Imposing Curvature Restrictions on a Translog Cost Function using a Markov Chain Monte Carlo Simulation Approach

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Abstract

Using Kansas Farm data from 1973 to 1998, curvature restrictions are imposed on a translog cost function. Using uninformative priors with indicator functions representing distribution and inequality constraints, a Markov Chain Monte Carlo Simulation method is used to estimate parameters and check curvature at each point. Comparison is made to the Cholesky factorization method commonly used with the normalized quadratic functional form.

Introduction:

Estimating cost functions using flexible functional forms is common in that they offer advantages in terms of reducing specification errors, increasing deduction, and obtaining price elasticities at a point without imposing stringent restrictions on input elasticities. Symmetry, homogeneity and curvature conditions are required for a function to be consistent with economic theory. But violation of curvature properties is one of the problems encountered with flexible functional forms. Satisfying global curvature conditions that are consistent with economic theory are extremely important when estimating functional forms of cost, profit and production functions. Further satisfying curvature restrictions without sacrificing the flexibility of the functional form is a challenging task. Though, prior studies have dealt with satisfying curvature conditions on flexible functional forms (Terrell 1996, Featherstone and Moss 1994, Talpaz *et.al* 1989, Gallant and Golub 1984) there are limited studies (Lau 1978, Geweke 1986, Griffiths *et al*. 2000) done to address the problem of imposing global curvature conditions without destroying the flexibility properties of the functional form.

Curvature has often been imposed using the Cholesky decomposition method (Lau). Though this method satisfies global curvature restrictions for the normalized quadratic functional form, it poses problems for the translog functional form. For the translog functional form, imposing curvature restrictions can only be done locally. In this paper we address curvature conditions by employing a Markov chain Monte Carlo (MCMC) simulation approach. Using the Metropolis Hastings Algorithm, we estimate a translog system of cost and share equations for Kansas Farm data from 1973-1998. Comparison is also made to the Cholesky Factorization Normalized Quadratic method. Thus, the main objectives of this paper are to:

a) empirically test the Markov Chain Monte Carlo Simulation Method for imposing curvature restrictions on a translog cost function.

b) compare estimates from the translog cost function with and without curvature restrictions imposed.

c) estimate a normalized quadratic cost function with curvature imposed using the Cholesky decomposition approach.

d) compare the Cholesky factorization method with the Markov Chain Monte Carlo Simulation Method.

e) compare economic estimates of the normalized quadratic cost function with the translog cost function.

The remainder of the paper is divided into six sections. Section one discusses the normalized quadratic cost function and curvature imposition using the Cholesky factorization method. Section two details the translog cost function used in this paper. Section three introduces the Markov chain Monte Carlo Simulation method to impose curvature restrictions. Section four discusses the data sources used in this paper. Section five compares the results obtained under different approaches while section six provides concluding comments.

1. The Normalized Quadratic Cost Function:

The normalized quadratic function estimated in this paper takes the following general form:

$$C^{*'} = b_0 + \sum_{i=1}^{m-1} b_i W_i^{'} + \sum_{i=m+1}^{n} b_i Y_i + 1/2\left(\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} b_{ij} W_i^{'} W_j^{'} + \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} b_{ij} Y_i Y_j \right) + \sum_{i=1}^{m-1}\sum_{j=m+1}^{n} b_{ij} W_i^{'} Y_j$$

where $C^{*'} = C^*/W_m$ *and* $W_i^{'} = W_i W_m$ and Wi are the input prices while Yi are the output quantities. Using Shephard's lemma we can obtain the factor demand equations as

$$\partial C/\partial W_i = X_i$$

Cross- equation symmetry restrictions are imposed by setting

$$b_{ij}=b_{ji} \text{ for all } i,j$$

and homogeneity is imposed by normalization.

Curvature retrictions on the input side are satisfied if the Hessian matrix of prices is negative semi-definite while on the output side curvature restrictions hold if the Hessian matrix of quantities is positive semi-definite. Curvature restrictions are first checked by calculating the eigen values for the Hessian matrix of input prices and output. Eigenvalues need to be negative for the matrix of prices to satisfy concavity and positive for the matrix of output to satisfy convexity.

If curvature restrictions do not hold, curvature is imposed using the Cholesky decomposition method. A negative semi-definite Hessian matrix ensures that appropriate curvature restrictions are met on the input side. We can ensure negative-semidefiniteness of the Hessian matrix by letting

$$\mathbf{B} \equiv -\mathbf{A}\mathbf{A}^{\mathbf{T}}$$

where B represents matrices of the parameters of the system we wish to estimate and A is a n*n lower triangular matrix. Using Cholesky decomposition we reparameterize the model and estimate the parameters in A instead of the parameters in B. This ensures that the Hessian matrix $B \equiv -\mathbf{AA}^{\mathbf{T}}$ is negative semi-definite. (Featherstone and Moss 1994). A similar approach is used to ensure positive semi-definiteness on the output side.

