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# A GRAPHICAL ILLUSTRATION OF THE DULOY PARADOX

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As a guide to the reorganization of resources, the MVP/MFC comparison can result in the adjustment of resources to new levels which are actually further away from their optimal levels than their original values. Duloy illustrated this paradox mathematically in the context of fitted agricultural production functions in a paper published in 1959. His result is of considerable importance to agricultural micro-economists; however, it is not easy to teach to students of poor mathematical background. The need to be able to teach in literary form the Duloy Paradox—that increasing the application of a factor with a relatively high MVP can result in a movement away from the optimum input-mix—led to the following analysis.

## *Duloy's Main Finding*

Duloy's 1959 article had as its main thrust the following message:

The commonly used method of computing marginal value productivity of resources at some observed level of all resources, and comparing these with the market prices of resources can lead to incorrect conclusions about the direction of resource shifts required to bring about an optimum combination of resources.<sup>1</sup>

Duloy demonstrated that this result can hold whether firm size is limited by diminishing returns to scale or by capital rationing. It is the latter case which is taken up here.<sup>2</sup> The particular functional forms investigated were the Cobb-Douglas and Transcendental production functions. He examined the cases in which all resources were variable, as well as the case in which some resources were fixed.

## *Background Assumptions*

Although the aim of this note is to illustrate geometrically the breakdown of the MVP/MFC comparison as a basis for the reorganization of resources, it should be pointed out at the start that the analysis here is *not* directly equivalent to Duloy's. As noted above, we shall here be concerned with the case in which a budget constraint on the total value of variable resources operates, and *effectively* sets a limit to the scale of

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<sup>1</sup> J. H. Duloy, 'Resource Allocation and a Fitted Production Function', *Australian Journal of Agricultural Economics*, Vol. 3, No. 2 (December 1959), pp. 75-85.

<sup>2</sup> The traditional marginal indicators—the MVP/MFC comparison, for instance—take no account of the capital constraint. This means that they are doubly unreliable: first, in the sense demonstrated by Duloy (and taken up here); second, because of their failure to take into account the restrictions on working capital. According to the theory of the second best [R. G. Lipsey and K. Lancaster, 'The General Theory of the Second Best', *Review of Economic Studies*, Vol. 24 (1): pp. 11-32 (1956-57)], this failure may mean that the marginal indicators cannot even be relied upon to make ameliorating *marginal* adjustments.

the firm. We also take as given the existence of a global profit maximum which would be reached by the firm in the absence of capital rationing. Thus we examine the special case in which the size of the firm would be finite *even if* there were no capital constraint; this is achieved by assuming diminishing returns to scale to the variable factors.<sup>3</sup> Finally, whilst Duloy laid most stress on the analysis of the Cobb-Douglas production function, for convenience we will assume instead a hybrid form

$$(1.1) \quad Y = F(x_1, x_2, x_3),$$

$$(1.2) \quad = \alpha x_1^{\beta_1} x_2^{\beta_2} + \gamma x_3^{\beta_3};$$

( $x_1, x_2, x_3$  variable;  $\alpha$  and  $\gamma$  to take account of any fixed factors).

#### *Uniqueness of the Iso-profit Map*

We choose the above form because, as will be now shown, the MRS between  $x_1$  and  $x_2$  in the production of *profit* is independent of the level at which  $x_3$  is applied.<sup>4</sup>

We recall that Duloy defined the resource levels  $\{x_i\}$  in money terms, valued at current prices. The resource constraint was that

$$(2) \quad \sum_{i=1}^3 x_i = T,$$

$T$  being the extent of the liquid funds available. The profit function is

$$(3) \quad \pi = F(x_1, x_2, x_3) \cdot P_Y - \sum_{i=1}^3 x_i.$$

Although above  $\sum x_i$  is constant (equal to  $T$ ), the traditional marginal indicators do not take account of this. Thus in examining the marginal profitability of an additional unit of the  $i$ -th resource from the traditional viewpoint, one does *not* take into account that a marginal increment in  $x_1$  (for example) can be achieved only at the expense of a marginal diminution in either or both of  $x_2$  and  $x_3$ . Because the traditional textbook exposition does not take this into account, the marginal profitability of the  $i$ -th variable resource is reckoned as

$$(4) \quad \frac{\partial \pi}{\partial x_i} = F'_i \cdot P_Y - 1,$$

where  $F'_i \equiv \frac{\partial Y}{\partial x_i}$ . Again, we stress that (4) does *not* give the marginal profitability of the  $i$ -th resource when there is a constraint on working capital ( $\sum x_i \leq T$ ). Also, we note that from a traditional marginalist point of view,

<sup>3</sup> The scale behaviour of the non-homogeneous production function (1.2) varies with the level of  $x_3$ . The sum of the partial production elasticities of  $Y$  is a function which declines with increasing values of  $x_3$  if and only if  $(\beta_1 + \beta_2)$  exceeds  $\beta_3$ . We implicitly assume this to be so in all that follows. Thus, in the positive orthant, returns to scale are globally of the diminishing variety (though the intensity of diminution varies from point to point on the map).

