



*The World's Largest Open Access Agricultural & Applied Economics Digital Library*

**This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.**

**Help ensure our sustainability.**

Give to AgEcon Search

AgEcon Search

<http://ageconsearch.umn.edu>

[aesearch@umn.edu](mailto:aesearch@umn.edu)

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

*No endorsement of AgEcon Search or its fundraising activities by the author(s) of the following work or their employer(s) is intended or implied.*

# THE AUSTRALIAN JOURNAL OF AGRICULTURAL ECONOMICS

---

VOL. 11

JUNE, 1967

NO. 1

---

## DEVELOPMENT AND ANALYSIS OF INPUT-OUTPUT RELATIONS FOR IRRIGATION WATER

J. C. FLINN and W. F. MUSGRAVE\*

*University of New England*

Recent developments in climatology mean that economists now have a highly acceptable physical theory which can underlie their analysis of the economic aspects of water as an input to the production process, as a source of production instability, and as a major component of error in their estimated crop production functions. This paper presents a model and a procedure for synthesizing and analysing irrigation water crop input-output relations based on this theory. The importance of time of application of water as well as quantity is shown. Different frequencies of irrigation are optimal at different times of the growing season.

Economic evaluation of the potential use and development of water resources requires, *inter alia*, estimates of the marginal value product of water. For example, marginal values are needed to assess the benefits from potential projects to supply irrigation water. They are also needed in determining the optimum allocation of existing water supplies among competing uses. Estimates of the marginal product of water imply some knowledge of the production function for water. The existence of such knowledge holds out promise, not only for the study of irrigation policy, but also for systematically handling climate as a major component of error in crop production functions and as a source of production instability in agriculture. This paper demonstrates how recently developed concepts of plant-water relationships allow estimation of the productivity of water. The demonstration and subsequent analysis are in terms of irrigation but it is believed that the implications of this development extend well beyond that area.

The object of this paper is to demonstrate that the use of physically derived plant-water relationships allows a meaningful plant-water production function to be estimated. Within such a framework it will be

\* This paper reports on part of a study which has received assistance from the Rural Credits Development Fund. Many people have assisted us in this work but we would particularly like to acknowledge the contributions of R. O. Slatyer and his colleagues of the Land Use and Regional Survey Division of the C.S.I.R.O. and the interest and assistance of J. L. Dillon and M. M. Kelso. However, above all, our debt is greatest to the C.S.I.R.O. scientists of the Irrigation Research Laboratories, Griffith, of whom P. M. Fleming must be singled out. As usual, errors and omissions are our responsibility.

shown that any specified time period of an irrigation season has a unique plant-water production function. Consequently there will be an optimal irrigation programme for each time period during the crop's growing period.

Though it is not the purpose of this paper to present a comprehensive review of the literature of plant-water production functions, a brief appraisal is needed to place the current work in perspective. Four broad approaches are discernible. They are the estimation of:

- (a) a single-valued crop-water requirement;
- (b) a function based on physical input-output data;
- (c) a synthetic function based on physical criteria; or
- (d) a simulated function based on biological and physical relationships.

#### *Single-Valued Crop-Water Requirements*

By far the majority of published literature considering returns from irrigation assumes that any crop has a unique water requirement. In such work the possibilities for substitution between water and other inputs, particularly land, are ignored. Such a procedure needs explicit defence, particularly when the price of water or any other good is varied in the analysis. This principle is generally assumed in most farm budgets, as also in most other work where an estimate of the relationship between water input and resulting crop output is necessary. Examples of the use of unique water inputs are given by the work of Lee<sup>1</sup> and Miller *et al.*<sup>2</sup>

The widespread use of unique water requirements for crops probably reflects two things. First, the inflexibility of water price policy in most irrigation areas and, second, the opinion of many physical scientists that plants require a certain invariable quantity of water for desirable (usually maximum) growth. The work of Blaney<sup>3</sup> in the United States has done much to entrench this second position. In the light of present knowledge, agricultural economists should be anxious to break free from such a tradition.

#### *Function Based on Physical Yield Data*

This approach is in the tradition of conventional production function estimation and thus has the advantages and disadvantages of that approach. The main data sources have been irrigation experiments with varying water/land input ratios, and crop yield data as related to rainfall or irrigation intensities. Papers published using such physical input data vary widely in complexity and sophistication.

Engineers have been interested in the estimation of crop-water input-output relationships because of the resulting implications for the design of irrigation works. Clyde *et al.*<sup>4</sup> presented a theoretical analysis of

<sup>1</sup> Lee, I. M. Optimal Water Resource Development: A Preliminary Statement of Methodology of Quantitative Analysis. California Agric. Expt. Stn., Giannini Foundtn. Res. Rept. No. 206, Berkeley, 1958.

<sup>2</sup> Miller, S. F., Boersma, L. L., and Castle, E. N. Irrigation Water Values in the Willamette Valley: A Study of Alternative Valuation Methods. Oregon State Univ., Agric. Expt. Stn. Tech. Bul. 85, Corvallis, 1965.

<sup>3</sup> Blaney, H. F. Climate as an Index of Irrigation Needs. In *Water, the Yearbook of Agriculture 1955*, U.S.D.A., Washington, 1955.

<sup>4</sup> Clyde, H. S., Gardner, W., and Israelson, O. The Economical Use of Irrigation Water Based on Tests. *Eng. News Record* 91: 549-52, 1923.

input-output relations for irrigation in terms of a production function in which output was in terms of a single crop, the production factors being land and water. The use of this type of analysis, and its application to engineering and economic analysis, is discussed at some length by Roe.<sup>5</sup>

It was not until 1959 that Beringer<sup>6</sup> drew the attention of agricultural economists to the fact that irrigation water was valuable, not of itself, but through its impact on soil moisture and that it differs from most other farm inputs because its time of application is often of far more importance in determining productivity than the quantity applied. On physiological grounds the use of total water quantity as the only water input variable can have but limited usefulness since: (1) total water quantity fails to consider water distribution over the irrigation season as an important variable; and (2) total irrigation water applied cannot be directly correlated with plant water use.

It is well established that changes in the soil moisture regime during the plant's growing period result in corresponding changes in yield of the irrigated crop. As a result it is possible to vary the level of yield by controlling the soil moisture level, or more precisely, by attempting to control the incidence of moisture stress. This attempt at control is made through manipulation of the frequency of irrigation, the quantity of water applied per irrigation, and other associated irrigation variables.

Having shown the limitations of the water quantity approach, Beringer suggested the use of a moisture stress index called Integrated Moisture Stress,<sup>7</sup> a concept based on similar principles to the soil moisture deficiency index of the climatologists. Both indices are based on the principle of aggregation of some measure of moisture deficiency over the growing season. Both indices fail to recognize the importance of the time of occurrence of moisture deficiency within the growing season.

Beringer's soil moisture stress index is of limited usefulness because of estimation difficulties. It would seem that his main contribution has been to introduce economists to the moisture deficiency concepts which have been used by physical and biological scientists for a number of years. For example, in 1952 Taylor<sup>8</sup> and others observed that as mean integrated moisture stress increased, the resulting crop yield decreased.

