INTERNATIONAL TRADE AND MATHEMATICAL PROGRAMMING

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In this paper an attempt is made to formulate one type of international trade problem within a class of mathematical programming models. A new approach—from the dual side—to the proofs of existence and characteristics of the solution of the Koopmans-Hitchcock transportation cost minimization linear programming problem is presented, and the same approach is applied to the quadratic programming formulation of the problem.

Introduction

In the course of the development of economics as a branch of applied science, a simple theory of comparative advantage proposed by Ricardo [15] and a location theory proposed by von Thünen [23] have been modified, sharpened, and generalized especially in the field of international trade economics. A line of development by Heckscher [8], Ohlin [14], Graham [6], Yntema [24], Mosak [13], McKenzie [12], Samuelson [17], Arrow and Debreu [1], Lefeber [11], Uzawa [22] and Takayama and Judge [21], represents a general equilibrium approach. On the other hand, that by Beckmann and Marschak [2], Koopmans [10], Enke [4], Samuelson [16], and Takayama and Judge [18] [20], represents a partial equilibrium approach. From the mathematical programming point of view the general equilibrium approach is subject to an intractable programming difficulty. The difficulty lies in the balance of payment condition. The partial equilibrium approach is free from this difficulty but can be extended so that it eventually faces the same balance of payment problem as is shown in the last section of this paper.

To avoid unnecessary complication the mathematical programming models discussed are restricted to transportation cost oriented models such as the Koopmans-Hitchcock transportation cost-minimization linear programming model which is used to facilitate understanding of one of the Takayama-Judge models to be discussed.

The structure of the paper is as follows: Section 1 sets out the problems around which the discussion is centred, as well as the assumptions of the models to be dealt with. In this section we compare the two programming models mentioned above and show the internal structure of the models. New theoretical aspects of the two models are revealed. In Section 2, examples and their solutions are given. In Section 3, extensions and modifications of the model and a few programming difficulties are dealt with.

Section 1

Linear programming methods can be used to solve special multilateral trade problems in circumstances where each country's total demand quantity, total supply quantity, and the transportation cost between any pair of countries is known; and where there are no legal restrictions to limit the actions of the profit seeking traders in each country. Under these circumstances we wish to ascertain the quantity and direction of trade between each possible pair of countries. Under normal trade conditions the above model is not acceptable, since it completely ignores price information. Since this shortcoming will be overcome when we deal with the Takayama-Judge model, we will be content with a formal treatment of the linear programming model.

Definitions and notation to be employed are as follows:

i, j denote the national demand and supply points where $i, j = 1, 2, \ldots, n$.

$X = [x_{ij}]$ denotes a column vector of the n^2 possible flow activities between the demand and supply points.

$T = [t_{ij}]$ denotes a column vector of the n^2 transport costs per unit between the supply and demand points.

$P_d = [p_i]$ denotes a column vector of the country's demand prices standardized by, say, the Australian dollar at the demand points.

$P_s = [p^i]$ denotes a column vector of the country's supply prices, standardized in the same way as P_d.

$D = [d_i]$ denotes a column vector of the fixed demand quantity in each country.

$S = [s_i]$ denotes a column vector of the fixed supply quantity in each country.

The Koopmans-Hitchcock transport cost-minimization linear programming problem is formulated as:

Problem I

To minimize

(1) $$T'X = \Sigma\Sigma t_{ij}x_{ij}$$

subject to

(2) $$GX \geqslant \begin{bmatrix} D \\ -S \end{bmatrix}$$

and

(3) $$X \geqslant 0,$$

where G is a $2n \times n^2$ matrix of the following form:

(4) $$G = \begin{bmatrix} +1 & & & +1 & & & \cdots & +1 & & \\ & +1 & & & +1 & & & & +1 & \\ & & \ddots & & & \ddots & \cdots & & & \ddots \\ & & +1 & & & +1 & & & & +1 \\ -1 & -1 \ \ldots & -1 & & & & & & & \\ & & & -1 & -1 \ \ldots & -1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & -1 & -1 \ \ldots & -1 \end{bmatrix}$$

