool constraint	1·9	1·0							—1						⩽	0
	—2·2	0·5								—1					⩽	0
	—1·0	1·2									—1				⩽	0
	0·3	—0·6										—1			⩽	0
Standard deviation identities					1·21	1·21	1·21	1·21					—1		⩽	0
									1·21	1·21	1·21	1·21		—1	⩽	0

* From mean output per unit of activity.

coefficients associated with intermediate activities producing feed for livestock have been assumed to be stochastic. *A priori*, this aspect of variability in the input-output coefficients of the linear programming matrix would be expected to be very important in farm planning on the Northern Tablelands. While livestock demands for feed can reasonably be treated in a deterministic manner, the production of feed from pastures, forage crops and feed grain crops, because of its direct dependence on weather conditions, must be treated as stochastic in any realistic model of farm planning in this region.[14]

Four new sets of constraints were therefore included in the models to account for the variability of contributions to seasonal feedpools in spring, summer, autumn and winter, respectively. With feed production and utilization constraints for each season of the year, and an indication of the proportion of years in which farmers are typically willing to provide supplementary feed for each season (over and above normal purchases and provisions), the major additional requirement was the need to estimate the parameters of the distributions of total seasonal feed supply. These naturally vary with the optimal composition of farm activities. The MOTAD approach was therefore utilized to obtain these changing mean and standard deviation estimates required by the chance constraints. The feasibility of this approach depended upon the availability of simultaneous samples of observations on all the stochastic output coefficients.

Available sample data commonly reflect either the average pattern of seasonal production [16] or, if for a number of years, annual production [19]. Consequently, such data are unsuitable for representing quarterly variability in production. For this reason a simulation approach based on the work of Keig and McAlpine [17], Smith and Johns [33] and Smith and Stephens [34] was utilized to generate appropriate data. This involved the simulation of seasonal production

[14] Basic matrix structure followed that of the earlier version of APMAA described by Monypenny and Walker [24], with the exceptions that behavioural constraints and related activities were excluded and other sectors were refined. Grazing land was split into three categories, dependent on the extent to which development had taken place. These were land on which forage crops could be grown, land which had been cleared and sown to pastures, and land on which no clearing had taken place. The area of each of these types of land was fixed but the composition of activities on any land type allowed to vary. Improvement was permitted to land in the final category through the application of superphosphate. Also, in the original formulation, only two feedpools, critical and non-critical, were defined. As this implied a certain level of prior knowledge for each situation, four feedpools (spring, summer, autumn and winter) were defined for the revised version.

Other modifications, relevant only to this analysis, were made to the structure of the representative farm matrices as follows. Three transfer activities were introduced to allow feed to be transferred between any pair of successive seasons other than winter to spring, a period for which feed transfers would never occur in this area. The coefficients used for these transfers were obtained from Rickards and Passmore [30].

Additional constraints were imposed to limit expansion of sheep and cattle breeding units to not more than 20 per cent of the level initially set in the matrix. When the problems were solved, only cattle proved to be limited by these constraints. All variation in livestock units occurred in sheep. These solutions are consistent with post 1970/71 developments. Improved native pasture was limited to 50 per cent of the total area of native pasture, a proportion characteristic of the aggregate of the Northern Tablelands.

by various grazing and pasture activities for a given set of years from readily available climatic data. To contain the size of the final linear programming matrix, whilst ensuring a reasonable number of observations for the MOTAD-style sample, estimation of dry matter production was restricted to the ten years up to spring 1970. Activities for which coefficients were estimated in this way were improved sown pasture, improved native pasture, unimproved native pasture, grazing oats and lucerne.[15]

Results

Solution of the problems was complicated by lack of empirical evidence concerning farmers' attitudes to both the riskiness of total gross margin (represented by the risk aversion coefficient ϕ) and the riskiness of not being able to feed livestock (represented by the risk aversion coefficients η_{ii}, $i = 1, \ldots, 4$). Thus in order to illustrate the spectrum of possible results, and also to obtain possibly one corresponding reasonably well to reality, several solutions were required. In order to contain this number at a manageable level, the simplifying assumption was made that one value be common to the pasture risk aversion coefficients (η_{ii}) in all seasons. Further, extension of the analysis to include the riskiness of gross margin was restricted to the medium representative farm. Then, in common with other similar analyses [12, 25, 27], solutions were obtained for seven different values of ϕ, and η_{ii}, namely, 0·0, 0·39, 0·78, 1·17, 1·56, 1·95 and 2·33.

