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CONGESTION MODELS WITH CONSISTENT CONJECTURES

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This paper demonstrates that, in situations in which a cumulative externality exists, the basic nature and extent of resource misallocation may be substantially less than we imagine. This conclusion stems from deriving consistent conjectures in a unified framework in which congestion is present. Experiments support the conclusion that, when numbers of agents are small, when there is little heterogeneity among them, and when they have the opportunity to observe each other during repeated experiment, the market allocation may be efficient.

1. Introduction

Apparently unrelated concepts sometimes share a common structure, and this simple observation can often be used to solidify a body of knowledge. This paper exploits an equivalence between the separate theories of common-property allocation, environmental pollution, the private provision of public goods, and oligopoly in order to argue that, in many situations in which a cumulative externality exists, the basic nature and extent of resource misallocation may be substantially less than we imagine. This conclusion stems from deriving consistent conjectures in a unified framework in which congestion is present. In this context, the conventional Nash equilibrium has received criticism due, primarily, to the inapplicability of its fundamental assumption,

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which is that agents take the actions of rivals as given. In response, authors have sought generalizations in models of extensive-form or evolutionary games, or through the application of conjectural variations. Due, no doubt, to its algebraic tractability, the use of conjectural variations is now widespread. However, we know little about the efficiency of the market allocation when agents form conjectures.

This paper shows formally and unequivocally that, when agents have consistent conjectures, the market allocation is efficient. In the allocation of common-property, in the generation of pollution, in the private provision of public goods, and within oligopoly, the desire to be consistent yields efficient outcomes. The empirical validity of this conclusion is strengthened by the results of experiments which suggest that, when agents are few, when there is little heterogeneity among them, and when they have the ability to observe each other during repeated experiment, an efficient allocation prevails. Motivating this conclusion is the main objective of this paper.

Section two presents the common analytical framework and section three motivates its application by presenting four examples. Section four characterizes Pareto-efficient, Nash, and conjectural-variations allocations and section five discusses equilibrium. Section six initiates the search for consistent conjectures and section seven derives key propositions. Section eight presents the results of experiments and section nine concludes.

2. *A General Framework*

Consider a set of individuals, $\{i \; i=1..N\}$, who make independent decisions and take private actions, $\{x_i \; i=1..N\}$. The private actions generate values, $\{\upsilon_i \; i=1..N\}$, from a set of processes, $\{\Phi_i(\cdot) \; i=1..N\}$, and, in turn, generate a public action, x. The public and private actions are related through the aggregation condition

$$(1) \quad x = \sum_{i-1}^{N} x_i \, ,$$

which is the key relationship in the economy. It follows that the private actions affect the processes in two distinct ways. One is direct and the other, which is indirect, is through the public action. In addition, we allow the processes to be conditioned by a set of characteristics, $\{\sigma_i \; i=1..N\}$, which may, in certain circumstances, be common. Accordingly,

$$\text{Problem 1:} \begin{cases} \max\limits_{x_i}: & \upsilon_I = \Phi_i(x_i x, \sigma_i) \qquad i = 1..N, \\ \text{subject to:} & x = \sum\limits_{j=1}^{N} x_j \end{cases}$$

characterizes choice.

3. Applications

Several situations can be depicted within this framework. Four examples follow.

Allocation of Common Property

Let the individuals be firms and interpret $\{x_i \; i=1..N\}$ as levels of effort. Since effort can be applied elsewhere, it has an opportunity cost. Therefore, let $\{\sigma_i \; i=1..N\}$ denote the real per-unit costs of effort, and assume that output is obtained from applying units of effort to another resource which is essential for production, is available in fixed and finite supply, and is excludable common property among the N competing firms. Thus, increasing applications of effort lead to congestion. There is, thus, an externality in which individual returns depend on both the public and private levels of effort. Accordingly, the production possibilities of each of the firms are given by the set of functions $\{f_i(x_i, x) \; i=1..N\}$. In terms of Problem 1, $\{\upsilon_i \; i=1..N\}$ denote levels of profit, $\{\Phi_i(\cdot) \equiv f_i(x_i, x) - \sigma_i x_i \; i=1..N\}$ denote the profit processes and, at the point of equilibrium, $\{\partial\Phi_i(\cdot)/\partial x_i > 0 \; i=1..N\}$ and $\{\partial\Phi_i(\cdot)/\partial x < 0 \; i=1..N\}$.[1] The original theory is due to Dasgupta and Heal (1979). Examples are numerous. Several are identified by Gordon (1954), Haveman (1973), Weitzman (1974), and Brown (1974). They include gathering natural harvests through hunting or fishing; sharing common pools for resources such as petroleum; allocating access to commercial transport lines; and queuing.

Externalities in Consumption

Individuals consume two sets of goods. The first set has only private effects, but the second, namely $\{x_i \; i=1..N\}$, has both private and public effects due to a cumulative externality. Let $\{\sigma_i \; i=1..N\}$ denote income endowments, and assume that the budget constraint holds with strict equality. Then $\{\upsilon_i \; i=1..N\}$ denote utility indices and $\{\Phi_i(\cdot) \equiv U_i(\sigma_i - x_i, x_i, x) \; i=1..N\}$ denote utility functions wherein $\{\partial\Phi_i(\cdot)/\partial x_i > 0 \; i=1..N\}$ and $\{\partial\Phi_i(\cdot)/\partial x < 0 \; i=1..N\}$.[2] Examples include contamination of public waterways, littering scenic places, and smoking (Segerson, 1988; Shaw and Shaw, 1991).

1 Although perhaps unclear at the moment, in each of the examples presented in this section this condition is actually necessary in order for an equilibrium to prevail. We elaborate subsequently.

2 Of course, we could have just as easily modeled a positive externality by assuming $\{\partial\Phi_i(\cdot)/\partial x > 0 \;\; i = 1..N\}$. A negative externality is adopted because it provides a nice contrast to the situation that follows.

The Private Provision of a Public Good

Individuals consume a private good and contribute to the level of a public good. As above, let $\{\sigma_i \ i=1..N\}$ denote income endowments. Then, $\{\upsilon_i \ i=1..N\}$ denote utility indices and $\{\Phi_i(\cdot) \equiv U_i(\sigma_i - x_i , x) \ i=1..N\}$ denote the utility functions, wherein $\{\partial\Phi_i(\cdot)/\partial x_i < 0 \ i=1..N\}$ and $\{\partial\Phi_i(\cdot)/\partial x > 0 \ i=1..N\}$. Examples are numerous. They include community activities that fund the disadvantaged, private donations to charities, and service among academics and professional societies (Sugden, 1985; Scafuri, 1988; Costrell, 1991; MacAulay, 1995).

Oligopoly

Finally, consider an oligopoly. There are N firms. They produce outputs $\{x_i \ i=1..N\}$ and face common demand $p = D(x)$, where p denotes price and x denotes aggregate output. Let $\{c_i(x_i|\sigma_i) \ i=1..N\}$ denote variable costs, in which the parameters $\{\sigma_i \ i=1..N\}$ denote, for example, a set of fixed factors, or a set of firm-specific factor prices. Then $\{\upsilon_i \ i=1..N\}$ denote levels of profit, $\{\Phi_i(\cdot) \equiv D(x)\, x_i - c_i(x_i \mid \sigma_i) \ i=1..N\}$ denote the profit processes, and $\{\partial\Phi_i(\cdot)/\partial x_i > 0 \ i=1..N\}$ and $\{\partial\Phi_i(\cdot)/\partial x < 0 \ i=1..N\}$.

In this example, the private effect is assumed to be positive. There are, however, both positive and negative components: Increasing output increases revenue, but it also raises costs. We assume that the revenue effect dominates. Later, we show that this condition is tantamount to assuming that firms price above marginal cost and, indeed, that it is necessary for an equilibrium to exist. Turning attention to the public effect, observe that it is unambiguously negative. That is, increases in industry output lead to reductions in price. The amount by which price declines depends on two factors. One is the responsiveness of price to changes in quantity; the other is the responsiveness of rivals to adjustments in own output. Traditional Nash equilibrium presupposes the absence of such adjustments, yet it seems highly plausible that these adjustments occur. It is precisely this issue that has prompted authors to seek an alternative conceptual framework that circumvents the Nash assumption, but which is equally as tractable. Such a framework is offered by the theory of conjectural variations and, since the early 1980s, a wealth of applications have arisen in the separate literatures on common property, consumption externalities, public goods, and oligopoly.[3] Although apparently unrelated, these theories share many similarities and can be united within a framework in which congestion exists (Porter, 1978). In the remainder of the paper, we exploit their similarities through the common framework in Problem 1.