Further we assume that summed over all supply and demand points:

(5) $$\Sigma d_i = \Sigma s_i.$$

For this problem we know that a finite minimum feasible solution, X^*, always exists [5]. The dual of Problem I can be specified as follows:

Problem II

To maximize

(6) $$\begin{bmatrix} D \\ -S \end{bmatrix}' \begin{bmatrix} P_d \\ P_s \end{bmatrix} = \Sigma p_i d_i - \Sigma p^i s_i$$

subject to

(7) $$G' P \leqslant T$$

where P equals $[P_d P_s]'$ and

(8) $$P \geqslant 0.$$

By the duality theorem of linear programming [3] we know that Problem II has a finite maximum feasible solution P^* equals $[P_d{}^* P_s{}^*]'$ and

(9) $$\begin{aligned} T' X^* &= [P^*]' \begin{bmatrix} D \\ -S \end{bmatrix} \\ &= \Sigma p_i{}^* d_i - \Sigma p^{i*} s^i. \end{aligned}$$

We also know that the optimum demand price is always equal to the optimum supply price in each country. This reduces the right hand side of equation (9) to $\Sigma p_i{}^*(d_i - s_i)$, which is a rather troublesome expression. Let us assume there exists another solution of the form $p_i{}^* + r$, for all i, where r is a finite constant. Unfortunately this solution satisfies all the conditions that P^* satisfies. For instance, equation (9) is satisfied since

$$\begin{aligned} \Sigma(p_i{}^* + r)(d_i - s_i) &= \Sigma p_i{}^*(d_i - s_i) + r\Sigma(d_i - s_i) \\ &= \Sigma p_i{}^*(d_i - s_i) + r(\Sigma d_i - \Sigma s_i) \\ &= \Sigma p_i{}^*(d_i - s_i). \end{aligned}$$

Thus the solution for the dual problem is not unique. From the applied programming point of view the non-uniqueness of the optimum prices is not satisfactory.

With this much background about the simplest type of transportation cost oriented trade model, we move to a more interesting, and more realistic, problem of the Enke-Samuelson [4] [16] spatial (partial) equilibrium type which plays an important role in international trade theories [9].

Consider n (> 2) countries trading homogeneous products. Each country constitutes a single and distinct market separated from the others but not isolated because transportation activities are carried on beween them with a transportation cost per physical unit of product which is independent of volume. Assume there are no legal restrictions against the profit-seeking traders in each country. For each country a demand function and a supply function are given as a linear function of the domestic price of the product. Given these conditions, we wish to ascertain:

(i) the real net price in each country;
(ii) the quantity of exports or imports for each country; and
(iii) the volume and direction of trade between each possible pair of countries.

Takayama and Judge [18] formulated this problem as follows:

Problem III

To maximize

$$(10)\qquad F(P) = \Sigma\alpha_i p_i - \tfrac{1}{2}\Sigma\beta_i(p_i)^2 - \Sigma\theta_i p^i - \tfrac{1}{2}\Sigma\gamma_i(p^i)^2$$
$$= \begin{bmatrix} \alpha \\ -\theta \end{bmatrix}' P - \tfrac{1}{2}P'\begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix} P$$

subject to restrictions (7) and (8), and where

$$(11)\qquad D = \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} - \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \beta_n \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix} = \alpha - \beta P_d,$$

and

$$(12)\qquad S = \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \theta_n \end{bmatrix} + \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \gamma_n \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix} = \theta + \gamma P_s.$$

To show the similarity between this problem in quadratic form and Problems I and II, let us reformulate the problem in the following way, assuming that there exists a solution P equals P^*:

Problem IV

To maximize

$$(13)\qquad \left\{\begin{bmatrix} \alpha \\ -\theta \end{bmatrix} - \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix} P^* \right\}' P = \begin{bmatrix} D^* \\ -S^* \end{bmatrix}' \begin{bmatrix} P_d \\ P_s \end{bmatrix}$$
$$= \Sigma d_i^* p_i - \Sigma s_i^* p^i$$

subject to restrictions (7) and (8), where, as a conclusion of our primal-dual programming formulation if a regular solution exists for Problem III [18], Σd_i^* equals Σs_i^*.