Several economists have related crop yield to soil moisture deficiency indices using experimental results. An outstanding example of such work is that of Reutlinger and Seagraves<sup>9</sup> who, using covariance analysis to remove inter-year effects, expressed experimental tobacco yield as a function of a computed soil moisture deficiency index. Using long-term estimates of the soil moisture deficiency index derived from

<sup>5</sup> Roe, H. B. *Moisture Requirements in Agriculture*. McGraw Hill, N.Y., 1950.

<sup>6</sup> Beringer, C. Some Conceptual Problems Encountered in Determining the Production Function for Water. In *The West in a Growing Economy*, Proc. Western Farm Econ. Assn. Thirty-second Annual Meeting held at Logan, Utah, July 1959.

<sup>7</sup> Beringer, C. An Economic Model for Determining the Production Function for Water in Agriculture. California Agric. Expt. Stn., Giannini Foundtn. Res. Rept. No. 240, Berkeley, 1961.

<sup>8</sup> Taylor, S. A. Estimating the Integrated Soil Moisture Tension in the Rootzone of Growing Crops. *Soil Sci.* 73: 331-40, 1952.

<sup>9</sup> Reutlinger, S. and Seagraves, J. A. Estimating Returns on Investment in Irrigation Systems for Flue-cured Tobacco. North Carolina Agric. Expt. Stn. Tech. Bul. No. 153, Raleigh, 1962.

weather records, they demonstrated how a probability distribution of moisture deficits could be combined with their estimated function to obtain expected yield and the variance of yield resulting from different irrigation policies. Once again, a serious limitation to this otherwise admirable technique is its failure to allow for the effect of time of incidence of moisture deficiency. However, this does not go to say that estimates of the effect of incidence of stress on eventual yield cannot be incorporated in the Reutlinger and Seagraves methodology. Should this be possible, it seems that this approach would then be an acceptable alternative to the procedure suggested in this paper in those instances where the necessary data are available.

### *Synthetic Function Based on Physical Criteria*

The need to recognize the soil moisture deficiency variables in crop water production functions, together with the general paucity of data, has led some agricultural economists to attempt to synthesize a production function. Certain plant, water, soil interrelationships are postulated, basic data on weather is fed into the model and conclusions as to plant growth are drawn. This approach has its genesis in the soil water budgets of soil scientists and climatologists. A major limitation is that these models are better able to estimate crop-water use than resulting crop production.

The most significant contribution of this type, to date, has been that of Moore.<sup>10</sup> He postulated a definite relationship between crop yield and available moisture in the soil profile. He suggested the use of the moisture release curve<sup>11</sup> for a specific soil as an indicator of the actual growth rate of a plant grown in that soil. From the relationship between "potential" growth (i.e. growth when the soil moisture level is not limiting) and actual growth, an index of plant growth is derived. Multiplying this index by estimates of "potential" yield<sup>12</sup> provides an estimate of actual yield under specified irrigation practices. The specified irrigation practices are included as components in a water budget table to obtain the labour and water requirements for the crop under consideration. By attaching prices to these inputs and outputs, a value is established for the water used in the irrigation cycle.

Though Moore's analysis of the plant-water production function is outstanding, it is limited on plant physiological grounds. The postulated moisture release curve is a specific case, and would not generally apply in semi-arid irrigation areas. Further, the concept is too vague in view of current knowledge and thought. As Lowrey<sup>13</sup> has shown, actual moisture release will depend on the atmospheric demand for water and the plant cover as well as the soil moisture level. Clearly it is misleading to talk about a two-dimensional release curve under real-world circumstances. A more generalized theory is necessary. Also, the assumption that plant evapotranspiration will always be at the potential rate (and

<sup>10</sup> Moore, C. V. A General Analytical Framework for Estimating the Production Function for Crops Using Irrigation Water. *J. Farm Econ.* 43: 876-88, 1961.

<sup>11</sup> A moisture release curve represents the rate at which water is released from the soil between field capacity and permanent wilting point.

<sup>12</sup> The "potential" yield is that yield which would result with fully adequate supplies of water, all other crop practices remaining the same.

<sup>13</sup> Lowrey, W. P. The Falling Rate Phase of Evaporative Soil Moisture Loss. *Bul. Amer. Met. Soc.* 40: 605-8, 1959.

thus independent of the soil moisture level) now seems to be incorrect and likely to result in underestimation of the soil moisture level. Moore must be commended, however, for recognizing that growth can be retarded before soil moisture falls to permanent wilting point, for recognizing the importance of decisions about irrigating by cycle and not by total growing season, and for developing the use of synthetic procedures in studying the productivity of water.

Plant growth is not a function of the soil moisture level alone. Rather it is a function of the parameters associated with plant-moisture stress. The relevant parameters, as pointed out by Fleming,<sup>14</sup> are the rate of moisture intake by the plant roots from the soil, and the rate of moisture loss of the plant leaves to the atmosphere. These parameters in turn will be influenced by the soil moisture tension, the atmospheric demand for water, and the physiological characteristics of the plants involved. A production function must recognize these parameters if it is to be realistic in biological terms.

#### *Simulated Function Based on Biological and Physical Relationships*

A simulated plant-water model used to synthesize crop-water production functions must fulfil two requirements. They are: (i) provide a rational accounting method to estimate periodic, preferably daily, soil moisture levels; and (ii) relate the daily soil moisture level to evaporative parameters so an index of plant growth may be derived.

The first requirement is dealt with by the use of a soil moisture budget, in a similar fashion to that of Moore, but with greater recognition of the complexity of the plant rootzone and soil profile. The principle of a soil moisture budget is that the plant's rootzone is a reservoir for soil moisture. Additions to the reservoir are in terms of irrigation and rainfall, losses are in terms of evapotranspiration, deep percolation and runoff.

The second requirement, i.e. deriving the index of plant growth, is met via the biological relationship between potential and actual evapotranspiration. Denmead and Shaw<sup>15</sup> have argued that on any day when evapotranspiration falls below the potential rate it is reasonable to assume that growth on that day is zero. Dale and Shaw<sup>16</sup> have shown that on such days evapotranspiration is at a rate set by the ability of the crop to extract moisture from the drying soil and that when this happens the plant is "stressed". The rate at which a plant can extract moisture from the soil as it dries is independent of the atmospheric conditions, but it appears that, on any day on which potential evapotranspiration exceeds the rate of moisture extraction appropriate for the prevailing level of soil moisture, stress occurs. In other words, net growth can be said to take place on any day when evapotranspiration

<sup>14</sup> Fleming, P. M. A Water Budget Method to Predict Plant Response and Irrigation Requirements for Widely Varying Evaporative Conditions. *Sixth Internat. Cong. Agric. Eng.*, Switzerland, 1964, Sect. 1.7, pp. 1-12.