3 Space limitations prevent appending a five-page selected bibliography covering these areas, but it may be obtained from the author on request.

Further Concordance

The concordance between the theories is further highlighted with reference to some standard diagrams (Figure 1) and by making use of the concept of marginal rates of substitution. For this purpose, we define

$$(2) \qquad MRS(x_1, x) \equiv -\partial\Phi_i(\cdot)/\partial x \div \partial\Phi_i(\cdot)/\partial x_i,$$

as the marginal rate of substitution between private and public actions in the welfare of agent i.

Figures 1a and 1b depict typical welfare contours, and Figures 1c and 1d depict the corresponding rates of substitution. Consider, first, the common-property, pollution, and oligopoly situations. For given levels of private activity, increases in the level of the public variable lower welfare; for given levels of the public action, increases in private activity increase welfare. Consequently, the iso-welfare contours in Figure 1a are increasing in a north-west direction. The contours bound convex sets of preferred combinations of public and private activities. This follows from the assumption that, along each contour, successively greater increments of private effort are required in order to offset increases in public activity, or 'congestion' (Figure 1a). This, in turn, leads to increasing marginal rates of substitution over levels of the public action (Figure 1c).

In the case of the private provision of a public good the situation is reversed. Welfare increases in the level of the public good, but declines as private contributions increase, and successively smaller decrements of private contributions are required to keep welfare constant as the level of the public good expands. Thus, welfare increases in a south-east direction (Figure 1b) and marginal rates of substitution over levels of the public action are declining (Figure 1d).

4. *Comparing Allocations*

Having familiarized ourselves with the basic framework, we wish to compare the allocations arising in three alternative regimes; namely, the allocation derived from egalitarian social planning, the allocation obtained in Nash equilibrium, and the allocation that evolves when agents have conjectural variations. This comparison is facilitated with reference to the definitions in (2) and by exploiting two standard results in the literature on cumulative externalities.

Pareto-Efficiency versus Nash-Equilibrium

Relegating derivations to the Appendix, the set of efficient allocations is characterized by the condition

$$(3) \qquad \sum_{i=1}^{N} MRS(x_i, x) = 1,$$

FIGURE 1
Iso-welfare Contours and Marginal Rates of Substitution

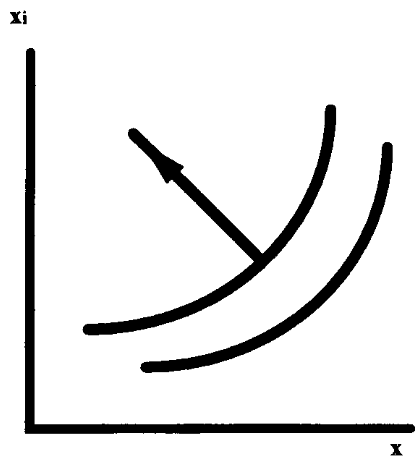

Figure 1a. Iso-welfare contours— common property, pollution and oligopoly cases.

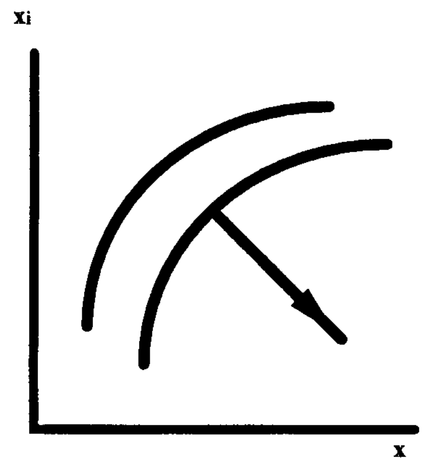

Figure 1b. Iso-welfare contours— public goods case.

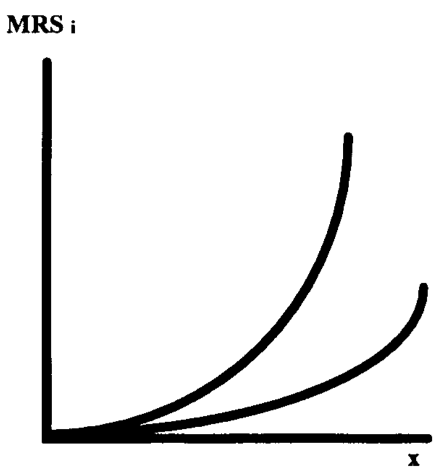

Figure 1c. Marginal rates of substitution— common property, pollution and oligopoly cases.

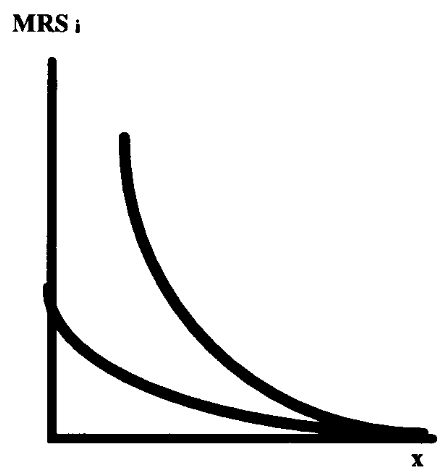

Figure 1d. marginal rates of substitution— public goods case.

or that marginal rates of substitution sum to one. In contrast, the set of Nash-equilibrium allocations is characterized by the condition

$$(4) \quad \sum_{i=1}^{N} \text{MRS}(x_i, x) = N,$$

which states that marginal rates of substitution sum to the number of individuals exploiting the public domain. Hence, in Nash equilibrium, the extent of departure from an efficient allocation rises monotonically with the number of agents within the community. This well-known result lends itself to two important policy implications, namely that the market allocation can be made more efficient by restricting access, and that absolute efficiency can be achieved by granting property rights to a single agent.[4]

Conjectural Variations

In what follows, it is important to recognize that the policy implications above follow directly from the magnitude of the right-hand side of equation (4) and that this, in turn, arises as a consequence of the fundamental Nash assumption; namely, that each agent takes the actions of rivals as given. Suppose, instead, that agents form conjectural variations. That is, when deciding on the set of private actions, $\{x_i\ i=1..N\}$, the agents form a set of perceptions, $\{\{x_j = \chi_{ij}(x_i)\ j=1..N\}\ i=1..N\}$, about the relationships between own actions and those of each of the other members of the community. For simplicity, we include $\{\chi_{ii}(x_i) \equiv x_i\ i=1..N\}$, depicting agents' perceptions of themselves. If agents observe all actions and know how to count, then a set of correspondences, namely $\{x = \chi_i(x_i) \equiv \Sigma_j\chi_{ij}(x_i)\ i=1..N\}$, is implied by equation (1). In other words, the inter-agent conjectures give rise to a set of conjectures about the relationship between own actions and the cumulative externality in the community. Working with these aggregate conjectures is convenient because it enables substantial algebra to be avoided. This convenience is further enhanced by deriving a set of elasticities, $\{\hat{\theta}_i \equiv (\partial\chi_i(\cdot)/\partial x_i)(x_i/\chi_i(\cdot))\ i=1..N\}$ which depict perceived rates of change in the public variable in response to adjustments in own actions.

With these elasticities at hand, we are in a position to characterize the market allocation under conjectural variations. Relegating details to the Appendix, we obtain the condition

$$(5)\qquad \sum_{i=1}^{N} MRS(x_i,x) = \sum_{i=1}^{N}\frac{x_i/x}{\hat{\theta}_i},$$

or that marginal rates of substitution sum to a share-weighted sum of the inverse of the conjectures. The appearance of the conjectures in the right-hand side reflect a departure from the usual Nash rule in which firms equate the absolute values of public and private rates of response. Under conjectural variations, these effects differ by a magnitude that

4 This is also the case in oligopoly, but the result is less familiar. It stems from the fact that we have only considered the supply side of the market. In essence, Problem 1 ignores consumer welfare.

depends on the conjecture. This, of course, has attendant consequences for resource allocation, as equation (5) and the following discussion demonstrate.

The Allocative Consequences of Alternative Regimes

Comparing equations (4) and (5), it is readily observed that conjectural variations simulate Nash equilibrium through the set of conjectures $\{\hat{\theta}_i = x_i/x \; i = 1..N\}$. At this point it is instructive to compare alternative equilibria with this benchmark situation.