The above problem has a solution[1] at P equals P^* [18]. Problem IV is exactly the same as Problem II except that the optimum price vector P^* is already specified in this problem.

The dual of Problem IV provides us with the quantity solution we wished to ascertain. The dual is as follows:

[1] Due to the observation made on the solution of Problem II, the optimum solution for Problem IV is not unique. However, we can always find a solution which is the same as P^*.

Problem V

To minimize

(1) $$T' X$$

subject to

(14) $$GX \geqslant \begin{bmatrix} \alpha \\ -\theta \end{bmatrix} - \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix} P^* = \begin{bmatrix} D^* \\ -S^* \end{bmatrix}$$

and restriction (3).

Because of the nature of the G matrix, (14) can be written as

(15) $$\sum_{i=1}^{n} x_{ij} \geqq \alpha_j - \beta_j p_j^* = d_j^* \text{ for all } j,$$

and

(16) $$-\sum_{j=1}^{n} x_{ij} \geqq -\theta_i - \gamma_i p^{i*} = -s_i^* \text{ for all } i.$$

Inequalities in (15) and (16) will be replaced by equalities corresponding to (11) and (12) in our primal-dual quadratic programming formulation [18] which is presented as follows:

Problem VI

To maximize

(17) $$-V' X = -\Sigma\Sigma v_{ij} x_{ij} \leqq 0$$

subject to

(18) $$G' P + V = T,$$

(19) $$GX = \begin{bmatrix} \alpha \\ -\theta \end{bmatrix} - \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix} P,$$

and

(20) $$P \geqq 0, \quad X \geqq 0, \quad V \geqq 0,$$

where V is a slack vector with n^2 components v_{ij} arranged in the same order as the X vector.

Problem VI, once solved for the optimum P and X, provides such information as:

(i) the real net price in each country, P^*;
(ii) the quantity of exports and imports for each country[2]; and
(iii) the volume and direction of optimum trade between each possible pair of countries, i.e. x_{ij}^* for all i and j.

Equivalent short-cut formulations of Problems I, II, III can be given by using an excess supply or demand concept. An equivalent short-cut formulation of Problem I is given as:

Problem VII

To minimize

(21) $$T_o' X_o$$

subject to

[2] I.e. Σx_{ij}^* for all i with $j \neq i$, and Σx_{ij}^* for all j with $i \neq j$.

(22) $$G_oX_o \geqq [D - S] = [d_i - s_i] = [e_i] = E$$

and

(23) $$X_o \geqslant 0$$

where E is a column vector, X_o is the $(n^2 - n)$ column vector formed by removing x_{ii}, for all i, from the vector X, G_o is a $n \times (n^2 - n)$ matrix of the following form:

(24) $$G_o = \begin{bmatrix} -1 & -1 & \ldots & -1 & +1 & & & & \ldots & +1 & \\ +1 & & & & -1 & -1 & \ldots & -1 & \ldots & & +1 \\ & +1 & & & & +1 & & & & & \cdot \\ & & & & & & & & & & \cdot \\ & & & & & & & & & & \cdot \\ & \cdot & & & & \cdot & & & & & \\ & & \cdot & & & & \cdot & & \ldots & & +1 \\ & & & +1 & & & & +1 & -1 & -1 \; \ldots & -1 \end{bmatrix}$$

and T_o is the $(n^2 - n)$ column vector formed by removing t_{ii}, for all i, from the vector T and arranging the elements in the same order as in X_o. The dual is given by:

Problem VIII[3]

To maximize

(25) $$H(P) = E' P_d = \Sigma e_i p_i$$

subject to

(26) $$G_o' P_d \leqslant T_o$$

and

(27) $$P_d \geqslant 0.$$

The quadratic programming counterpart of Problem VIII is given by:

Problem IX

To maximize

(28) $$\begin{aligned} F(P) &= (\alpha - \theta)' P_d - \tfrac{1}{2} P_d'(\beta + \gamma) P_d \\ &= \Sigma(\alpha_i - \theta_i)p_i - \tfrac{1}{2}\Sigma(\beta_i + \gamma_i)(p_i)^2 \end{aligned}$$

subject to

(29) $$G_o' P_d \leqslant T_o$$

and

(30) $$P_d \geqslant 0.$$

Problem IX has the same constraint set as Problem VIII. However, because of the nature of the objective function, the optimum price set is given uniquely so long as $\beta_i + \gamma_i > 0$ for all i. Complete information on the optimum prices and export and import quantities of each country

[3] It is easy to see that the feasibility set $P = \{P_d \mid G'_o P_d \leqslant T_o \text{ and } P_d \geqslant 0\}$ is a polyhedral parallepiped containing a line $L(P) = \{P_d \mid p_1 = p_2 = \ldots = p_n\}$ passing through the origin, if $T_o \geqslant 0$. Thus the feasibility set P is non-empty. Since $\Sigma e_i = [e_1 \, e_2 \ldots e_n] \, [1 \; 1 \ldots 1]' = 0$, vector $E = [e_1 \, e_2 \ldots e_n]'$ is perpendicular to $L(P)$ and the whole hyperplane of the form (25) is parallel to the edges of the feasibility set P. This proves the existence and non-uniqueness of the solution for Problem VIII.

with other countries can be obtained by solving the quadratic primal-dual formulation [18].

The quadratic programming formulation of partial equilibrium models of international trade presented in Problems III and IX can be modified to incorporate import tariffs, export subsidies, import quotas, etc. Some such problems will be presented in the following section in the form of simple numerical examples along with their solutions.

Section 2

A very simple problem in the form of the Koopmans-Hitchcock transportation cost-minimization linear programme will be dealt with first, followed by the Takayama-Judge quadratic formulations.

Let us assume that there are three countries involved in trading one commodity under the following conditions:

(i) Excess supply or demand condition $(d_i - s_i)$:

country 1	country 2	country 3
34·7619	—69·0476	34·2857

(ii) Transportation costs:

$$t_{12} = t_{21} = 2;\; t_{13} = t_{31} = 2;\; t_{23} = t_{32} = 1.$$

Problems VII and VIII can be written as follows:

Problem VII

To minimize

(31) $$T_o' X_o = 2x_{12} + 2x_{13} + 2x_{21} + x_{23} + 2x_{31} + x_{32}$$

subject to

(32) $$\begin{bmatrix} -1 & -1 & 1 & & 1 & \\ 1 & & -1 & -1 & & 1 \\ & 1 & & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{21} \\ x_{23} \\ x_{31} \\ x_{32} \end{bmatrix} \geqslant \begin{bmatrix} 34{\cdot}7619 \\ -69{\cdot}0476 \\ 34{\cdot}2857 \end{bmatrix}$$

and

(33) $$X_o \geqslant 0.$$

Problem VIII

To maximize

(34) $$H(P) = 34{\cdot}6719 p_1 - 69{\cdot}0476 p_2 + 34{\cdot}2857 p_3$$

subject to

(35) $$\begin{bmatrix} -1 & 1 & \\ -1 & & 1 \\ 1 & -1 & \\ & -1 & 1 \\ 1 & & -1 \\ & 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \leqslant \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

and

(36) $$p_i \geqslant 0, \text{ for all } i = 1, 2, 3.$$

The exact solution is

$$p_1^* = 2 + p_2^*;\ p_2^* = r \geqslant 0;\ p_3^* = 1 + p_2^*;$$
$$x_{21}^* = 34{\cdot}7619;\ x_{23}^* = 34{\cdot}2857.$$