<sup>15</sup> Denmead, O. T. and Shaw, R. H. Availability of Soil Water to Plants as Affected by Soil Moisture Content and Meteorological Conditions. *Agron. J.* 45: 385-90, 1962.

<sup>16</sup> Dale, R. F. and Shaw, R. H. The Climatology of Soil Moisture, Atmospheric Evaporative Demand, and Resulting Moisture Stress Days for Corn at Ames, Iowa. *J. Applied Met.* 4: 661-9, 1965.

is not limited by soil dryness—i.e. the plant is able to transpire at the rate demanded by the atmosphere. Relating days of actual growth to the number of days in the time period gives an index of plant growth.

Knowledge of the biological relationships allowed a plant-soil moisture simulation model to be developed. Inputs to the simulation model were parameters reflecting the water-holding capacity of the soil in question, parameters related to the type of crop grown,<sup>17</sup> and weather data. The weather inputs used were daily recordings of rainfall and pan evaporation supplied by the Bureau of Meteorology for the 40 years between 1920 and 1960 at Griffith, N.S.W. Using elements of the water budget (i.e. tracing actual evapotranspiration, rainfall, deep percolation and surface runoff), soil moisture levels and the presence or absence of growth were calculated for each day of these 40 seasons. On the basis of a prespecified array of soil moisture levels, irrigation was simulated to return the soil to "field capacity". Aggregating the daily results over the irrigation season then allows the derivation of an index of plant growth. This index is an estimate of the expected outcome of the use of the irrigation rule: "return the soil to field capacity whenever soil moisture falls to a particular terminal level". An estimate of the expected value and the variance of the outcome attached to each of the terminal soil moisture levels nominated is generated.

Such a simulation model differs significantly from previous models in three respects: (1) the model uses estimates of soil moisture, atmospheric conditions and the stage of crop growth to ensure a reliable estimate of actual evapotranspiration; (2) the index of plant growth is dependent on the relationship between actual and potential evapotranspiration, and so relates days of actual growth to the number of days in the irrigation cycle; and (3) actual daily atmospheric and soil moisture conditions are identified so the model approaches reality, as opposed to the more static analyses employed in the past.

As in Moore's study, estimates of maximum crop yield when moisture is not limiting were obtained from physical scientists. Multiplication of the estimated maximum yield by the index of plant growth then gives the estimate of actual crop yield under any given irrigation regime.

The model further highlights the importance of the time of application of irrigation water. Early in the irrigation season when evaporative demand is low, growth will continue at lower moisture levels than in midsummer when evaporative demand is high. That is, each irrigation cycle will have its own production function for soil moisture. *Ipsa facto*, on biological grounds there will be an optimal soil moisture level to maintain within any irrigation cycle or time period.

To take account of this intracycle effect, the irrigation season was subdivided into eight 30-day time periods (or stages) numbered 1 to 8 respectively. The choice of such stages was a compromise between having 240 one-day stages, or one stage taking in the entire irrigation season. The 30-day stage was chosen for several reasons: (i) the variance of the index of relative growth and irrigation water input was sufficiently low to make statistical analysis of results meaningful; (ii) eight irrigation stages were adequate to allow logical economic and

<sup>17</sup> The crop considered in this analysis is a hypothetical one, emerging at the start and being harvested at the end of the irrigation season. Although an irrigated crop with an eight month growing season is an abstract from reality, the crop parameters used in the simulation model are biologically sound.

biological interpretations of the results obtained; and (iii) a 30-day stage covers two irrigation cycles (the usual roster in the irrigation area being a 15-day cycle).

### Analysis of Results

In estimating the soil moisture production function for a single crop, the following general specification was applied within each stage:

$$(1) \quad d_i = f(X_{1i})$$

where  $d_i$  is days of growth<sup>18</sup> in stage  $i$ ; and

$X_{1i}$  is terminal soil moisture level in stage  $i$ .

TABLE 1

*Least Squares Estimates of Equation (1): Days of Growth in Stage  $i$  ( $d_i$ ) as a Quadratic Function of Terminal Soil Moisture Level in Stage  $i$  ( $X_{1i}$ )*

i	Estimated coefficients(a)			$R^2$
	Constant	$X_{1i}$	$X_{1i}^2$	
1	28.078	0.057†	-0.001	0.10
2	10.622	0.351	-0.002	0.35
3	0.432†	0.518	-0.004	0.67
4	1.330†	0.521	-0.004	0.70
5	-2.252†	0.554	-0.004	0.74
6	-1.331†	0.553	-0.004	0.70
7	1.775†	0.516	-0.004	0.58
8	12.016	0.444	-0.004	0.32

(a) † denotes non-significance at 5 per cent level. All other coefficients are significant at the 1 per cent level.

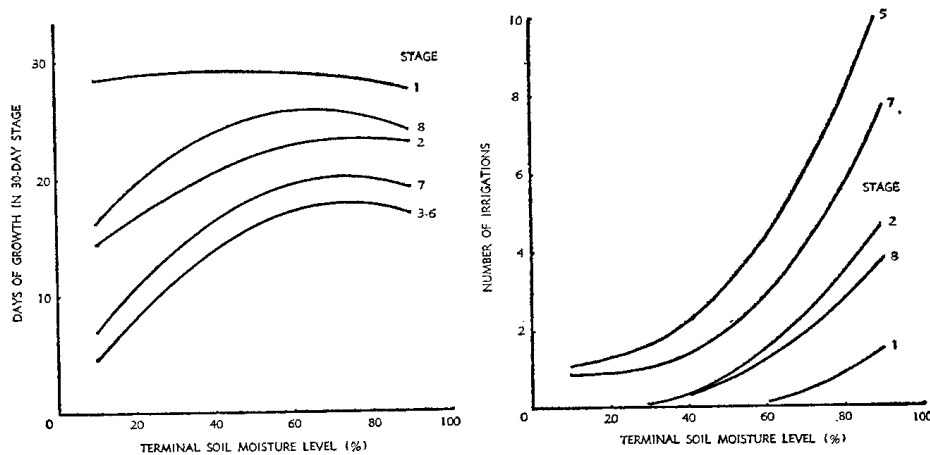


FIG. 1

- (a) Relationship between terminal soil moisture level and crop production in various stages.  
 (b) Number of irrigations as a function of terminal soil moisture level in various stages.

<sup>18</sup> Days of growth in a time period is comparable with the index of plant growth since days of growth divided by the number of days in the period is the index of plant growth for the period.



Table 1 and Figure 1(a) present estimates of the soil moisture production function for a summer crop grown in the Murrumbidgee Irrigation Area. The production functions presented in Figure 1(a) are not the same as appear in traditional economic analysis. In contrast to normal usage, they relate to soil moisture depletion, and only indirectly to a controllable input. None the less, the principles involved in input-output analysis still hold true.