In the common-property setting, the optimal level of own effort depends on two factors. One is the real, per-unit cost of effort, which is unaffected by any action. The other is the marginal rate of return to effort. As the theory suggests, when own effort changes, there may be simultaneous adjustment among others in the cohort; if not, the Nash assumption would be appropriate, and firms would generate Nash allocations if they conjectured thus. When the Nash assumption is inappropriate, the optimal level of effort will depend on the rates of response among the other agents. When the firm anticipates that rivals adjust similarly, each agent conjectures responses that exceed the so-called Nash conjectures. That is, conjectures $\{\hat{\theta}_i > x_i/x \; i=1..N\}$ prevail. These conjectures lead to allocations of effort that lie below the ones in Nash equilibrium. In contrast, when the firm anticipates that rivals will contract effort, conjectures $\{\hat{\theta} < x_i/x \; i=1..N\}$ prevail, and they generate allocations that lie above the ones in Nash equilibrium.

In oligopoly a similar pattern emerges. Here, conjectures above Cournot-Nash lead to less output being supplied, and to equilibria that are, in this sense, more collusive. Conjectures below Cournot-Nash lead to greater levels of output and to equilibria that are more competitive. Similar interpretations prevail when there is an externality in consumption. However, in the case of public-goods the situation is reversed. Here, a conjecture above Nash leads to more of the public good being supplied, and to less when the converse occurs. It follows that, with Nash as a dividing line, less collusive outcomes lead to commonly observed phenomena, such as overcrowding common property and under-provision of public goods, while more collusive equilibria mitigate these effects.

Indexing Inefficiency in the Market Allocation

The expression on the right side of (5) will reappear at an important, subsequent juncture. We will refer to it as the function $\Gamma(\cdot)$ and assume $\Gamma(\cdot) \in (0,\infty)$. The range follows naturally from acknowledging that the shares of public effort are strictly positive fractions, and that the conjectures are defined over the set of open intervals $\{\hat{\theta}_i \in (0, \infty)\ i = 1..N\}$.[5] Hence, $\Gamma(\cdot)$ approaches its extrema when the conjecture of a single individual approaches its extrema. Specifically, when a conjecture approaches zero from above, $\Gamma(\cdot)$ approaches positive infinity. In the common-property, pollution, and oligopoly scenarios there is over-allocation of effort; in the public-goods setting there is under-allocation. Conversely, when a single conjecture approaches infinity, $\Gamma(\cdot)$ approaches zero from above. In the public-goods setting there is over-allocation of effort; in the other situations there is under-allocation. Therefore, comparing equations (3) and (5), it is convenient to interpret $\Gamma(\cdot)$ as an index of inefficiency in the market allocation when agents have conjectural variations—but several caveats should be kept in mind.[6] First, a better index may exist. Locating such a measure would require a detailed exploration into the theory of index numbers and, thus, lies outside the scope of the present discussion. Nevertheless, it would seem desirable in any extension of this work to construct a money metric of the welfare loss associated with alternative allocations. This measure should parallel conventional notions of welfare change such as equivalent and compensating variations in income. The latter measures, we should note, are cardinal; the function $\Gamma(\cdot)$ is, of course, ordinal. As an ordinal measure, $\Gamma(\cdot)$ is unable to distinguish between situations in which the conjectures of one or more agents approach extreme values. Presumably, the extent of misallocation depends on the number of agents who conjecture extreme values. Another limitation arises from the fact that $\Gamma(\cdot)$ is monotonically increasing over its domain. Hence, it is unable to assign a preference ordering for misallocations of effort on either side of the efficient level, which is $\Gamma(\cdot) = 1$. This need, of course, is negated in Nash equilibrium, which, recall, may be simulated through conjectures $\{\hat{\theta}_i = x_i/x\ \ i=1..N\}$. In this case, the lower bound for $\Gamma(\cdot)$ is the efficient level and, hence, it follows that the extent of inefficiency in the market allocation rises monotonically with increases in $\Gamma(\cdot)$.

With these caveats in mind, we will interpret $\Gamma(\cdot) \in (0,\infty)$ as an ordinal measure of inefficiency in the market allocation. Different modes of conduct yield alternative measures in $\Gamma(\cdot)$ and, without

5 The lower-bound on the domains of the conjectures is readily motivated with reference to the oligopoly example, and by rewriting the first-order conditions

$$p\,(1+\varepsilon\,\hat{\theta}_i) - \partial c_i(x_i|\sigma_i)\,/\,\partial x_i = 0,\ \ i = 1..N$$

where $\varepsilon \equiv \partial D(x)\,/\,\partial x)(x\,/\,p) \in (-\infty,0)$ denotes the price flexibility of demand (see, for example, Appelbaum, 1982). In this case, restricting attention to conjectures $\{\hat{\theta}_i \in (0,\infty)\ \ i = 1..N$ is consistent with firms pricing above marginal costs.

6 I thank one of the reviewers for drawing my attention to an error in a previous draft.

further specialization of the conjectures, little more can be said. Three particular specifications are, however, noteworthy. The first is the value obtained in Nash equilibrium, which we have already discussed; the second is the value observed when there is concordance among the beliefs of all agents; and the third is the value derived from a simple transformation of the Nash conjectures.

As an alternative to Cournot-Nash, suppose that agents have homogeneous beliefs. In other words, suppose that there exist concordances of the form $\{\hat{\theta}_i = \hat{\theta} \; i=1..N\}$. How such a situation comes to be realized, we leave unanswered for the moment. Inserting these conditions into equation (5), and using the fact that the shares sum to one, the value $\hat{\theta}^{-1}$ is obtained. It follows, therefore, that the extent of inefficiency in the market allocation depends on the extent to which this common conjecture departs from the value one.

Finally, consider a transformation in which agents conjecture values that are N times the ones in Nash equilibrium. That is, suppose conjectures $\{\hat{\theta}_i = Nx_i/x \; i=1..N\}$ prevail. Inserting these conditions into the right side of (5), we find that these conjectures yield efficient allocations. Later we show that, under certain circumstances, the community converges to this particular set of conjectures.

Summary

This discussion highlights the importance of three features of the equilibrium that affect efficiency in the market allocation. The first is the distribution of the set of private activities, $\{x_i/x \; i=1..N\}$; the second is the distribution of the set of private beliefs, $\{\hat{\theta}_i \; i=1..N\}$; and the third is the number of agents that generate the externality. The extent of departure from an efficient allocation depends on the magnitude of each agent's conjecture, and the function $\Gamma(\cdot)$ provides an ordinal ranking of inefficiency between allocations. The condition

$$(6) \qquad \sum_{i=1}^{N} \frac{x_i / x}{\hat{\theta}_i} \equiv \Gamma(\cdot) = 1,$$

characterizes efficient allocations. In the common-property, pollution, and oligopoly situations, the condition $\Gamma(\cdot) > 1$ implies over-allocations of resources; in the case of public goods, it implies under-allocations. The situation is reversed whenever the condition $\Gamma(\cdot) < 1$ prevails. Figures 2a and 2b illustrate optimal, Nash-equilibrium, and arbitrary conjectural-variations allocations.

5. *The Conjectural-Variations Congestion Game*

The interesting feature of conjectural variations is that they can predict any type of behavior between pure competition and complete

FIGURE 2
Nash-equilibrium and Conjectural-Variations Allocations

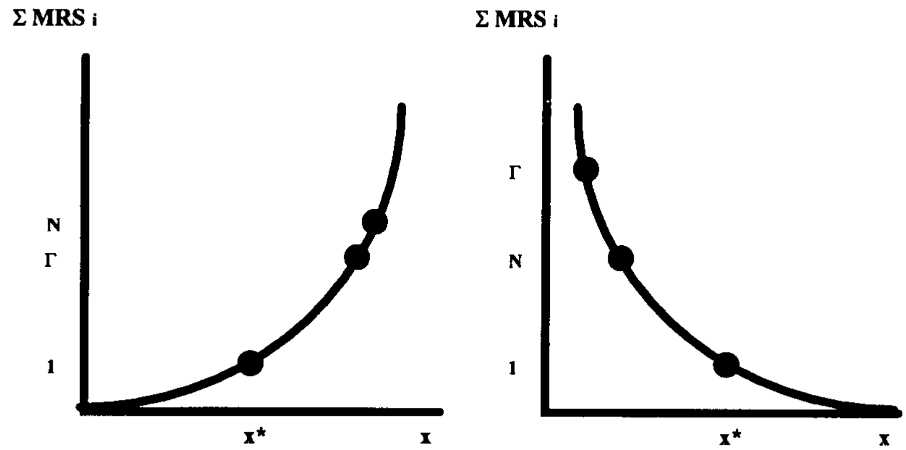

Figure 2a. Common property, pollution, and oligopoly allocations.

Figure 2b. Public goods allocations.

collusion. This of course—if nothing else—is the redeeming feature of the model, but it also leads to an 'embarrassment of riches'. Which of the many possible outcomes will prevail in equilibrium? In other words, what are the values of the conjectures, $\{\hat{\theta}_i \ i=1..N\}$, and the shares in public effort, $\{x_i/x \ i=1..N\}$, that are consistent with equilibrium?