Model I

Let us turn to a more interesting problem in which both the demand and the supply functions for each country are given as linear functions, such as in the following case:

country 1	country 2	country 3
$d_1 = 200 - 10p_1$	$d_2 = 100 - 5p_2$	$d_3 = 160 - 8p_3$
$s_1 = -50 + 10p_1$	$s_2 = -50 + 20p_2$	$s_3 = -50 + 10p_3.$

The excess supply conditions are derived from the above as:

country 1	country 2	country 3
$e_1 = d_1 - s_1$	$e_2 = d_2 - s_2$	$e_3 = d_3 - s_3$
$= 250 - 20p_1$	$= 150 - 25p_2$	$= 210 - 18p_3.$

The transportation costs are assumed to be the same as those for the previous example. The quadratic primal formulation takes the following form:

Problem IX

To maximize

(37) $$F(P) = 250p_1 - 10(p_1)^2 + 150p_2 - 12{\cdot}5(p_2)^2 + 210p_3 - 9(p_3)^2$$

subject to (35) and (36).

The quadratic primal-dual formulation takes the following form:

Problem X

To maximize

(38) $$-X_o' V_o = \Sigma x_{ij} v_{ij} \leqslant 0,\ i \neq j$$

subject to (35) where V_o is the slack vector associated with (35) and

(39) $$\begin{bmatrix} -1 & -1 & 1 & & 1 & \\ 1 & & -1 & -1 & & 1 \\ & 1 & & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{21} \\ x_{23} \\ x_{31} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 250 - 20p_1 \\ 150 - 25p_2 \\ 210 - 18p_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

or

(39′) $$\begin{bmatrix} -1 & -1 & 1 & & 1 & \\ 1 & & -1 & -1 & & 1 \\ & 1 & & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{21} \\ x_{23} \\ x_{31} \\ x_{32} \end{bmatrix} + \begin{bmatrix} 20 & & \\ & 25 & \\ & & 18 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 250 \\ 150 \\ 210 \end{bmatrix}$$

and

(40) $$X_o \geqslant 0,\quad P_d \geqslant 0,\quad V_o \geqslant 0.$$

The solution is:

$$\begin{array}{llllll} p_1^* & = 10 \cdot 7620; & p_2^* & = 8 \cdot 7620; & p_3^* & = 9 \cdot 7620; \\ x_{11}^{**} & = 57 \cdot 6190; & x_{12}^* & = 0; & x_{13}^* & = 0; \\ x_{21}^* & = 34 \cdot 7619; & x_{22}^{**} & = 56 \cdot 1905; & x_{23}^* & = 34 \cdot 2857; \\ x_{31}^* & = 0; & x_{32}^* & = 0; & x_{33}^{**} & = 47 \cdot 5190; \end{array}$$

where the double starred figures are computed from the assumed demand and supply equations.

In the following examples we will consider situations involving import tariffs, quotas and export subsidies. It is in these cases that the quadratic primal-dual formulation shows its power and flexibility.

Model IIa

We use the same demand and supply functions and transportation costs as in Model I but assume that country 1 imposes import tariffs, denoted by δ_{21} and δ_{31}, both equal to $1 \cdot 0$ per unit of product. The problem is as follows:

Problem XIa

To maximize (37) subject to

$$(41) \quad \begin{bmatrix} -1 & 1 & \\ -1 & & 1 \\ 1 & -1 & \\ & -1 & 1 \\ 1 & & -1 \\ & 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \leqslant \begin{bmatrix} t_{12} \\ t_{13} \\ t_{21} + \delta_{21} \\ t_{23} \\ t_{31} + \delta_{31} \\ t_{32} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2+1 \\ 1 \\ 2+1 \\ 1 \end{bmatrix}$$

and (39′) and (40). The solution is as follows:

$$\begin{array}{llllll} p_1^* & = 11 \cdot 1587; & p_2 & = 8 \cdot 1487; & p_3^* & = 10 \cdot 1587; \\ x_{11}^{**} & = 51 \cdot 5873; & x_{12}^* & = 0; & x_{13}^* & = 0; \\ x_{21}^* & = 26 \cdot 8254; & x_{22}^{**} & = 59 \cdot 2064; & x_{23}^* & = 27 \cdot 1429; \\ x_{31}^* & = 0; & x_{32}^* & = 0; & x_{33}^{**} & = 51 \cdot 5873. \end{array}$$