Own price elasticities are calculated as follows

$$Z_{ii} = (\partial X_i / \partial Wi)(W_i /X_i)$$

while cross price elasticities are calculated as follows

$$Z_{ii} = (\partial X_i/W_j \ )(W_j /X_i)$$

2. The Translog Cost Function:

We next estimate the translog cost function without curvature restrictions imposed. Letting *w* denote the price of input *i* and *y* denote the output *j*. Thus the general form for the translog cost function with *n* inputs is as follows:

$$\ln c(w,y) = a0 + \sum_{i=1}^{n} a_i \ln w_i + \sum_{i=1}^{n} a_y \ln y + 1/2 \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \ln w_i \ln w_j + \sum_{i=1}^{n}\sum_{y=1}^{m} a_{iy} \ln w_i \ln y + 1/2 \sum_{y=1}^{m}\sum_{y=1}^{m} a_{yy} \ln y \ln y$$

where $a_{ij} = a_{ji}$ for all *i,j*

$$\sum_{i=1}^{n} a_i = 1$$

$$\sum_{j=1}^{n} a_{ij} = 0 (i = 1,...,n)$$

$\sum_{j=1}^{n} a_{ij} = 0$ ensures that homogeneity of degree one in factor prices is imposed in the translog cost function. Symmetry is imposed by setting $a_{ij}=a_{ji}$ for all *i,j*. The parameters of the translog cost function are estimated as a system of equations which includes the log cost function and *n*-1 share equations. By applying Shepherds lemma, the *n* share equations in the translog cost function are as follows:

$$s_i\,(w,q) = a_i + \sum_{j=1}^{n} a_{ij} \ln w_j + a_{iy} \ln y$$

Monotonicity in input prices for the translog cost function requires non-negative shares. Concavity restrictions on the input side can be checked by ensuring that the Hessian matrix is negative semi-definite. Alternatively, the Allen partial matrix can be used to check whether curvature restrictions hold. The Allen partial matrix is defined as follows:

$$Z_{ij} / s_j$$

where Z_{ij} is the elasticity and s_j is the share equation.

If the Allen partial matrix is concave, then the Hessian matrix is also concave. Curvature on the output side is checked by ensuring that the Hessian matrix is positive semi-definite. For the translog cost function curvature needs to be checked at each point. own price elasticities are calculated as:

$$Z_{ii} = B_{ii} / s_i + s_i - 1$$

where Z_{ii} is the own price elasticity and s_i is the i^{th} share equation.

Cross price elasticities are calculated as:

$$Z_{ii} = B_{ij} / s_i + s_j$$

In this paper we estimate a system of 7 share equations and the log cost function. The 8^{th} share equation and the resulting parameters are recovered by homogeneity.

3. Markov Chain Monte Carlo Simulation Approach (MCMC):

Due to the highly non-linear nature of the translog cost function, it is necessary to check curvature at each point. We use the Bayesian methodology to impose curvature restrictions on the translog cost function. The Bayesian approach is being increasingly used in the recent years. This method uses uninformative priors with indicator functions representing distribution and inequality constraints. Using a Markov Chain Monte Carlo simulation method, parameters are estimated. Curvature can then be checked at each point. If curvature restrictions hold, the parameter estimates are retained otherwise they are discarded and re-sampling is done. This approach can be extremely useful in obtaining reliable elasticity estimates for studies that require the use of flexible functional forms. It is further useful to test the robustness of the estimates within the observed data range as well as outside the data range.

The Bayesian Approach is based on Bayes Theorem which states that

$$f(\beta,\Sigma|Y,X) \propto L(Y,X|\beta,\Sigma)p(\beta,\Sigma)$$

where $\propto$ denotes 'proportional to', $f(\beta,\Sigma|Y,X)$ represents the posterior joint density function for β and Σ given Y and X, $L(Y,X|\beta,\Sigma)$ is the likelihood function and $p(\beta,\Sigma)$ the prior density function for β and Σ.

Using this approach and assuming that the distribution of residuals is multivariate normal, the likelihood function can be written as:

$$L(Y,X|\beta,\Sigma) \propto |\Sigma|^{-N/2}\exp[-0.5\mathrm{tr}(R\Sigma^{-1})$$

where R denotes the a symmetric matrix and N is the number of observations.