To answer this question we will use the concept of *consistent conjectures*. That is, given a set of observed responses, $\{\hat{\theta}_i \ i=1..N\}$, about which the agents form conjectures, $\{\hat{\theta}_i \ i=1..N\}$, we will use the conditions $\{\hat{\theta}_i = \theta_i \ i=1..N\}$ to identify equilibrium.

The Traditional Approach

The consistent-conjectures problem received a good deal of attention in the oligopoly literature in the 1980s. However, both a general characterization of the problem, and a general solution to it, proved elusive. The proposed methodology is to compute the response between rival firms, *ceteris paribus*, by applying the implicit function theorem to one of the firm's first-order conditions. Unfortunately, this procedure is flawed. The reason stems from an independence engendered by the conjectures and, specifically, from the definitions $\{\chi_{ij}(x_i)\ j=1..N\ j\neq i\}$. Since agent i has *conjectures* about the action of each of its rivals, the *levels* $\{x_j\ j=1..N\ j\neq i\}$ no longer appear as arguments in agent i's objective function. Consequently, the first-order conditions are also independent of the actions of rivals, and it follows naturally from this simple observation that the implicit-function theorem is inapplicable when firms have conjectural variations. More generally, it is this aspect of the paradigm that represents its main departure from traditional Cournot-Nash. Authors of the modern literature appear unaware of this problem. However, an early contributor— Stackelberg—was well aware of the independence engendered by the conjectures. In reviewing the original theory proposed by Bowley he notes:

> 'The Mathematical Groundwork of Economics' ... talks in the end of the paragraph on 'several manufacturers, one commodity' (i.e., a 'supply oligopoly') about the duopoly of supply. There, each duopolist looks at the quantity supplied by his competitor as being dependent on his own quantity supplied, i.e., as a function of his own quantity supplied. This function 'depends on what each producer thinks the other is likely to do.' It is obvious that in reality both functions cannot exist simultaneously. Here, each duopolist has the 'position of independence.' That is, the market situation described by Bowley is the form of duopoly that we have already analyzed as 'Bowley's Duopoly'. ... After all, priority for the idea of 'quantity independence' has to be given to Bowley, and therefore we have named the market situation where oligopolists strive for 'quantity independence' after Bowley (pp. 82-83).
>
> *—Heinrich von Stackelberg, Marktform und Gleichgewicht, 1934. (Translation by Barbara Hegenbart.)*

This feature of conjectural variations is important, historically, because it led Stackelberg toward his solution to the duopoly problem—a solution that he proposed as an alternative to the competing theories of Cournot and Bowley. But the independence engendered by the conjectures is important for another reason: It explains why traditional methodology cannot be applied. Traditional methodology requires us to perturb the private action of a rival in the first-order condition of another agent when only the private action of this latter agent appears. Any approach that perturbs the action of a rival in the first-order condition of another agent is inconsistent with the conjectural-variations paradigm. An alternative methodology is required.[7]

A Traditional Qua Non-Traditional Approach

Although he does not pursue it, an alternative procedure, suggested by Bresnahan, is more akin to traditional comparative statics:

> The comparative statics of equilibrium give firms enough information to recover one anothers' behavior. Suppose that some variable exogenous to the oligopoly (say, costs, the location of the demand curve) is changed. Equilibrium prices and quantities will change whatever the nature of the equilibrium concept. Suppose that firms learn nothing about one another from the dynamic process from which the new

7 This feature of the theory can be observed more explicitly from the first-order conditions presented in the Appendix.

equilibrium is obtained. They can still learn one anothers' reactions from the location of the new equilibrium ... The natural experiment, the movement of exogenous variables, can reveal to firms that their conjectures are inconsistent. This argument does not depend in any critical way on the belief that the equilibrium comparative statics 'actually happen' ... If firms have inconsistent conjectures and it is possible for them to learn how their industry reacts to exogenous shocks, they will learn that their conjectures are wrong. If they have consistent conjectures, nothing in the comparative statics of equilibrium will reveal those conjectures to be wrong. By what dynamic process the conjectures will come to be consistent is an unsolved problem, as is the possibility of an informationally consistent, stable dynamic for oligopoly prices and quantities (pp. 942-43).

In terms of Problem 1, these comments are interpreted as follows: Given the set of characteristics, $\{\sigma_i \; i=1..N\}$, which are exogenous, equilibrium values for the private actions, $\{x_i \; i=1..N\}$, and the public variable, x, are determined by a system of N+1 equations consisting of the N first-order conditions of each of the agents and the aggregation condition, (1). When the characteristics change, we compute the resulting change in each of the private actions and, subsequently, through (1), compute the corresponding adjustment in the public variable. When the observed adjustment in the public variable corresponds to the ones conjectured in the initial equilibrium we say that the conjectures are consistent.

A more formal definition is facilitated by expressing adjustments in proportional-change terms.[8] For some variable, say v, let $\tilde{v} \equiv \Delta v / v$ denote a change relative to its value in initial equilibrium. Accordingly, we study proportional adjustments in the private actions, $\{\tilde{x}_i \; i=1..N\}$, and in the public variable, $\tilde{x}$, in response to proportional adjustments in the characteristics, $\{\tilde{\sigma}_i \; i=1..N\}$. Using (1) and the N first-order conditions (see Appendix), the set of endogenous adjustments are determined as solutions to the N+1 equation system:

$$(7) \qquad \tilde{x} = \sum_{i=1}^{N} \frac{x_i}{x} \tilde{x}_i \, ,$$

$$(8) \qquad \eta_i \tilde{x}_i + \mu_i \tilde{\sigma}_i = 0 \qquad i = 1..N;$$

where the effects $\eta_i \equiv x_i \partial\xi_i(\cdot) / \partial x_i \equiv \eta_i(\hat{\theta}_i)$ and $\mu_i \equiv \sigma_i \partial\xi_i(\cdot) / \partial\sigma_i \equiv \mu_i(\hat{\theta}_i)$ depend, implicitly, on the conjectures; and the function $\xi_i(\cdot)$—defined in the Appendix—denotes the first-order partial derivative of agent i's objective function. Normalizing repeatedly on the set of private adjust-

8 This technique, which is frequently employed by agricultural economists, is referred to as 'displacement modelling'. For an interesting discussion and list of applications see Piggott (1993).

ments, the observed effects about which the agents form conjectures are the set of ratios $\{\theta_i \equiv \tilde{x} / \tilde{x}_i \;\; i=1..N\}$. Note, of course, that the individual adjustments are conditioned, through (8), by the values of each agent's conjecture. Accordingly, a consistent-conjectures equilibrium is a set of adjustments in the public and private actions that satisfy equations (7) and (8); and a set of conjectures, $\{\hat{\theta}_i \;\; i=1..N\}$, that solve the fixed-point problem:

$$(9) \qquad \hat{\theta}_i = \frac{\tilde{x}(\hat{\theta}_1..\hat{\theta}_N)}{\tilde{x}_i(\hat{\theta}_i)} \;\; i=1..N.$$

The Equilibrium Concept

Before initiating the search for equilibrium, a few comments are in order. The first observation is that the equilibrium is determined by comparative-static experiment. Consequently, unlike conventional Nash equilibrium, the consistent-conjectures equilibrium is not static—but nor is it dynamic. While it is true that confirmation of the conjectures can only occur through repeated experimentation, these experiments need not occur through time. They could, conceivably be made in cross-sections of the population that are deemed to possess the same market structure.[9] A more formal indication of the non- dynamic structure of the model is available from the objective functions of the agents (Problem 1). They optimize with reference to only two states; one is the current static situation and the other is the ensuing comparative-static phase that emanates from the initial state. The agents do not set up and solve a dynamic optimization problem defined over an infinite horizon, as they would in an extensive-form or evolutionary game. In comparison to these benchmarks, conjectural-variations agents are myopic.[10]

This feature of the model is worth emphasizing because it is essentially this aspect of the theory that is unconventional and controversial. When consistency is sought, there are now two phases to the game. In the first stage, agents form conjectures and take private actions, conditional on the responses they expect from their rivals. This part of the equilibrium is encompassed by equation (1) and the N first-order conditions presented in the Appendix. The subsequent stage—the comparative-static phase— is depicted by equations (7) and (8). When adjustments occur, agents observe rivals' responses, and they compare these responses to the ones they conjectured in the initial equilibrium. When the conjectures and the adjustments conform, we say that conjectures are *consistent*.

9 This point is particularly relevant to empirical work in which displacement modelling is used. Wohlgenant (1989) provides a nice example. In this context, time series are usually employed. However, pooled time series or cross-sectional data are equally applicable provided that the 'controls' are consistent across the experiments.