Model IIb

Let us assume that country 2 provides export subsidies, denoted by $\tau_{21}(= 0 \cdot 5)$ and $\tau_{23}(=0 \cdot 5)$ per unit of product to help the industry. We can formulate and solve this problem as follows:

Problem XIb

To maximize (37) subject to

$$(42) \quad \begin{bmatrix} -1 & 1 & \\ -1 & & 1 \\ 1 & -1 & \\ & -1 & 1 \\ 1 & & -1 \\ & 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \\ \\ \end{bmatrix} \leqslant \begin{bmatrix} t_{12} \\ t_{13} \\ t_{21} - \tau_{21} \\ t_{23} - \tau_{23} \\ t_{31} \\ t_{32} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 - 0 \cdot 5 \\ 1 - 0 \cdot 5 \\ 2 \\ 1 \end{bmatrix}$$

and (39′) and (40). The solution is:

$$\begin{array}{lll} p_1{}^* = 10{\cdot}5635; & p_2{}^* = 9{\cdot}0635; & p_3{}^* = 9{\cdot}5635; \\ x_{11}{}^{**} = 55{\cdot}6349; & x_{12}{}^* = 0; & x_{13}{}^* = 0; \\ x_{21}{}^* = 38{\cdot}7302; & x_{22}{}^{**} = 54{\cdot}6825; & x_{23}{}^* = 37{\cdot}8571; \\ x_{31}{}^* = 0; & x_{32}{}^* = 0; & x_{33}{}^{**} = 45{\cdot}6349. \end{array}$$

Consistency of the solutions so far obtained with commonsense in the so-called "partial equilibrium short-run" framework of international trade theory is rather obvious.

The next model provides an analytical tool for solving a problem with import quotas imposed by some importing country or countries.

Model III

Imposition of an import quota by some importing country or countries takes us out of a rather comfortable world, in which the existence and uniqueness of the solutions are known, to a challenging one where we know very little. A typical semi-definite quadratic programming problem of this more realistic type is as follows:

Assume that country 1 and country 2 restrict their imports of the product to 30 units and 20 units respectively. This makes us modify our formulation to the following:

Problem XII

To maximize

$$(43) \qquad F(P) = 250p_1 - 10(p_1)^2 + 150p_2 - 12{\cdot}5(p_2)^2 + 210p_3 - 9(p_3)^2 - 30p_4 - 20p_5$$

subject to

$$(44) \qquad \begin{bmatrix} -1 & 1 & & & \\ -1 & & 1 & & -1 \\ 1 & -1 & & -1 & \\ & -1 & 1 & & -1 \\ 1 & & -1 & -1 & \\ & 1 & -1 & & \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} + \begin{bmatrix} v_{12} \\ v_{13} \\ v_{21} \\ v_{23} \\ v_{31} \\ v_{32} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

and

$$(45) \qquad V_o \geqslant 0, \text{ and } P_a \geqslant 0$$

where P_a equals $(p_1, p_2, p_3, p_4, p_5)'$ and p_4 and p_5 can be considered administrative costs of the imposition of the import quota by country 1 and country 2 respectively.

For completeness, the quadratic primal-dual formulation of Problem XII is given below.