These values yielded seven solutions involving only feed supply risk for each of the three representative farms and a further forty-two solutions involving feed supply and gross margin risk for the medium representative farm. Tables 2, 3 and 4 show the main results obtained for the case of risk in feed supply alone for small, medium and large representative farms respectively, together with summary statistics of the actual 1970/71 situation. Some caution should be exercised when making comparisons since linear programming results reflect the long-run partial equilibrium situation, assuming expected gross margins remain at the level set for the solutions. Even so, useful comparisons can be made between the solutions and the actual situation, especially with respect to the total number of livestock units and the areas of various grazing crops.

Simultaneous examination of these three tables reveals several points

[15] Modification of the coefficients for seasonal dry matter production to reflect the actual situation was required, since the simulation model generated maximum potential growth with superphosphate non-limiting, subject to the climatic variables. Dr R. C. Smith (University of New England) in a personal communication provided the basis for the necessary modification, as well as the data on native pasture species which were not discussed in [34]. A further adjustment was required to account for losses during grazing [39], after which the data were converted into seasonal livestock feed units. Expected yields for both the feedpool constraints and estimation of deviations were calculated as the ten-year averages of these 'actual' data. The only exception to this process was hay, where aggregate production data could be obtained from the relevant Statistical Registers [9]. Use of aggregate hay production data was preferred because of the relative ease of utilizing this information to derive yield deviations. However, the problems associated with using such information for estimating standard deviation of yield should be acknowledged, since it is highly unlikely that the conditions for avoiding bias as outlined by Hazell [12] are fully satisfied.

TABLE 2

Estimated Results and Actual Data for the Small Representative Sheep-Grain Farm (200 ha)

Activity	Coefficient of risk aversion for pasture η_{ii} 0·0	0·78	1·17	1·56	2·33	Actual 1970/71
Crops (ha)						
Wheat	47	47	47	47	47	14
Grain oats						33[c]
Grazing oats						} 14[d]
Hay		6	6	4	1	
Lucerne for green fodder	26	20	20	22	25	12
Livestock (livestock units)						
Total sheep[a]	635	539	435	348	178	387
Total cattle[b]	111	111	111	111	111	96
Gross margin ($)						
Expected	5321	4766	4491	4212	3914	na.
Standard deviation	1688	1592	1490	1418	1268	na.
Mean absolute deviations (differences between modelled and actual values)						
Crops	15·7	13·7	13·7	14·3	15·3	
Livestock	131·5	83·5	31·5	27·0	112·0	

[a] Total number of sheep estimated directly from the composition of Merino and First Cross Lamb activities. One sheep is equivalent to one livestock unit.
[b] Total number of cattle estimated directly from the appropriate activities and multiplied by 8 to give livestock units.
[c] Defined in the CBCS data as 'Other Grain Crops'. The principal one only of these is included in the model.
[d] These two categories are not separate in our analysis of the Agricultural Census data.
na. Not available.

of interest. First, the only grain crop in any of the plans is wheat. It seems likely that the lack of diversification in cash cropping is at least in part a consequence of the implied assumption of risk neutrality with respect to the gross margins. Second, examination of the grazing crop and livestock components of the model shows (on the basis of the mean absolute deviations) the greatest correspondence between modelled and actual values to occur for the small and medium representative farms at an η_{ii} value of 1·17, and for the large representative farm at an η_{ii} value of 1·95. These results respectively correspond to an aversion to having to provide additional feed more than one year in six, and one year in sixteen. *A priori*, such values appear highly plausible. Third, as the values of each of the coefficients η_{ii} increase, the representative farmer's response is both towards diversification in feed supply production and reduction in the total number of livestock units carried.