10 I am grateful to Tim Besley for beneficial discussions on this point.

The Fixed-Point Problem

The formidable task in identifying equilibrium is solving the fixed-point problem posed by equations (9). It is precisely a similar computation that has proved so difficult in traditional searches for consistent conjectures. In the traditional methodology, authors have sought direct solutions to this problem. Its inherent intractability, however, has forced them to impose a number of restrictive assumptions, for example, that their are two agents within the community (e.g., Bresnahan, 1981), or that all agents are identical (e.g., Perry, 1982). In this regard, equations (9) contain two key features. First, they are not readily amenable to further manipulation. Second, they identify conditions that are both *necessary* and *sufficient* for equilibrium.

The strategy adopted below is to identify a set of conditions that are only *necessary* for consistency, but which are more amenable to further manipulation, and which lead ultimately to the main conclusions of the paper. Since these conditions are only necessary, they admit a potentially larger set of conjectures than the ones we ultimately seek. Therefore, in the parlance of econometrics, we term these conditions *admissibility criteria* and the conjectures that meet them *admissible conjectures*. Subsequently, we use experiment to determine a set of sufficient conditions under which consistency prevails.

Admissible Conjectures

The notion of admissibility that we will use is that agents make correct conjectures about aggregate adjustments, but that these adjustments need conform only to equation (7). There is no economic intuition underlying this restriction; it is merely a technical detail that facilitates the search for equilibrium. By ignoring equations (8) we incur a cost, but accrue a significant benefit. The cost is that we are unable to derive explicit values for the conjectures. This however, should not concern us, for two reasons. First, it is clear that, without further specialization of the objective function in Problem 1, the derivatives in (8) have no explicit form. Consequently, no solution is apparent without compromising the current level of generality. The second reason, which was alluded to earlier, is that the particular values of the conjectures that make the equilibrium consistent have no bearing on the main result of the paper. Accordingly, an admissible-conjectures equilibrium is a set of adjustments in the private actions, $\{\tilde{x}_i \ \ i = 1..N\}$, and an adjustment in the public action, $\tilde{x}$, that satisfy (7); and a set of conjectures, $\{\hat{\theta}_i \ \ i=1..N\}$, that satisfy:

$$(10) \quad \hat{\theta} = \frac{\tilde{x}}{\tilde{x}_i} \quad i = 1..N.$$

Note the essential difference between the conditions in (9) and (10). The definition in (10) is much weaker than the one in (9). In (10) we require only that the observed adjustments must correspond to the counting rule that relates private and public activity. This requires only that there is accurate recording of private and public actions. It imposes no restrictions across the responses of each agent. Consequently, conjectures that satisfy (9) must also satisfy (10), but not the converse. Hence, admissibility is necessary, but is not sufficient for consistency.

6. *Search*

A single restriction evolves in four steps. The first consists of rewriting the N criteria in (10) in the form:

$$(11)\quad \tilde{x} = \hat{\theta}_i \tilde{x}_i \qquad i = 1..N.$$

The second is to combine these equations with the one in (7) to form the system:

$$(12)\quad \Psi\tilde{\mathbf{x}} = \mathbf{0} \quad ,$$

where, $\tilde{\mathbf{x}} \equiv (\tilde{x}, \tilde{x}_1 .. \tilde{x}_N)^T$, $\mathbf{0}$ denotes the N+1 null vector, and Ψ is defined

$$(13)\quad \Psi \equiv \begin{pmatrix} 1 & -x_1/x & -x_2/x & .. & -x_N/x \\ 1 & -\hat{\theta}_1 & & .. & \\ 1 & & -\hat{\theta}_2 & .. & \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & & & .. & -\hat{\theta}_N \end{pmatrix}$$

The third step is to observe that the system in (12) is homogeneous. Consequently, Ψ must be singular in order to force non-trivial solutions $\tilde{\mathbf{x}} \neq \mathbf{0}$. Trivial solutions render the entire exercise uninteresting. Invoking the condition $|\Psi| = 0$ we obtain

$$(14)\quad \prod_{i=1}^{N} -\hat{\theta}_i + \sum_{i=1}^{N} x_i/x \prod_{j \neq i}^{N} -\hat{\theta}_j = 0\,.$$

Now, suppose we permit zero-valued conjectures. Then, inserting into (14) the set of conditions $\{\hat{\theta}_1 = 0 \;\; i=1..N\}$ we see that the so-called competitive conjectures satisfy the necessary condition to be consistent. Indeed, from (14) it is easy to see that, as long as there are at least two agents that act competitively, the restriction in (12) is satisfied. A set of strictly competitive conjectures, however, can never be consistent. The mathematical argument is complicated; it is detailed in Holloway (1995). The key recognition is that, in the empirically relevant case of finite adjustments across agents, we require the net adjustment to be zero. Clearly, agents must adjust effort in opposite directions. This, however, implies a certain form of heterogeneity within the community which, in any practical situation, seems nonsensical. To illustrate, consider a duopoly. Suppose the location of the demand schedule changes. Each firm will adjust output. If the adjustments exactly offset each other then the net adjustment is zero. Hence if the firms conjectured zero in the original equilibrium, their conjectures would be consistent. There are two problems with this. First, it is difficult to construct a set of demand and marginal-cost configurations and a set of conjectures that would cause one firm to rationally expand and another to rationally contract output. Second, if the two firms actually behaved in a competitive manner (as opposed to having conjectures that synthesized the competitive situation) the industry would almost certainly expand output.

The first problem relates to the fact that, in any reasonable equilibrium, agents must be somewhat similar. Here, 'similar' means that the adjustments must occur in the same direction. This, of course, excludes a net adjustment of zero and, with it, the so called competitive conjecture. The second problem relates to the fact that, while it may seem reasonable, moreover useful, to synthesize competition through a set of zero-valued conjectures, the comparative statics that emanate from such an equilibrium are almost certainly different from the ones that occur when the agents consider market variables (say, price) to be parametric. This, of course, says nothing more than different economic models generate different qualitative effects. But when such effects are needed to confirm the consistency of conjectures, a good deal of care should be exercised.

Restricting attention to the set of strictly positive real numbers $\{\tilde{\theta}_i > 0 \ i=1..N\}$, we can derive a transformation of equation (14) that is more amenable to visual inspection. Dividing through with the product $\prod_{i=1}^{N} \hat{\theta}_i \neq 0$, we obtain[11]

$$(15) \quad \sum_{i=1}^{N} \frac{x_i/x}{\hat{\theta}_i} = 1.$$

Every set of conjectures that satisfies this condition is admissible as a consistent-conjectures equilibrium.

7. *Admissibility, Consistency and Pareto-Efficiency*

Two important conclusions follow from comparing equations (15) and (6), namely:

LEMMA: Admissible-conjectures allocations are Pareto-efficient.
COROLLARY: Consistent-conjectures allocations are Pareto-efficient.

The lemma follows from the observation that equation (6)—the condition characterizing efficiency in the conjectural-variations allo-

11 Note in (14) the change of sign between products of terms in N and N-1, respectively.

cation—and equation (15)—the condition characterizing admissible-conjectures equilibria—are the same. The corollary follows from the fact that admissibility is necessary for consistency.

The policy implications of these two results are striking. They imply, for consistent-conjectures economies, the absence of grounds for intervention. When agents conjecture consistently, their actions engender efficiency in the market allocation. An 'expectation-internalizing effect' evolves from a special form of foresight among the agents. When each predicts accurately the behavior of the group, a form of self enforcement prevails in which agents internalize the externality that stems from congestion. In this way, they make decisions on behalf of the community as a whole and take actions accordingly. When their conjectures are correct, each agent's action replicates those of a social planner and it follows, naturally, that when conjectures are consistent the effects of the externality are completely internalized. In the allocation of common-property, in the generation of pollution, in the private provision of public goods, and in oligopoly, the desire to be consistent yields efficient outcomes.

8. *Repeated Play, Learning and Convergence*

Since admissibility is only necessary for consistency an important question now arises: In which types of communities are consistent conjectures likely to prevail? To shed light on this issue, this section presents the results of experiments which suggest that, when agents are few, when they are not too dissimilar, and when they have the opportunity to observe each others' actions during repeated experiment, consistent conjectures are likely to evolve.