Scrutiny of the fitted curves shown in Figure 1(a) indicates that, with the exception of stage 1, the slopes of the curves are much the same, the only real difference being in their overall elevation and this difference is quite minor for the curves fitted for stages 3, 4, 5 and 6. The differences in elevation between stages may be explained in terms of differences in evaporative demand and other climatic variables. This is reasonable. Very early in the season (stage 1) evaporative demand is not limiting. In stage 2 this demand is higher, rising to a maximum over the mid-summer periods, then falling again in autumn.

### *Optimizing Model*

The question still remains, what is the optimal quantity and time of application of irrigation water within any stage? In terms of the present analysis, this would be when the marginal return from an increment in the terminal soil moisture level is equal to the marginal cost of maintaining that increment.

For the purpose of economic analysis, the terminal soil moisture level is only an index and therefore does not itself have a price which can be compared with marginal value product. Before an economic optimum can be determined it is necessary to express the terminal soil moisture level as a function of irrigation water applied and other associated inputs. The general relationship between irrigation inputs and terminal soil moisture level is given by:

$$(2) \quad X_{1i} = f(X_{2i}, X_{3i}, X_{4i}, X_{5i}, X_{6i})$$

where  $X_{1i}$  is terminal soil moisture maintained in stage  $i$ ;

$X_{2i}$  is potential evapotranspiration in stage  $i$ ;

$X_{3i}$  is effective rainfall in stage  $i$ ;

$X_{4i}$  is number of irrigations in stage  $i$ ;

$X_{5i}$  is quantity of irrigation water applied in stage  $i$ ; and

$X_{6i}$  is runoff and deep percolation in stage  $i$ .

The inputs over which the farmer has some measure of control are the number of irrigations and the quantity of water applied per irrigation. The relationship of operational interest is thus:

$$(3) \quad X_{1i} = f(X_{4i}, X_{5i}).$$

As the terminal soil moisture level increases (say from permanent wilting percentage toward field capacity) the irrigation frequency will also increase. In practice, with flood irrigation on heavy soils, water applied per irrigation is relatively fixed compared to light soils, due to soil infiltration characteristics. This results in the maximum amount being invariably applied at each irrigation. Consequently, at high terminal soil moisture levels excess water will be applied in the sense that less than the usually applied quantity of water would have restored the soil to field capacity. While this leads to a greater intensity of water

use than is necessary, it is an accurate picture of what happens in practice. Also, as evaporative demand increases towards the middle of the irrigation season, more frequent irrigations are necessary to maintain the same minimum soil moisture level than in the low evaporative demand periods of the irrigation season.

Assuming labour, capital and irrigation water inputs are constant per irrigation, each particular irrigation decision rule has a single-valued cost attached to it in each time period. This cost is determined by function (4):

$$(4) \quad c_i = g(X_{1i})$$

where  $c_i$  is cost of irrigation in stage  $i$ ;

$X_{1i}$  is terminal soil moisture level in stage  $i$ .

TABLE 2  
*Least Squares Estimates of Equation (4): Cost of Irrigation in Stage  $i$  ( $c_i$ ) as a Quadratic Function of Terminal Soil Moisture Level in Stage  $i$  ( $X_{1i}$ )*

i	Estimated coefficients <sup>(a)</sup>			$R^2$
	Constant	$X_{1i}$	$X_{1i}^2$	
1	0.6869	-0.0514	0.0007	0.64
2	0.7101	-0.0567	0.0012	0.80
3	2.2079	-0.0903	0.0018	0.90
4	1.7782	-0.0568	0.0016	0.90
5	1.9427	-0.0629	0.0017	0.91
6	1.8174	-0.0650	0.0017	0.90
7	1.9568	-0.0827	0.0016	0.84
8	0.5404	-0.0815	0.0013	0.71

(a) All coefficients are significant at the 1 per cent level.

Table 2 presents estimates of the cost function for soil moisture as used by the crop in the study area. Some of these cost functions are graphed in Figure 1(b). They indicate that as evaporative demand increases towards the middle of the irrigation season, the number of irrigations required to maintain any given terminal soil moisture level will also increase.

By attaching a cost to each irrigation decision rule and a revenue to the plant growth generated, the optimal irrigation frequency within any one stage may be calculated. The optimal frequency will be the frequency which maintains a terminal soil moisture level such that the marginal cost of maintaining the soil moisture level at that point is equal to the marginal revenue generated by not allowing soil moisture to fall below that level. Thus, using hypothetical data, the intersection of the marginal cost and marginal revenue curves of Figure 2 gives the optimal soil moisture level. Note how, although the marginal revenue curve is assumed to remain unchanged between irrigation stages, the marginal cost of maintaining any given soil moisture level increases as atmospheric demand for plant moisture increases. Thus the optimal terminal soil moisture level will be lower towards the middle than at the start or end of the irrigation season. However, this does not infer less frequent irrigation during the mid-summer high atmospheric demand stages of the irrigation season.

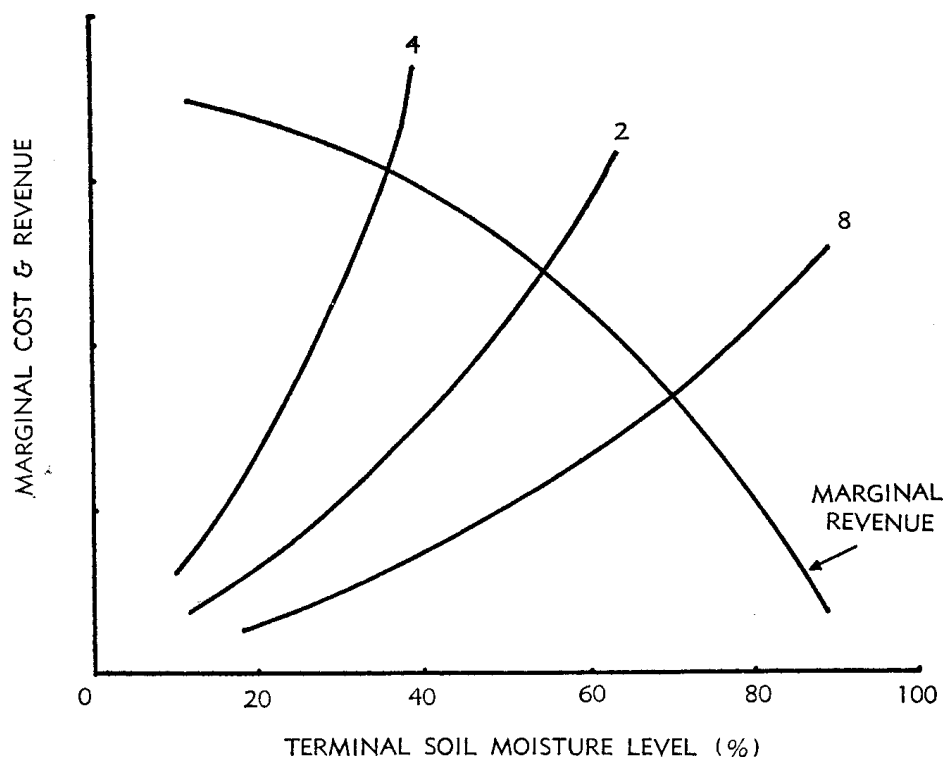


FIG. 2—Optimal soil moisture depletion levels illustrated by hypothetical cost and revenue data.