Inclusion of stochastic gross margins in the model produces the

TABLE 3

Estimated Results and Actual Data for the Medium Representative Sheep-Grain Farm (553 ha)

Activity	Coefficient of risk aversion for pasture η_{ii} 0·0	0·78	1·17	1·56	2·33	Actual 1970/71
Crops (ha)						
Wheat	105	105	105	105	105	58
Grain oats						47[c]
Grazing oats						} 30[d]
Hay		15	15	12		
Lucerne for green fodder	43	36	36	36	43	21
Livestock (livestock units)						
Total sheep[a]	1517	1308	1011	796	329	929
Total cattle[b]	416	416	416	416	416	344
Gross margin ($)						
Expected	12998	11643	10861	10087	7919	na.
Standard deviation	3829	3622	3340	3160	2796	na.
Mean absolute deviations (differences between modelled and actual values)						
Crops	25·7	20·7	20·7	21·7	25·7	
Livestock	330·0	225·5	77·0	102·5	336·0	

[a], [b], [c] and [d]: Footnotes as in Table 2.
na. Not available.

results for the medium representative farm shown in Table 5 (this table lists some only of the pivot point solutions obtained). In spite of the complexity associated with interpreting the volume of information, a few salient points can again be made. First, tabulation of the mean absolute deviations for crops and for livestock, as shown in Table 6, suggests that, at least for the farm type considered here, the risk of not being able to feed livestock is more critical in deriving an acceptable farm plan than the risk associated with the gross margin.

Second, the solutions which have the lowest values for the mean absolute deviation also exhibit highly plausible values for ϕ and η_{ii} Clearly the solution in which $\phi = 1 \cdot 17$ and $\eta_{ii} = 1 \cdot 56$—corresponding to a desire not to have to provide additional feed more than one year in ten—is easily the best when considered in terms of the goodness of fit for both crops and livestock. The $\phi = 1 \cdot 17$, $\eta_{ii} = 1 \cdot 95$ solution is slightly better with respect to crops but far worse for livestock, whereas the $\phi = 0 \cdot 39$, $\eta_{ii} = 1 \cdot 17$ solution—that is, one year in six—is slightly worse on both counts. Any conclusions about relevant values of ϕ and the η_{ii} must be tentative at this stage, awaiting further research in this area.

Third, reference to Figure 1 shows the tradeoff between expected gross margin (E) and standard deviation of gross margin (σ) for various

TABLE 4

Estimated Results and Actual Data for the Large Representative Sheep-Grain Farm (1687 ha)

Activity	Coefficient of risk aversion for pasture η_{ii} 0·0	0·78	1·56	1·95	2·33	Actual 1970/71
Crops (ha)						
Wheat	191	191	191	191	191	111
Grain oats						80[c]
Grazing oats						} 66[d]
Hay			49	49	41	}
Lucerne for green fodder		125	76	76	84	59
Livestock (livestock units)						
Total sheep[a]	3775	3775	2639	1934	1192	2273
Total cattle[b]	1000	1000	1000	1000	1000	832
Gross margin ($)						
Expected	29452	29115	24073	21503	18207	na.
Standard deviation	7848	7848	6751	6163	5663	na.
Mean absolute deviations (differences between modelled and actual values)						
Crops	68·3	48·7	32·3	32·3	35·0	
Livestock	835·0	835·0	267·0	253·5	624·5	

[a], [b], [c] and [d]: Footnotes as in Table 2.
na. Not available.

levels of aversion to the possibility of not being able to feed livestock. Clearly a considerable amount of trade-off is possible. Using the measures of goodness of fit given in Table 5, it is clear that one should anticipate the actual situation to be one of farmers being willing to receive a lower expected gross margin, for a given standard deviation in gross margin, in return for a reduction in the possibility of having to provide additional feed. It seems feasible that this additional dimension of risk, which was not considered by Lin, Dean and Moore [20], may have been a factor in their finding that, although Bernoullian utility performed best, no form of utility measure was able to predict actual farmer behaviour well.

Conclusions

In this paper an alternative method for solving non-sequential stochastic linear programming problems, where the stochasticity is in the input-output coefficients, has been developed. The method used extends the MOTAD approach of Hazell to incorporate other important aspects of risk in farm planning. The solutions obtained represent estimated partial equilibria for given expected gross margins, resource avail-

TABLE 5

Estimated Results for the Medium Representative Sheep-Grain Farm for Selected Attitudes Towards Risk in Pasture Production and Gross Margin