The experiments have the following format. In an initial period we endow agents with shares of the public variable and a corresponding set of Nash conjectures. The endowments and, thus, the conjectures are selected at random from a uniform distribution. We then consider a sequence of iterations, $\{t\ t=0...T\}$, in which agents receive a common shock, $\tilde{\sigma}(t)$, and make adjustments in each of their private actions. These adjustments are conditioned by a set of conjectures formed at the beginning of each period, namely $\{\hat{\theta}_i(t)\ i=1..N\}$. At the end of each period the agents compare their conjectured responses to the set of observed responses, $\{\theta_i(t)\ i=1..N\}$, and they update their conjectures accordingly. At the beginning of period t+1 the firms receive another shock, $\hat{\sigma}(t+1)$, and the game is repeated until convergence, if ever, is achieved.[12]

12 The experiments were executed in Mathematica™, Version1.2/33 enhanced, on a Macintosh Power PC. The programmes are available from the author.

The experiments are conducted over four dimensions. Respectively, we consider variations in (a) the number of agents within the community, (b) the degree of asymmetry in the initial distribution of effort, (c) the relative magnitude of the shocks, and (d) the learning rule by which agents update their conjectures. Convergence rests crucially on the transition of the function

$$(16)\quad \Gamma(t) \equiv \sum_{i=1}^{N} \frac{x_i(t)/x(t)}{\hat{\theta}_i(t)},$$

which, in turn, depends on the transition of the conjectures in relation to the shares of public effort. In this regard, it is informative to consider whether the industry has become more or less 'concentrated' as the iterations progress. Several indices exist for this purpose, but the index

$$(17)\quad f(t) \equiv \prod_{i=1}^{N} Nx_i(t)/x(t),$$

is desirable for two reasons. First, it is conveniently defined over the unit interval. It has a well defined maximum at the value one, which occurs when the distribution is symmetric, and approaches, asymptotically, its minimum of zero whenever the share of any single agent becomes negligible. Second, this index is stable whenever the shares of each of the agents are stable, but it adjusts whenever a single share adjusts. Consequently, the sequence $f(t)$ t=0..T proves convenient as a dual indicator of concentration and stability of the equilibrium.

Another measure, which proves useful in relation to the former one, is the index

$$(18)\quad g(t) \equiv \prod_{i=1}^{N} \hat{\theta}_i(t).$$

The two indices converge whenever the conjectures converge to the set of values that are N times the so-called Nash conjectures. That is, the right sides of (17) and (18) are identically equal whenever $\{\hat{\theta}_i(t) = Nx_i(t)/x(t) \quad i = 1..N\}$.

With these indices at hand, we consider convergence under three types of learning scheme. The first is the simple cobweb rule due to Ezekiel (1938):

$$(19)\quad \hat{\theta}_i(t) = \theta_i(t-1) \quad i = 1..N.$$

Figure 3 presents an example for a community of two individuals. Under this scheme learning occurs instantaneously (Figure 3a) and the community converges to conjectures N times the Nash conjectures (Figure 3b) after only a few iterations. Note, also, that the distribution of output remains asymmetric between the two agents. That is, the sequence $f(t)$ t=0..T (in Figure 3b) lies considerably below the maxi-

FIGURE 3

Convergence Under Hindsight

$\{\hat{\theta}_i(t) = \theta_i(t-1) \;\; i = 1..N\} \quad (N = 2)$

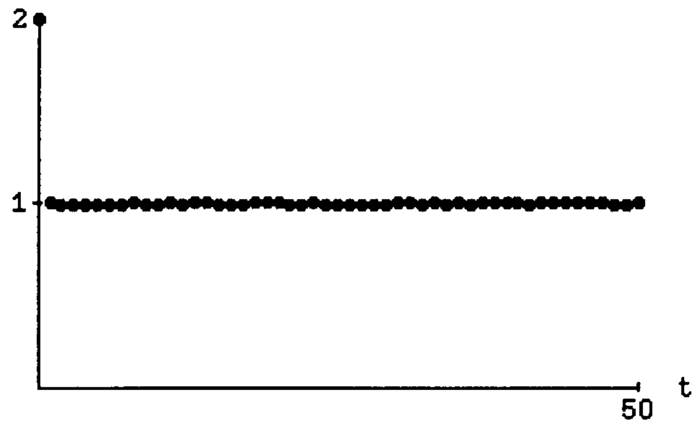

Figure 3a. — $\Gamma(t)$

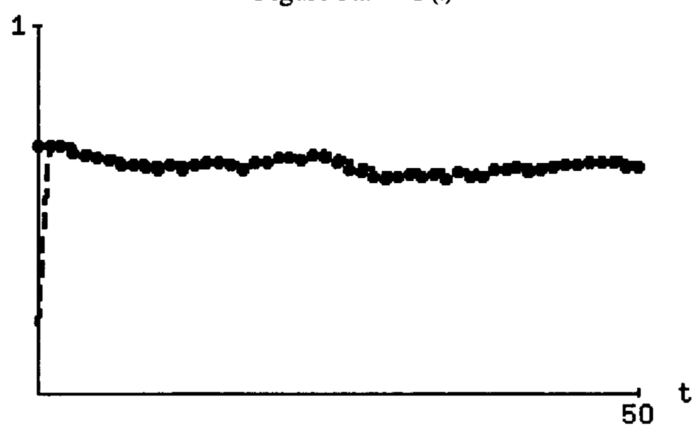

Figure 3b. — $f(t)$ • • and $g(t)$ --

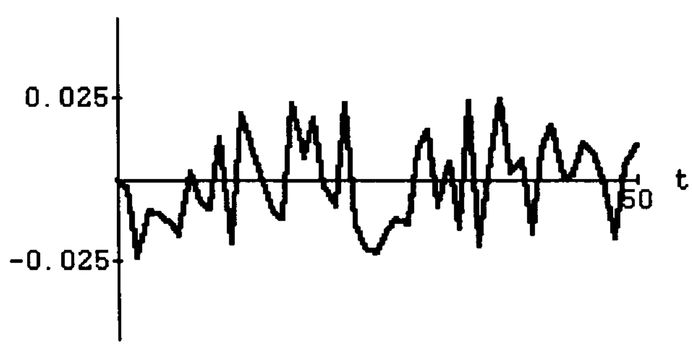

Figure 3c. — $\tilde{\sigma}(t)$

mum of one, obtained in symmetric equilibrium. Similar patterns emerge from experiments with five- and ten-agent communities.[13]

Next, we consider learning based on the simple mean of past histories:

$$(20)\quad \hat{\theta}_i(t) = \frac{1}{t}\sum_{s=0}^{t}\theta_i(s), \quad i = 1..N\ .$$

Figure 4 presents an example for a community of five agents. Like the results under the cobweb rule in (19), convergence is achieved instantaneously (Figure 4a) at which point the indices in (17) and (18) converge. Successive shocks cause the indices to diverge before, once again, converging at around 50 iterations. Similar patterns emerge within two- and ten-agent communities.

The third rule considered is adaptive expectations (Nerlove, 1958):

$$(21)\quad \hat{\theta}_t(t) = \hat{\theta}_i(t-1) + \alpha(\theta_i(t-1) - \hat{\theta}_i(t-1)),\ 0 < \alpha < 1,\ i = 1..N\ .$$

This rule updates conjectures by some proportion (α) of last period's forecast error. Figures 5 and 6 present experiments for two ten-agent communities, in which the adjustment coefficients assume the respective values α=0.1 and α=0.5. In the first case, the initial distribution of shares is quite asymmetric and convergence is rather sluggish. In the second example, the initial distribution of the shares is more symmetric and convergence is smooth and direct.

Additional experiments were conducted in order to assess the sensitivity of the results to three assumptions, namely the distribution of the shares in the initial period; the sign and magnitude of the shocks; and the number of agents exploiting the public domain. Accordingly, the initial endowments were made successively less symmetric; the shocks were made successively larger and were accorded particular sign patterns; finally, agent numbers were increased. Although convergence occurs in many situations, exceptions arise. Generally speaking, convergence appears to be unattainable when agents are numerous—typically, 20 or more; when the shocks that cause the perturbations are relatively large or of the same sign; and, in some cases, when the initial distribution of output is highly asymmetric. In all other situations the community approaches a stable equilibrium in which agents' conjectures converge to the set of conjectures N times the ones that simulate Nash equilibrium.