#### *A Dynamic Programming Formulation of the Irrigation Timing Problem*

The preceding analysis has assumed that the quantity of water used in each stage is independent of the quantity used in all other stages. Further, it assumes there is sufficient irrigation water available to irrigate each acre to the point where the marginal return from an increment in terminal soil moisture level is equal to the marginal cost of maintaining that increment. In many instances there is insufficient water available to irrigate the crop to that point in each stage of the crop production process, or it is technically infeasible to irrigate the crop as often as required in any stage. Because of the extreme importance of timing of application of irrigation water, the irrigator must determine how best to allocate a given total quantity of irrigation water over the crop's life span.

Many farm management decisions fall into this category of multi-stage decision processes. Such processes are characterized by the task of finding a sequence of decisions which maximize an appropriately defined objective function. In the present problem, the irrigator is faced with the problem of allocating a scarce resource (irrigation water) over the irrigation season in such a fashion as to maximize the net economic benefit of the irrigation water applied. Such a problem is eminently suited to dynamic programming.

*Formalizing the Problem*

Let  $S$  represent the total quantity of irrigation water available to the crop during any one irrigation season. The crop production process is divided into  $N$  stages, ( $i = 1, 2, \dots, N$ ), each stage having a revenue function  $g_i(x_i)$  resulting from allocating a quantity of water  $x_i$  to the  $i$ -th stage. The problem is to select the vector of irrigation water quantities  $(x_1, x_2, \dots, x_N)$  which will maximize the total return to the  $N$  stages of the crop production process. Such an *optimal* return will only depend on the number of stages  $N$ , and the initial quantity of water available,  $S$ . The optimal return is denoted by  $f_N(S)$  where:

$$(5) \quad f_N(S) = \text{Max } \sum g_i(x_i)$$

subject to  $0 \leq x_i, \sum x_i \leq S$ .

The solution to equation (5), i.e. selection of the optimal set of  $x_i$ , is by no means obvious. However, instead of considering a fixed quantity of water which might be available, it is possible to solve the problem for any variable quantity,  $s$ , which is allowed to vary from zero to  $S$ . This permits equation (5) to be transformed from one equation in  $N$  variables to  $N$  equations, each of one independent variable.

The rationale behind dynamic programming is given by the principle of optimality. Bellman's definition of this principle is:<sup>19</sup>

"An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

In our irrigation context, the term "stage" refers to the previously defined 30-day time period. The "state variable" is the quantity of water,  $s$ , available to be allocated at any stage. A decision at stage  $i$  constitutes allocating a quantity of water  $x_i$  to the  $i$ -th stage. The transformation function will thus be  $(s_i - x_i)$ , the value of which will be the state variable for the following stage.

Application of the principle of optimality to the present problem gives the recurrence relation:

$$(6) \quad f_N(s) = \text{Max } [g_N(x_N) + f_{N-1}(s - x_N)]$$

subject to  $0 \leq x_N \leq s$ .

That is, the optimal return from the entire process will be that which maximizes the sum of the return from the present stage and the optimal return from all remaining stages.

Solution of equation (6) is accomplished by computing  $f_1(s) = \text{Max } g_1(x_1)$ , since  $f_0(s) = 0$ , for all values of  $x_1$  ( $0 \leq x_1 \leq s$ ). This permits computation of  $f_2(s)$ , and so on until  $f_N(s)$  is obtained;  $s$  is then set equal to  $S$  in  $f_N(s)$  and the decision variables  $(x_N, x_{N-1}, \dots, x_1)$  traced out step by step for stages  $N, N-1, \dots, 1$ . But before these relationships can be applied to the allocation of irrigation water over the crop growing season, it is necessary to derive the net revenue equations,  $g_i(x_i)$ , for the various stages of the production process.

<sup>19</sup> Bellman, R. E. and Dreyfus, S. E. *Applied Dynamic Programming*. Princeton Univ. Press, 1962, p. 15.

### Revenue Functions

To make the analysis more meaningful, the assumption of linearity of growth over time assumed in the simulation model is relaxed. The potential growth curve of a crop is better reflected by a sigmoid growth curve. Consider the crop planted at the start of the irrigation season and harvested at the end of the final stage of the growing season. Assume that the contribution of crop production in the  $i$ -th stage,  $y_i$ , to final crop yield in each stage is additive so that:

$$(7) \quad Y = \sum_{i=1}^N y_i$$

where  $Y$  is crop yield at harvest.

Potential growth in each stage is weighted so final crop yield reflects the growth potential in each stage. Potential and hypothetical actual growth curves for annual plants are illustrated in Figure 3. An irrigation regime which does not ensure continuous plant growth will result in the actual growth curve being below the potential growth curve. These curves suggest that a moisture stress in the rapid growth and reproductive stages of plant growth will have a greater effect on final yield than stresses early or late in the irrigation season. This is in accordance with biological evidence. For example, Dale and Shaw<sup>20</sup> found that the moisture stress period which accounted for the greater proportion of yield reduction in corn was a 63 day period extending from six weeks before to three weeks after silking.

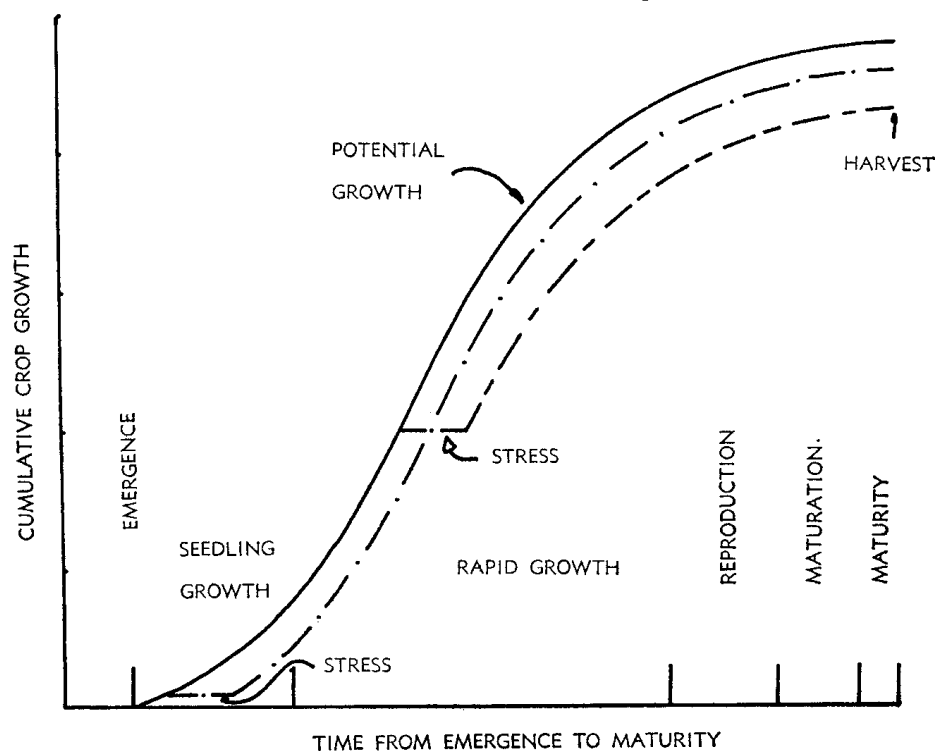


FIG. 3—Potential and actual growth curves for annual plants.