Coefficient of risk aversion for pasture, η_{ii}	0·0	0·39	0·78	1·17	1·17	1·17	1·17	1·56	1·56	1·56	1·56	1·95	1·95	2·33	
Coefficient of risk aversion for gross margin, ϕ	1·17	0·78	0·78	0·39	0·78	1·17	1·56	0·78	1·17	1·56	1·95	1·17	1·56	1·56	Actual 1970/71
Crops (ha)															
Wheat		67	12	105	24			105	48			66			58
Grain oats	25	39	93		81	91	88		57	45	40	40	62	51	47[c]
Grazing oats															} 30[d]
Hay				12				12	1	12	14			1	
Lucerne for green fodder	51	51	51	39	51	51	51	39	50	39	37	51	51	50	21
Livestock (livestock units)															
Total sheep[a]	1517	1517	1517	994	1234	1307	1253	779	933	832	815	634	636	363	929
Total cattle[b]	416	416	416	416	416	416	416	416	416	416	416	416	416	416	344
Gross margin ($)															
Expected	10434	11987	10575	10859	9916	9377	9309	10082	9384	7701	7609	8771	7066	5659	na.
Standard deviation	1323	2910	1608	3326	1636	1115	1059	3144	2521	1186	1132	2556	1262	953	na.
Mean absolute deviations (differences between modelled and actual values)															
Crops	36·7	13·0	25·3	13·8	21·3	29·3	27·7	21·7	13·0	26·0	27·0	12·7	29·3	27·7	
Livestock	330·0	330·0	330·0	68·5	188·5	225·0	198·0	111·0	38·0	84·5	93·0	183·5	182·5	319·0	

[a], [b], [c] and [d]: Footnotes as in Table 2. na. Not available.

TABLE 6

Tabulation of the Mean Absolute Deviation for Each of the Combinations of Values of ϕ and η_{ii} for the Medium Representative Sheep-Grain Farm

For Crops

ϕ	η_{ii} 0.0	0.39	0.78	1.17	1.56	1.95	2.33
0.0	25.7	25.3	20.7	20.7	21.7	23.0	25.7
0.39	25.7	25.3	25.3	13.8	21.7	23.0	25.7
0.78	25.7	13.0	25.3	21.3	21.7	23.0	25.7
1.17	36.7	29.3	29.3	29.3	13.0	12.7	16.3
1.56	36.7	29.3	29.3	27.7	26.0	29.3	27.7
1.95	36.7	29.3	29.3	27.7	27.0	31.3	34.0
2.33	36.7	29.3	29.3	27.7	39.7	37.7	34.0

For Livestock

ϕ	η_{ii} 0.0	0.39	0.78	1.17	1.56	1.95	2.33
0.0	330.0	306.0	225.5	77.0	102.5	220.5	336.0
0.39	330.0	304.0	143.0	68.5	111.0	220.5	336.0
0.78	330.0	330.0	330.0	188.5	111.0	220.5	336.0
1.17	330.0	330.0	330.0	225.0	38.0	183.5	313.0
1.56	330.0	330.0	330.0	198.0	84.5	182.5	319.0
1.95	330.0	330.0	330.0	198.0	93.0	233.5	363.0
2.33	330.0	330.0	330.0	198.0	202.0	292.0	386.0

abilities and attitudes to risk, and can be readily updated when any of these components change significantly.

Theoretical advantages of the formulation which individually are shared by some other approaches are as follows:

(a) There is no need to assume that the joint probability distribution of the stochastic coefficients is discrete. Each set of observations is assumed to be a single random sampling from a continuous multivariate population.

(b) If the decision maker's attitude to the forms of risk under study is known, the optimal strategy can be derived from a single linear programming solution. Even where this prior knowledge is lacking, only the diagonal elements of the matrix H together with ϕ need to be varied over a range of specific values to obtain a complete set of solutions. With most modern algorithms this can still be achieved with a single run of the linear programming algorithm.

(c) The decision-making sequence may be contrasted with that used in game theoretic approaches. In the present approach, as in other optimizing approaches, the decision maker's attitude is first determined, either explicitly or implicitly, and inserted into the linear programming framework to estimate directly the optimal solution. Thus a major objection to the game theoretic approach is overcome.