These results are important because they provide insight into the conditions under which efficient allocations arise. For this reason we should be careful to articulate an important feature of the equilibrium, that could go unnoticed. This is that, except for the adjustments among

13 The results of additional experiments are available upon request.

FIGURE 4

Convergence Under Means of Past Histories

$$\{\, \hat{\theta}_i(t) = \frac{1}{t}\sum_{s=0}^{t-1} \theta_i(s) \;\; i = 1..N \,\} \quad (N = 5)$$

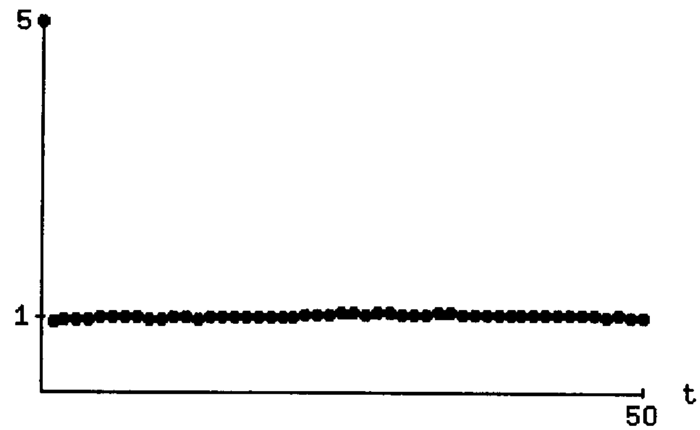

Figure 4a. — $\Gamma(t)$

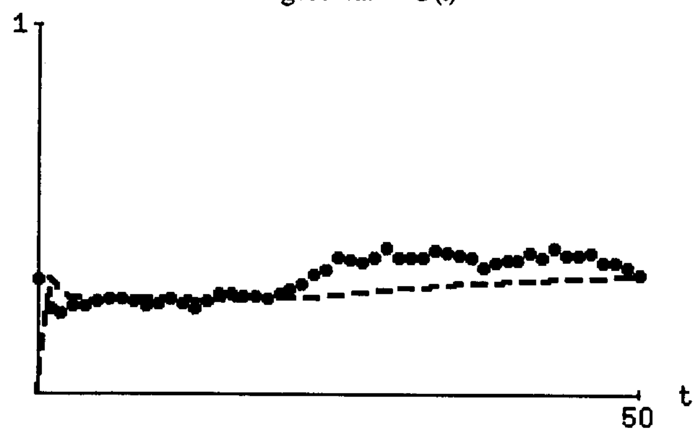

Figure 4b. — $f(t)$ • • and $g(t)$ --

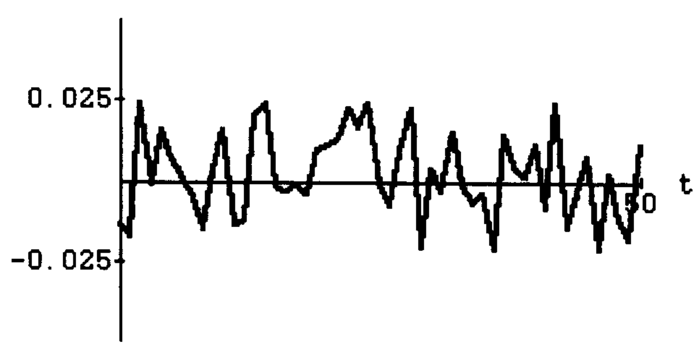

Figure 4c. — $\tilde{\sigma}(t)$

FIGURE 5

Convergence Under Adaptive Expectations

$$\{\hat{\theta}_i(t) = \{\hat{\theta}_i(t-1) + \alpha(\theta_i(t-1) - \hat{\theta}_i(t-1)) \;\; i = 1..N\} \quad (N = 10, \alpha = 0.1)$$

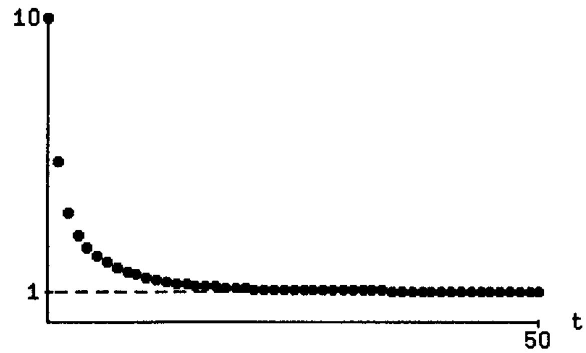

Figure 5a. — $\Gamma(t)$

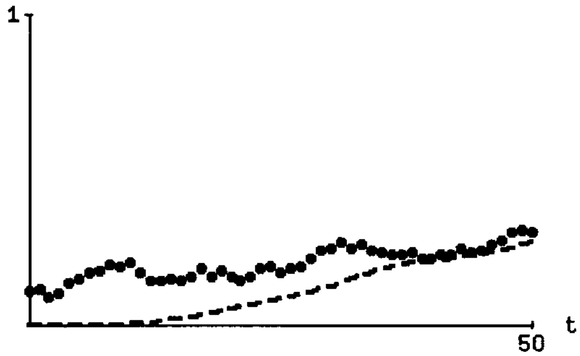

Figure 5b. — $f(t)$ • • and $g(t)$ --

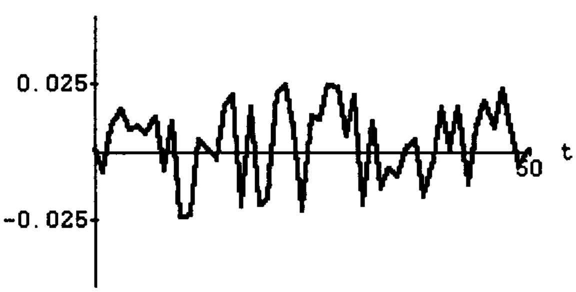

Figure 5c. — $\tilde{\sigma}(t)$

FIGURE 6

Convergence Under Adaptive Behavior

$\{\hat{\theta}_i(t) = \{\hat{\theta}_i(t-1) + \alpha(\theta_i(t-1) - \hat{\theta}_i(t-1)) \;\; i = 1..N\} \quad (N = 10, \alpha = 0.5)$

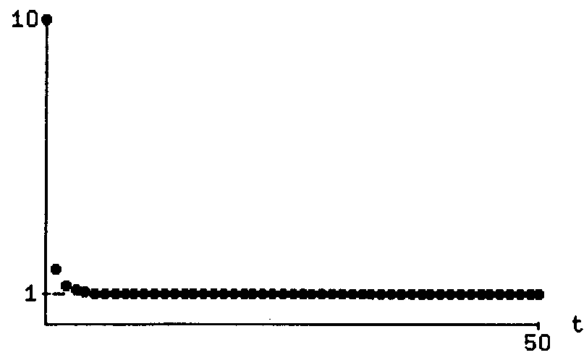

Figure 6a. — $\Gamma(t)$

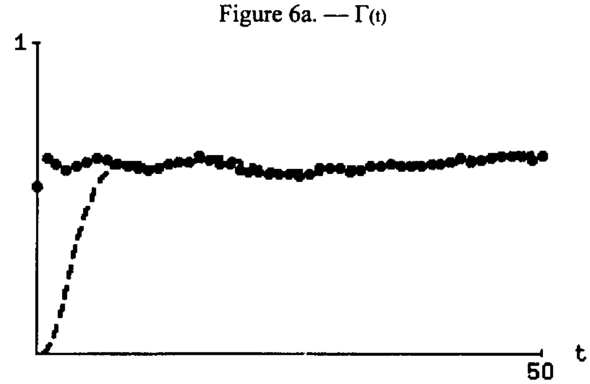

Figure 6b. — $f(t)$ • • and $g(t)$ --

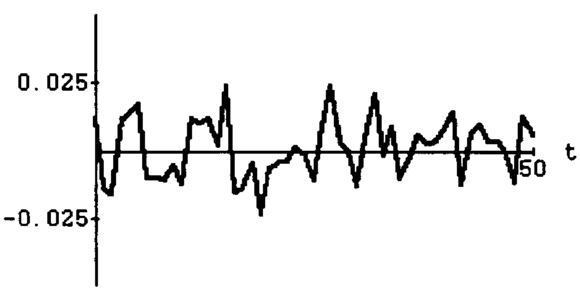

Figure 6c. — $\tilde{\sigma}(t)$

rivals in the cohort, the information structure is complete. This means, for example, that firms under oligopoly have precise knowledge about costs and demand, and changes that occur in these structural components between successive equilibria. This is a strong assumption. That the results are sensitive to its relaxation is open to question. Nevertheless, when the information structure is complete the results provide compelling evidence to suggest that consistency is attainable. Specifically, when numbers of agents are few, when they are not too heterogeneous, and when they have the opportunity to observe each another's actions during repeated experiments, it is possible, indeed probable, that an efficient allocation emerges.

9. *Concluding Comments*

An important feature of many environments is the presence of a cumulative externality. Market inefficiencies that stem from this phenomenum are exacerbated by myopic behavior. This leads to standard policy prescriptions, such as Pigouvian taxation or subsidization, and to regulation of entry. This paper has investigated the implications of an equilibrium concept, which provides a tractable alternative to the Nash equilibrium and has received a good deal of attention in the literature, but has not been formalized. It is the solution concept known as consistent conjectures. It leads to a key result that appears to have gone largely unnoticed to date, namely that, when agents have consistent conjectures, they take actions that lead to an efficient allocation of resources.