<sup>20</sup> Dale and Shaw, *loc. cit.*

Potential plant growth in each stage of the growing season is weighted by assigning the proportion of potential growth in each stage as it relates to final potential yield. The weights for each stage are read off potential growth curves such as that of Figure 3. Periods of low soil moisture demand correspond to stages where potential growth is low, while periods of high soil moisture demand coincide with stages where potential growth is a maximum.

Thus, by relating crop production to various terminal soil moisture levels, it is possible to develop a revenue function for each stage of the production process. Similarly, cost functions can be developed to relate the number of irrigations to the various terminal soil moisture levels in each stage. In the present context it is meaningful to relate crop production to inputs of soil moisture, i.e. rainfall and irrigation water. Rainfall is an exogenous variable; it is beyond the control of the farmer. In contrast, irrigation water is a decision variable—it is under the farmer's control. To retain realism in the model, it must be remembered that the quantity of irrigation water which enters the soil per irrigation is fixed. The number of irrigations will then, subject to some factor of proportionality, be equivalent to the quantity of water applied. In consequence, the following relationship is postulated:

$$(8) \quad y_i = f(x_i, r_i, w_i)$$

where  $y_i$  is the index of actual growth calculated as a percentage of the maximum attainable growth in stage  $i$ ;  
 $x_i$  is the number of irrigations in stage  $i$ ;  
 $r_i$  is the effective rainfall in stage  $i$ ; and  
 $w_i$  is the potential evapotranspiration in stage  $i$ .

TABLE 3

*Least Squares Estimates of Equation (8): Index of Actual Growth ( $y_i$ ) as a Function of Number of Irrigations in Stage  $i$  ( $x_i$ ), Effective Rainfall in Stage  $i$  ( $r_i$ ) and Potential Evapotranspiration in Stage  $i$  ( $w_i$ )*

i	Estimated coefficients <sup>(a)</sup>					R <sup>2</sup>
	Constant	$x_i$	$x_i^2$	$r_i$	$w_i$	
1	108.86**	—3.75*	0.43	0.15*	—14.01**	0.49
2	104.32**	12.16**	—1.57**	5.58**	—18.45**	0.51
3	48.08**	17.25	—1.38**	2.63**	—6.17**	0.69
4	27.87**	15.76**	—1.10**	7.42**	—3.90**	0.78
5	20.47*	15.69**	—1.05**	5.48**	—2.66*	0.79
6	47.39**	15.80**	—1.12**	5.64**	—6.66**	0.77
7	60.49**	14.58**	—1.14**	6.72**	—8.69**	0.66
8	95.99**	8.94**	—1.29**	6.00**	—15.24*	0.32

(a) \*\* denotes significance at the 1 per cent level; \* denotes significance at the 5 per cent level.

Table 3 presents the production relationships for the index of crop growth as it relates to irrigation water input, rainfall and potential evapotranspiration.<sup>21</sup> Scrutiny of Table 3 indicates that the production functions relating crop growth to irrigation water input are similar to

<sup>21</sup> Potential evapotranspiration is included in the model because with soil moisture, it is the main causal variable influencing crop growth.

the functions derived earlier relating crop growth to various soil moisture levels. That is, the index of crop growth in the early and final stages of the production season will be greater than that of the high evaporation stages in the middle of the growing season. For each stage, rain results in the functions being displaced upward, while evapotranspiration causes the curves to be displaced downward.

As rainfall and evapotranspiration are exogenous variables, they have been held at their means in each stage of the growing season to estimate the relationship between the index of plant growth and the number of irrigations in stage  $i$ . By including the observed values of rainfall and potential evapotranspiration, an expected plant growth function for the  $i$ -th stage would be generated instead of the deterministic functions employed in the present analysis. Burt,<sup>22</sup> for example, has illustrated how stochastic functions can be incorporated in dynamic programming analysis.

The quantity ( $q_i$ ) of crop produced in stage  $i$  is estimated by multiplying the index of plant growth ( $y_i$ ) in stage  $i$  by the crop weight assigned to stage  $i$ . Assume that the maximum crop yield is 150 bushels per acre, and the return per bushel is \$1.50, then the gross revenue function  $r_i(x_i)$  in the  $i$ -th stage will be:

$$(9) \quad r_i(x_i) = 1.5q_i.$$

#### *Cost Functions, $c_i(x_i)$*

The cost of applying irrigation water is a function of the quantity of irrigation water applied, labour and other inputs. Labour inputs will increase with an increasing frequency of irrigation as additional labour will be required for levee bank maintenance, control of drainage outflows, etc. Other inputs would remain constant irrespective of the frequency of irrigation. In fact the only other inputs used in the study area would be a shovel, and possibly a soil auger. In consequence, these inputs are neglected.

Equation (10) presents a typical irrigation water cost curve faced by large area farms in the study area.

$$(10) \quad c_1(x_1) = -0.009 + 0.648x_1 + 0.031x_1^2.$$

If water prices were to vary over the irrigation season, a separate cost curve would be necessary for each stage of the irrigation season. A feature of dynamic programming is that such price variability does not increase the complexity of the analysis.

#### *Net Revenue Functions*

Having derived the gross revenue functions  $r_i(x_i)$  and the cost function  $c_i(x_i)$ , the net revenue function for stage  $i$  of the growing season will simply be:

$$(11) \quad g_i(x_i) = r_i(x_i) - c_i(x_i).$$

Figure 4 illustrates these functions. Inspection of the curves for the various stages suggests that applying irrigation water in stages 1 and 8 will reduce net revenue. Irrigating in stages 2 and 7 will result in small increases in net income above a zero level irrigation. Periods 3,

<sup>22</sup> Burt, O. R. The Economics of Conjunctive Use of Ground and Surface Water. *Hilgardia* 36: 74-82, 1964.

4, 5 and 6 have the higher marginal product for any given level of irrigation. Such a relationship is reasonable since a shortage of irrigation water during the rapid growth and reproductive phases of plant growth will result in larger reduction in crop yield than a lack of moisture either very early or very late in the crop's growing season.