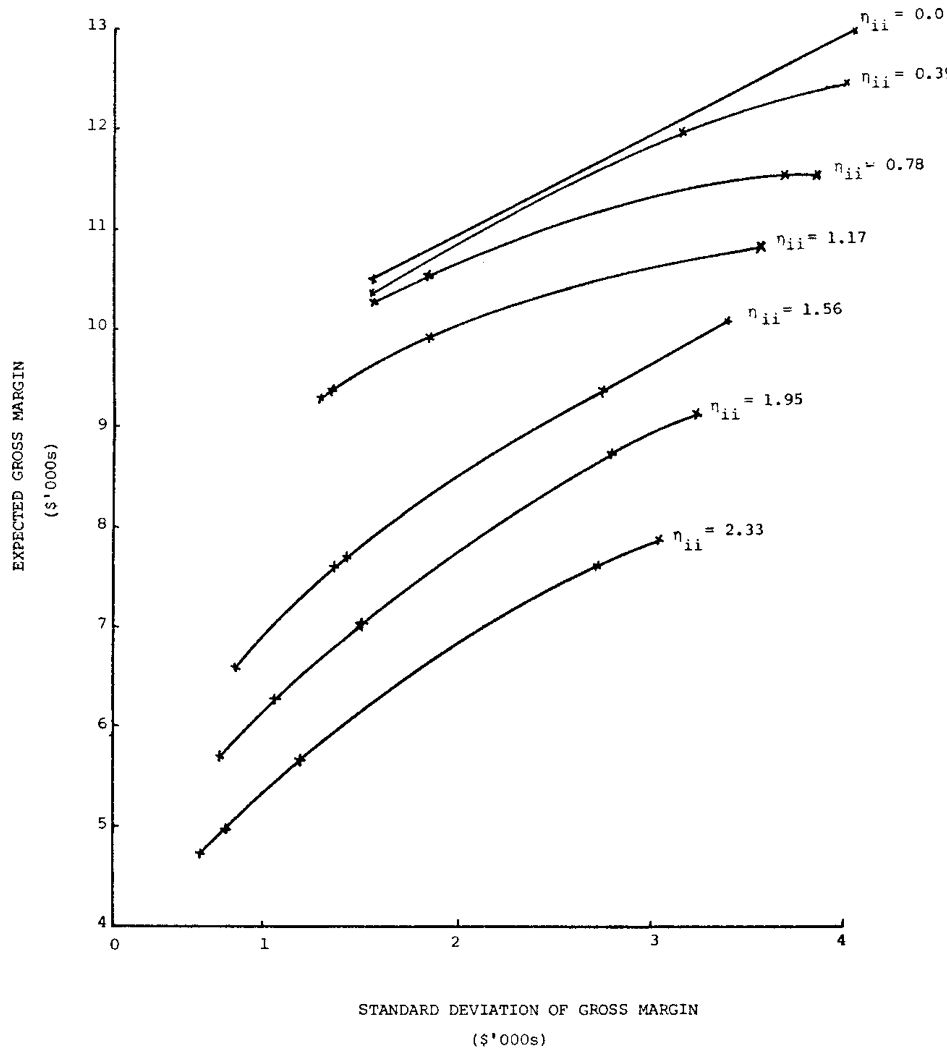

FIGURE 1—(E, σ) indifference curves for various levels of risk aversion for pasture production, η_{ii}.

(d) Estimation of risk attitude coefficients can often be directly related to concepts readily understood by the decision maker. For instance, in the example used in this paper, the coefficients are derived from the attitudes of farmers (the decision makers) toward having insufficient feed from normal sources for their stock. All that is required is an estimate of the frequency with which they are prepared to meet this situation, a fact which it should be reasonably easy to elicit.

Computationally, the method has two important advantages.

(1) Solution to the problem can be achieved by using a conventional linear programming algorithm. Further, in spite of the additional constraints and variables required, the formulation proved to be

highly tractable with respect to computing requirements.[16] This is considered to be of prime importance when determining the viability of any alternative approach.

(2) The only effect of extending the formulation to an additional constraint is to increase both the number of constraints and number of activities in the model by $(T + 1)$ for each new constraint with stochastic input-output coefficients (where T is the number of sets of observations). A similar addition is required to introduce stochasticity into the objective function. The solution procedure always remains unaltered.

From the practical viewpoint the formulation provides a method for solving an important class of risk problems. Further, it highlights the importance of including more than one aspect of risk within the farm planning context, particularly in situations involving livestock feeding based on weather-dependent fodder sources.

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[16] As a guide to computing times, with a matrix involving seventy-five rows and eighty-two columns (with fifty-five of each relating to the MOTAD with RINOCO requirements), and using the XDLA package on an ICL 1904A machine in 20K of core, two runs each generating twenty-eight solutions for varying values of ϕ and the η_{ii} on the medium representative farm took 341 and 365 mill seconds respectively of computing time. These times would be greatly reduced with more up-to-date computing facilities.

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