<sup>4</sup> Of course, the MRS between  $x_1$  and  $x_2$  in the production of  $Y$  is also independent of the level of  $x_3$ . But this is not the point of the present analysis.

$$(5) \quad \frac{\partial \pi}{\partial x_i} > 0$$

is in 1 : 1 correspondence with

$$(6) \quad MVP(x_i) > P_{x_i} .$$

The marginal rate of substitution of  $x_i$  for  $x_j$  in the production of profit—again, ignoring capital constraints—is

$$(7.1) \quad \left. \frac{\partial x_j}{\partial x_i} \right|_{\pi = \text{constant}} = - \frac{\partial \pi / \partial x_i}{\partial \pi / \partial x_j}$$

$$(7.2) \quad = -(F'_i P_Y - 1) / (F'_j P_Y - 1) .$$

For this MRS to be independent of the level of a third factor, we would require  $F'_i$  and  $F'_j$  not to be functions of the level of application of this third factor. In the case of the production function (1.2),

$\left. \frac{\partial x_1}{\partial x_2} \right|_{\pi}$  satisfies this invariance rule, since

$$(8.1) \quad F'_1 = \alpha \beta_1 x_1^{\beta_1 - 1} x_2^{\beta_2} ,$$

$$(8.2) \quad F'_2 = \alpha \beta_2 x_1^{\beta_1} x_2^{\beta_2 - 1} .$$

The above, then, ensures that we need only *one* map of profit contours in the  $x_1 - x_2$  plane, since the MRS values depend only on the levels of  $x_1$  and  $x_2$ . Altering the level of  $x_3$  will affect only the *numbers on the iso-profit contours*, not the position or shapes of the contours.

*Adjustments from Initial Disequilibrium*

Again, since we have assumed that a global maximum profit position exists—there are diminishing returns to scale—the profit map will have a hill-top (as in Figure 1).

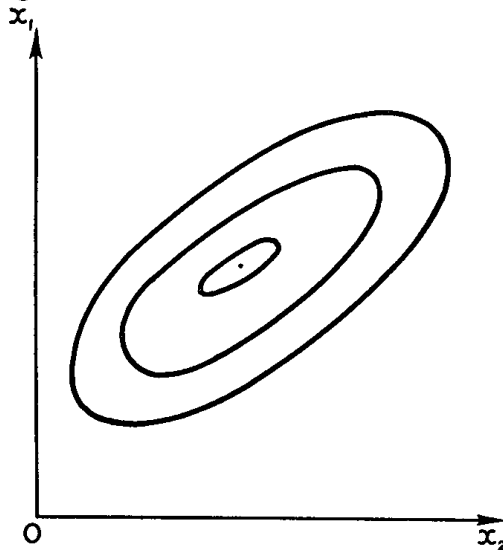


FIG. 1—The profit map with a global maximum profit position.

The situation in Figure 1 will prevail at all levels of application of  $x_3$ ; only when the 'right' level of  $x_3$  is applied, however, will the 'number' on the highest (point) contour be the maximum profit attainable.

The budget constraint can now be shown in the conventional manner. Between  $x_1$  and  $x_2$  there will be a family of budget lines, each conditional on the amount spent on  $x_3$ . The closer the budget line to the origin, the more of the total  $T$  dollars available is being spent on  $x_3$ .

Assume that the firm is initially misallocating by spending too little on  $x_3$ . (This means, of course, that it is spending too much on at least one of  $x_1$  and  $x_2$ .) In Figure 2, suppose that  $x_3 = x_3^*$  is the true optimal level of application of  $x_3$ . [That is, by varying  $x_3$  (and thus the  $x_1 - x_2$  budget lines), examining the 'number' on the relevant contour at the tangency point, and selecting that  $x_3$  value giving the highest such number, one could, in principle, determine  $x_3^*$ . Assume that this has been done!]

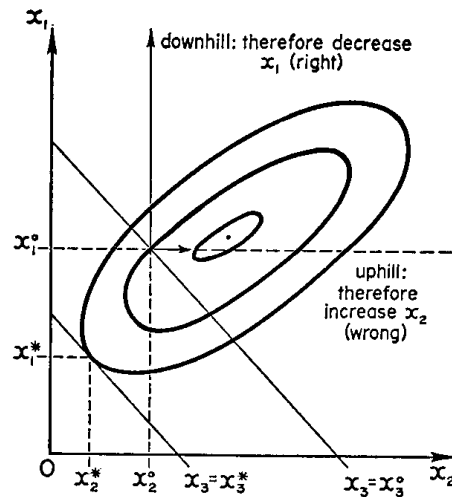


FIG. 2—Marginal adjustments from initial disequilibrium with capital rationing.

In Figure 2, let  $x_3 = x_3^o$ ,  $x_2 = x_2^o$ ,  $x_1 = x_1^o$  be initial position in which the firm is operating. The ordinary marginal indicators at this point would indicate that the firm is spending too much on  $x_1$  (marginal profit negative), and not enough on  $x_2$  (marginal profit positive). In fact, one is spending too much on both, the optimal  $x_1$  and  $x_2$  being  $x_1^*$  and  $x_2^*$  respectively. Thus using the naïve marginal diagnosis would, in the case of  $x_2$ , send you even further from the optimal level of  $x_2$ .

### Conclusion

It will be clear that one can only carry out this analysis on one diagram if the marginal profitabilities of  $x_1$  and  $x_2$  change equi-proportionately when  $x_3$  is varied; otherwise, one needs another set of contours.

Of course, for literary exposition one doesn't start with the production function (1.1), but rather with the assumption that marginal rates of substitution between  $x_1$  and  $x_2$  do not depend on the level of  $x_3$ . This leads to a considerable loss of generality in the demonstration; it does, however, convince graphically-minded students.