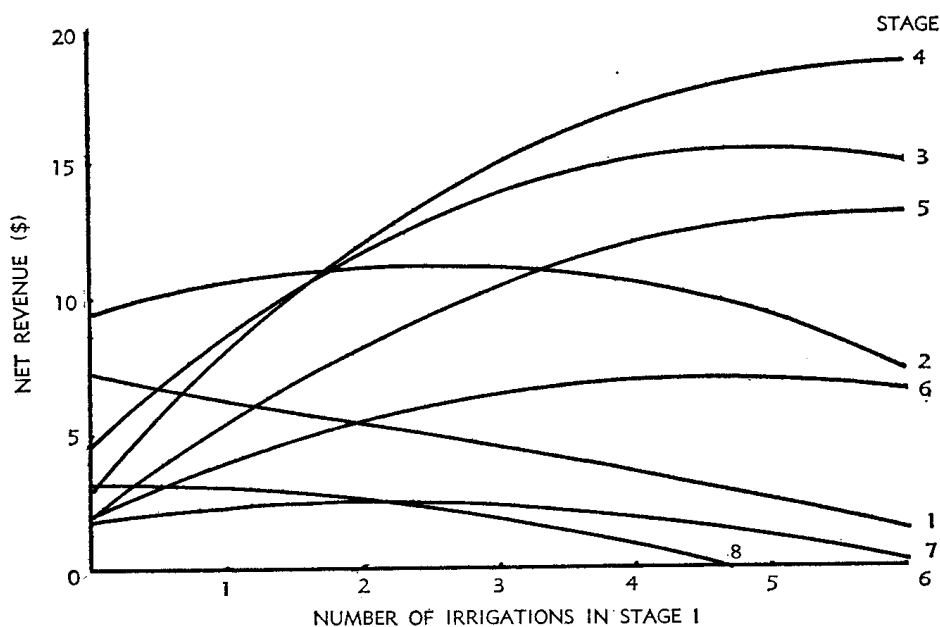


FIG. 4—Net revenue functions for stage  $i$  of the crop growing season.

#### *Dynamic Programming Solution*

Dynamic programming may now be used to specify the optimal allocation of irrigation water over the irrigation season as the net revenue functions for each stage of the irrigation season have been estimated. The quantity of water applied per irrigation is fixed, so the problem is an integer one of applying 1, 2 ...  $n$  irrigations in a 30-day period.

In the study area there are two rostered irrigations in a 30-day period (on day 1 and day 16). An irrigation lasts for five days, after which special waterings are applied for, and each "special" lasts five days. In consequence, there would be a maximum of six irrigations in a 30-day period commencing on days 1, 6, 11, 16, 21 and 26 of the stage. Apart from this administrative restraint there are three difficulties in that with more frequent irrigations (i) water entry is reduced below 5.5 cms. per irrigation, (ii) problems occur due to rises in the water table, and (iii) excessive surface and sub-surface drainage causes overtaxing of the drainage system. For these reasons, six irrigations in any stage of the irrigation season is both a technically and administratively feasible upper limit.

Part A of Table 4 presents the optimal number of irrigations in each stage of the growing season for diminishing total number of irrigations available. In Table 4A the maximum of up to six irrigations are possible in each stage. Under the specified costs and returns, the irrigator will



never irrigate more than 26 times. The crop will not be watered in stages 1 or 8 as irrigation in these stages would lead to a decline in net revenue. As the total number of irrigations available to be distributed over the irrigation season is reduced, irrigations will first be withdrawn from early and late stages of crop growth when water requirements are not so important. Higher irrigation frequencies will be maintained during the rapid growth, high transpiration stages of the production process.

Such a planned irrigation frequency would only be feasible if the irrigator was pumping from a dam, or was the only user of water from an irrigation channel. Because of the limited capacity of an irrigation channel if each irrigator wanted water, the maximum feasible number of deliveries in a 30-day period may only be three or four. An irrigation authority could achieve this restraint by limiting the number of irrigations to four (or three) in each stage, or by implementing a differential water pricing policy over the irrigation season.

TABLE 4  
*Optimal Number of Irrigations in Stage i for Various Total Quantities of Available Irrigation Water*

Irrigs. available	1	2	3	Stage 4	5	6	7	8	Irrigs. used	Net return
(No.)	(No.)							(No.)	(\$)	
<i>A: With a Maximum of Six Irrigations per Stage<sup>(a)</sup></i>										
48		2	5	6	6	5	2		26	78.37
26		2	5	6	6	5	2		26	78.37
25		2	5	6	6	5	1		25	78.27
20		2	4	5	5	4			20	76.73
15			4	5	4	2			15	72.64
10			3	4	3				10	64.99
5			1	3	1				5	52.32
1				1					1	37.79
0									0	32.91
<i>B: With a Maximum of Three Irrigations per Stage<sup>(a)</sup></i>										
32		2	3	3	3	3	2		16	69.32
16		2	3	3	3	3	2		16	69.32
15		2	3	3	3	3	1		15	69.22
10			3	3	3	1			10	64.80
5			1	3	1				5	52.32
1				1					1	37.79
0									0	32.91
<i>C: With Water Price Varied to restrain Irrigations to a Maximum of Four per Stage<sup>(b)</sup></i>										
48		2	4	4	4	4	2		20	63.13
20		2	4	4	4	4	2		20	63.13
19		2	4	4	4	4	1		19	62.03
15		1	4	4	3	3			15	61.97
10			3	3	3	1			10	58.67
5			2	2	1				5	49.11
1				1					1	36.85
0									0	32.91

<sup>(a)</sup> Irrigation water at a constant price of \$2.25 per acre foot.

<sup>(b)</sup> Irrigation water at a price of \$2.25, 2.25, 3.50, 5.75, 4.75, 2.50, 2.25, 2.25 respectively per acre foot through stages 1 to 8.

Part B of Table 4 presents the optimal number of irrigations when the maximum deliveries in each stage are restricted to three irrigations. The results follow the same form as for a maximum of six irrigations

in each stage, except that the maximum number of irrigations used are correspondingly reduced. Net returns for the total number of irrigations used are lower with the reduced irrigation inputs until the number of irrigations used in each stage is below the constrained maximum for each stage. These relationships between six, four, three and two possible irrigations in stage  $i$  are illustrated in Figure 5.