Problem XIII

To maximize (38) subject to (44) and

$$(46) \qquad \begin{bmatrix} -1 & -1 & 1 & & 1 & \\ 1 & & -1 & -1 & & 1 \\ & 1 & & 1 & -1 & -1 \\ & & -1 & & -1 & \\ & -1 & & -1 & & \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{21} \\ x_{23} \\ x_{31} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 250 - 20p_1 \\ 150 - 25p_2 \\ 210 - 18p_3 \\ -30 \\ -20 \end{bmatrix}$$

or

$$(46') \quad \begin{bmatrix} -1 & -1 & 1 & & 1 & \\ 1 & & -1 & -1 & & 1 \\ & 1 & & 1 & -1 & -1 \\ & & -1 & & -1 & \\ -1 & & & -1 & & \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{21} \\ x_{23} \\ x_{31} \\ x_{32} \end{bmatrix} + \begin{bmatrix} 20 & & & & \\ & 25 & & & \\ & & 18 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = \begin{bmatrix} 250 \\ 150 \\ 210 \\ -30 \\ -20 \end{bmatrix}$$

and

$$(47) \qquad V_o \geqslant 0, \quad P_a \geqslant 0, \quad V_o \geqslant 0.$$

The solution is:

$p_o^* = 11{\cdot}0$; $p_2^* = 8{\cdot}0$; $p_3^* = 10{\cdot}5555$;
$p_4^* = 1{\cdot}0$; $p_5^* = 1{\cdot}5555$;
$x_{11}^{**} = 50{\cdot}0$; $x_{12}^* = 0$; $x_{13}^* = 0$;
$x_{21}^* = 30{\cdot}0$; $x_{22}^{**} = 60{\cdot}0$; $x_{23}^* = 20{\cdot}0$;
$x_{31}^* = 0$; $x_{32}^* = 0$; $x_{33}^{**} = 55{\cdot}5555$.

By virtue of the non-vacuousness of the constraint set R of (35) and also (44), if the objective function is only concave or negative semi-definite, then a solution or a unique solution always exists for our price formulation, Models I through III. If further, we allow slack variables, say Y_o corresponding to Y in [18], for the second constraint such as (39′) and (46′) in the quadratic primal-dual formulation, then we will always find a solution if the objective function is negative semi-definite. This statement, given without rigorous proof, forms a strong basis for the efficiency of our programming formulation of the type of problem dealt with in this section. Further extensions to encompass multi-product multi-country cases, changes of exchange rates, administratively fixed prices, etc., can be solved effectively [18].

Section 3

There are nearly as many different problem formulations (models) as there are specialists in any economic field. In the framework of transportation cost oriented partial equilibrium models, an intertemporal partial equilibrium model [19] has opened up a way to solve one of the knotty problems in the field of spatial intertemporal analysis.

Another extension of spatial equilibrium models can be found in a quadratic programming formulation of an interregional activity analysis model proposed by Takayama and Judge [20]. Since it is difficult to treat this problem in a simple manner, I will not discuss this model in mathematical terms. Aside from the practical solvability of programming problems, it was proved by McKenzie [12] and Uzawa [22] that generally there exists a general equilibrium solution for a model in which either well-behaved demand functions for all commodities exist [12] or there is a well-behaved welfare function and the supply of all commodities is generated by the Koopmans-type activity analysis model [22].

From the mathematical programming point of view, proofs for the existence of a solution based on the so-called "fixed point theorem" employed by McKenzie [12], Arrow-Debreu [1] and Hadley and Kemp

[7] do not help in analytically approaching the equilbrium solution (if there is one) for problems which contain the "balance of payment" of each country as one necessary condition.

In most of the mathematical programming models, we face the situation in which the primal variables are the activity levels or the quantities of production and the dual variables are the shadow prices of resources (as in Problems I and II, for instance), or vice versa (as in Problems III and V). In such situations it is not possible to control the value represented by the sum of products of the primal variables and the dual variables of the model.[4] A method of attaining the balance of payments has been suggested [21] but has not been tested for its efficiency.

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[4] A simple example is a multi-commodity multi-country version of Model I. The balance of payment condition is:

$$\sum_{k=1}^{m} \sum_{i=1}^{n} p_j^k \, x_{ij}^k - \sum_{k=1}^{m} \sum_{i=1}^{n} p_j^k \, x_{ji}^k = 0$$

where k denotes commodity. Treating this condition as another constraint to be handled internally by introducing another Lagrangian variable does not seem to be operational.

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