Whether this finding is applicable to a broader set of circumstances remains open to question. Clearly, the predictions of the theory appear at odds with many open-access situations such as the spawning of deserts, exploitation of aquatic resources, and global warming. Nevertheless, the results of the experiments suggest that the circumstances surrounding the congestion of some public domains deserves further study. In the allocation of common property, in the generation of pollution, in the private provision of public goods, and in oligopoly, the desire to be consistent leads to outcomes that may be more efficient than we had imagined previously. At the very least, the empirical validity of this conclusion deserves closer scrutiny.

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Appendix

Efficient Allocations

A set of efficient allocations is a set of private actions $\{x_i\ i=1..N\}$ that solve:

$$\text{Problem 2:}\ \begin{cases} \max\limits_{x_i}: & \upsilon_i = \Phi_i(x_i, x, \sigma_i) \quad i=1..N \\ \text{subject to:} & \\ & x = \sum\limits_{j=1}^{N} x_j \\ & \Phi_j(x_j, x, \sigma_j) \geq \overline{\upsilon}_j \quad j=1..N\ \ j\neq i \end{cases}$$

where the values $\{\overline{\upsilon}_j\ j=1..N\ j\neq i\}$ are arbitrary. After substitution, the Lagrangean can be written:

$$L(x_i\ i=1..N; \lambda_j\ j=1..N\ j\neq i) \equiv \Phi_i(x_i, \sum_{j=1}^{N} x_j, \sigma_i) + \sum_{j\neq i}^{N} \lambda_j(\Phi_j(x_j, \sum_{i=1}^{N} x_i, \sigma_j) - \overline{\upsilon}_j),$$

where $\{\lambda_j\ j=1..N\ j\neq i\}$ denotes a set of multipliers on the second set of constraints. The corresponding Kuhn-Tucker conditions are:

$$\partial L(\cdot)/\partial x_i \equiv \partial\Phi_i(\cdot)/\partial x_i + \partial\Phi_i(\cdot)/\partial x + \sum_{j\neq i}^{N} \lambda_j \partial\Phi_j(\cdot)/\partial x \leq 0,$$
$$x_i \partial L(\cdot)/\partial x_i = 0,$$

$$\partial L(\cdot)/\partial x_j \equiv \partial\Phi_i(\cdot)/\partial x + \lambda_j \partial\Phi_j(\cdot)/\partial x_j + \sum_{j\neq i}^{N} \lambda_j \partial\Phi_j(\cdot)/\partial x \leq 0,$$
$$x_j \partial L(\cdot)/\partial x_j = 0, \qquad j=1..N\ j\neq i;$$

$$\partial L(\cdot)/\partial \lambda_j \equiv \Phi_j(\cdot) - \overline{\upsilon}_j \geq 0,$$
$$\lambda_j \partial L(\cdot)/\partial \lambda_j = 0, \qquad j=1..N\ \ j\neq i.$$

A maximum is ensured if the functions $\{\Phi_i(\cdot)\ i=1..N\}$ are concave in the set of private actions. Confining attention to interior solutions, define

$$\sum_{j\neq 1}^{N} \lambda_j \partial\Phi_j(\cdot)/\partial x \equiv \Omega,$$

so that the first N conditions can be rewritten

$$\partial\Phi_i(\cdot)/\partial x_i + \partial\Phi_i(\cdot)/\partial x = -\Omega,$$

$$\partial\Phi_i(\cdot)/\partial x + \lambda_j \partial\Phi_j(\cdot)/\partial x_j = -\Omega \qquad j=1..N\ j\neq i.$$

Consequently,

$$\lambda j = \frac{\partial\Phi_i(\cdot)/\partial x_i}{\partial\Phi_j(\cdot)/\partial x_j} \qquad j=1..N\ j\neq i.$$

Rewriting the first constraint in explicit form, and normalizing on the private effect of agent i yields

$$1+\frac{\partial\Phi_i(\cdot)/\partial x}{\partial\Phi_i(\cdot)/\partial x_i}+\sum_{j\neq i}^{N} \lambda_j \frac{\partial\Phi_j(\cdot)/\partial x}{\partial\Phi_i(\cdot)/\partial x_i}=0.$$

Substituting for the multipliers and transposing terms, we have

$$-\frac{\partial\Phi_i(\cdot)/\partial x}{\partial\Phi_i(\cdot)/\partial x_i}-\sum_{j\neq i}^{N} \frac{\partial\Phi_j(\cdot)/\partial x}{\partial\Phi_j(\cdot)/\partial x_j}=1,$$

which, by defining $MRS(x_i,x) \equiv -\partial\Phi_i(\cdot)/\partial x \div \partial\Phi_i(\cdot)/\partial x_i\ i=1..N$, can be rewritten

$$\sum_{i=1}^{N} MRS(x_i,x)=1.$$

Nash-Equilibrium Allocations

Repeated maximization in Problem 1, yields the N conditions that are necessary for a Nash equilibrium, namely

$$\partial\Phi_i(\cdot)/\partial x_i + \partial\Phi_i(\cdot)/\partial x \equiv \phi_i(x_i|x_j\ j=1..N\ j\neq i;\sigma_i)=0 \qquad i=1..N.$$

The functions $\{\phi_i(x_i|x_j\ j=1..N\ j\neq i;\sigma_i)\ i=1..N\}$ make explicit the fact that, when each agent takes its private action, it takes the private actions of each of the remaining agents as given. Using the definition in text equation (2), these conditions can be rewritten

$$MRS(x_i,x)=1, \qquad i=1..N.$$

Summing over the N agents yields the allocation rule

$$\sum_{i=1}^{N} \text{MRS}(x_i, x) = N.$$

Conjectural-Variations Allocations

Conjectural-variations agents solve

$$\text{Problem 3:} \begin{cases} \max_{x_i}: & \upsilon_i = \Phi_i(x_i, x, \sigma_i) \quad i = 1..N, \\ \text{subject to:} & \\ & x = \sum_{j=1}^{N} x_j \\ & x_j = \chi_{ij}(x_i) \qquad j = 1..N \end{cases}$$

where $\{ x_j = \chi_{ij}(x_i) \;\; j = 1..N \}$ denote the conjectures. Using $\{\chi_i(x_i) \equiv \Sigma_j \chi_{ij}(x_i) \;\; i = 1..N\}$, the first-order conditions can be written

$$\partial\Phi_i(\cdot) / \partial x_i + \partial\Phi_i(\cdot) / \partial x \times \partial\chi_i(\cdot) / \partial x_i \equiv \xi_i(x_i | \sigma_i) = 0 \quad i = 1..N.$$

where the functions $\{ \xi_i(x_i | \sigma_i) \; i = 1..N \}$ make clear the fact that, unlike the situation in Nash equilibrium, the private actions of rivals no longer condition optimizing behavior. The appearance of the terms $\{ \partial\chi_i(\cdot) / \partial x_i \;\; i = 1..N \}$ reflects a departure from the Nash rule, above, which equates the absolute values of the marginal returns to private and public actions. Transforming the conjectures into elasticities $\{ \hat{\theta}_i \equiv (\partial\chi_i(\cdot) / \partial x_i)(x_i / \chi_i(\cdot)) \; i = 1..N \}$ and transposing the resulting expression, we obtain

$$\text{MRS}(x_i, x) = \frac{x_i / x}{\hat{\theta}_i} \qquad i = 1..N,$$

where we have assumed $\{ \hat{\theta}_i \neq 0 \;\; i = 1..N \}$. Summing over the N agents yields the allocation rule:

$$\sum_{i=1}^{N} \text{MRS}(x_i, x) = \sum_{i=1}^{N} \frac{x_1 / x}{\hat{\theta}_1}$$

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Congestion Models with Consistent Conjectures

Garth J Holloway

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p. 265, para 3: *Change* $\left\{\tilde{\theta}_i > 0 \ i = 1..N\right\}$ *to* $\left\{\hat{\theta}_i > 0 \ i = 1..N\right\}$

p. 266, last sentence: *Change* $\hat{\sigma}$ *to* $\tilde{\sigma}$

p. 269, eq. (21): *Change* $\hat{\theta}_t\,(t) =$ *to* $\hat{\theta}_i(t) =$

p. 275, problem 2: *Delete* 'i = 1..N'

p.277, last line: *Change* $\sum_{i=1}^{N} \frac{x_1/x}{\hat{\theta}_1}$ *to* $\sum_{i=1}^{N} \frac{x_i/x}{\hat{\theta}_i}$