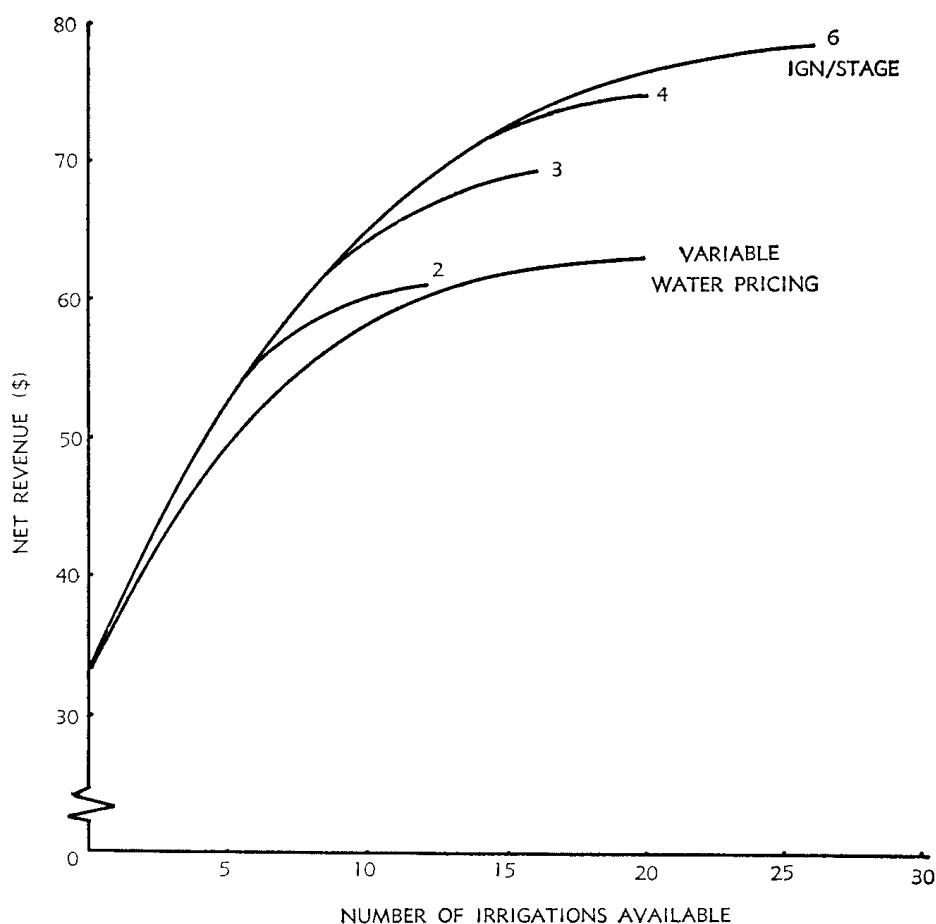


FIG. 5—Relationship between net returns and the total quantity of water available for various maximum deliveries in stage  $i$ .

Presuming farmers allocate resources efficiently and have unlimited capital, i.e. increase the level of the input until the marginal cost and marginal return from the resource are equal, the same restrained irrigation regime could be achieved by the irrigation authority varying water price over the irrigation season. By varying the water price used in the dynamic programming analysis, the minimum prices the authority would need to charge in the high demand periods of the irrigation season to restrain maximum water use to four irrigations per stage are derived. As anticipated, water prices could be raised highest in stage 4 followed by stages 5, 3 and 6, declining as the plants' demand for soil moisture declines. The minimum prices the authority would need to

charge to restrain water input to four in each stage are listed in footnote (b) of Table 4.

Part C of Table 4 also outlines the number of irrigations used in each stage as reduced quantities of total water are available over the irrigation season. A maximum of six irrigations in each stage is allowable. However, due to water pricing policy, four irrigations in each stage are the maximum quantity used. With the varying water price, marginal returns between stages are more closely equated; hence water use between periods is less variable, irrespective of the total number of irrigations available. None the less, with a reduced number of total irrigations available, water allocation will be reduced at the extremes of the irrigation season before being reduced in the rapid growth and reproductive phases of crop growth.

As would be expected, net return to any irrigation regime will be less with the varying water price policy than with a regime restrained by administrative procedures to some maximum number of irrigations in any stage. The net return function for the varying water price policy is included in Figure 5 for comparison with the previously discussed irrigation policies.

#### *Economic Implications of the Model*

The dynamic programming solution to the allocation of irrigation water over the crop growing season estimates the optimal allocation of any given quantity of water over the crop's growth. The optimal allocation of water to any stage will be a function of the total quantity of water available and the marginal product of water in each stage of the production process.

Two factors will reduce the optimal number of irrigations below the level which would ensure maximum physical crop production. They are: (i) increased expected cost when applying frequent irrigations, and (ii) diminishing marginal returns from additional quantities of water used in each stage as quantities of water used per stage become large. By weighting those factors which tend to reduce irrigation water use by the return resulting from irrigation, the optimal allocation of water over the production process can be quantitatively derived.

The analysis supports the view that the time of application of irrigation water is more important in determining its productivity than the total quantity used. The marginal value of an irrigation is determined by its time of application. It will be highest in the rapid growth and reproductive phases of crop growth, and lowest at the crop establishment and maturity phases of the crop production process. In consequence, to maximize net return from a crop under limited irrigation inputs, it is optimal to withhold irrigation water early and late in the season so as to allow the water to be applied during the stages of crop growth which maximize total expected return from the irrigation water.

The model would be useful to determine optimal policies for reservoir and irrigation system management for both the farm and the irrigation authority. From the farm point of view, the model specifies the optimal allocation of irrigation water over the irrigation season for any total quantity of water available. Further, it provides the irrigator with a technique to determine the optimal watering policy if faced with known different costs of applying water over the irrigation season. Varying

water cost need not be that imposed by an irrigation authority. For example, an irrigator pumping from a tube well will be faced with a rising cost curve over the irrigation season as pumping costs increase with aquifer drawdown.

By the use of this model, an irrigation authority could estimate the optimal water release policy for the irrigation season for any given quantity of water available in the storage dam. Further, the authority could estimate the effect of a differential water pricing policy over the irrigation season on water demand in any stage, assuming profit maximizing users. A differential water pricing policy would result in a more even demand for water over the irrigation season; also it would increase revenue from water sales for any given total quantity of water available.

### *Limitations of Results*

The results presented in this analysis are limited on both agronomic and economic grounds. The model has assumed that crop production in each stage is independent of production in all other stages. This is an over-simplification of reality as a water stress at critical stages of growth may have a major effect on final crop yield. The parameters used in the simulation model to simulate the crop and its environment can only be regarded as approximate. Water entry per irrigation has been considered fixed; in reality water entry will be influenced by plant cover, roughness of the soil surface, soil moisture level and the rate of water application, none of which have been considered in the model. The most obvious requirement for further refinement of data is in the area of the effect of moisture stresses at various phases of growth on final crop yield.

The production functions generated are deterministic and neglect the fact that crop production is stochastic in nature. Further, the analysis assumes that all inputs (except irrigation water) are held constant, no allowance being made for substitution between irrigation water and other inputs.

Finally, the results presented are normative, suggesting what the irrigator should do, as opposed to providing information on what he will do.

This approach, none the less, provides a model acceptable to biologists and economists. The model has the virtue that as more precise information is obtained about plant-soil moisture-atmospheric demand relationships, its assumptions can be replaced by more precise ones and the analysis improved accordingly.

### *Conclusions*

Irrigation water differs from many other farm inputs in that its time of application is extremely important in determining its productivity. This paper has shown that there is a production function for soil moisture during each period of an irrigation season. Using a simulation approach it is possible to determine the optimal number of irrigations in each time period relative to intra-seasonal variations in the cost of irrigation water and product price. The simulation is further suited for dynamic programming analysis. In consequence, estimation of the optimal allocation of irrigation water over the crop's growing season when there are restraints imposed on water deliveries is possible.