In addition, a non-informative prior is also used to permit better comparison of maximum likelihood results with Bayesian results irrespective of availability of information on monotonicity and concavity (Griffiths *et al*. 2000). Further, using a non-informative prior allows for a consistent algebraic form of the prior density function. Thus, the algebraic form does not alter upon availability of information on monotonicity and concavity despite the fact that the region over which the prior density function is defined varies. This also holds for the joint posterior density. We use the following non-informative prior:

$$p(\beta,\Sigma) = p(\beta)p(\Sigma)I(\beta \in h_s)$$

where I(.) denotes an indicator function which resumes a value of 1 if the argument holds and h_s represents the set of permissible parameter values when information on monotonicity and curvature (s = 2) is available and when (s = 1) it is not.

Thus, the posterior density assuming non-informative priors can be expressed as follows:

$$F(\beta,\Sigma|Y,X) \propto [|\Sigma|]^{-(N+I+1)/2}\ \mathrm{EXP}[0.5tr(R^*\Sigma^{-1})]I(\beta \in h_s) \quad s = 1,2.$$

We use the Metropolis-Hastings Algorithm to do the Bayesian estimation. This method has the advantage of drawing finite samples indirectly from the marginal probability density without derivation of the density itself. This approach allows us to impose monotonicity and curvature restrictions at a given set of prices. The procedure for the Metropolis-Hastings algorithm proposed by Griffiths et al. is described below:

Step 1: Specify an arbitrary starting value k^0 which satisfies the constraints of the translog cost function. and set the iteration level at i=0.

Step 2: Use the current value of k^i and a symmetric transition density transition density to generate the next candidate value in the sequence k^c.

Step 3: Use the candidate value generated k^c to test the monotonicity and curvature restrictions imposed. If any of the restrictions are violated then set $u(k^i,k^c) = 0$ and go to step five.

Step 4: Estimate $u(k^i,k^c)=\min(g(k^c)/g(k^i)),1)$ where g(k) is the kernel of f(k|Y,X). The kernel g(k) is acquired by integrating $\sum$ out of the joint posterior density function. Thus, g(k) is as follows (see Judge *et al*. 2000 for details):

$$f(k|Y,X) \propto |R|^{-N/2} I(k \in h_2) = g(k)$$

Step 5: Generate an independent uniform random variable U from the interval [0,1].

Step 6: Set $k^{i+1} = \{k^c$ if $U < U(k^i,k^c)$

Step 7: Set $i = i+1$ and go back to step 2.

This iteration results in a chain $k^1, k^2, \ldots$,which has a property that for a large i, k_i+1 is a sample point from f(k|Y,X). Thus, f(k|Y,X) can be regarded as the posterior joint density for k given Y and X which gives us all required information about k after Y and X have been observed from the sample. Essentially, the sequence $k^{i+1}, \ldots k^{k+m}$ can be regarded as a sample for f(k|Y,X) which satisfies monotonicity and curvature constraints. Curvature restrictions are checked in step 3 by using the maximum eigen value of the Hessian matrix evaluated. We chose starting values of $\alpha_i = 0.125$ ($i = 1,\ldots,7$) and $\alpha_{ij} = 0$ for all $i \neq j$. The starting values were chosen such that they satisfied monotonicity and curvature restrictions. The transition density we use $q(k^i,k^c)$ is arbitrary. The usual procedure is to assume multivariate normal distribution for the transition density which has mean k^i and a covariance matrix equal to the estimated covariance matrix of the restricted SUR estimator. In order to determine the rate at which the initial candidate value is accepted as the next value in the sequence, the covariance matrix is multiplied by a tuning constant *h*. This tuning constant was set at h=0.001. The value of *h* was chosen by trial and error. We found that a smaller tuning generally raises the acceptance. With the tuning constant set at h= 0.001 we obtained an acceptance rate of approximately 64 percent.

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<u>4. Data Sources</u>:

Kansas farm data for a period from 1973-1998 is used in this analysis. The data comprises of observations for 106 farms over a period of 26 years amounting to 2756 observations. In the translog model zero output quantities for livestock were substituted with a value of 10 percent of the mean to eliminate missing observations and estimation problems when taking the natural logarithm. There are eight inputs (seed, fertilizer, pesticides, seed, energy, labor, land and machinery) and two output quantities for crop and livestock production. The normalized quadratic cost function was estimated for the entire sample size, i.e. 2756 observations. The estimation was done in SHAZAM 9.0. The translog cost function was estimated for a subset of the sample due to the size of the data set and the length of time involved to run the entire data set. Only 200 observations were used in the estimation. This estimation was done in GAUSS 3.2.

<u>5. Results</u>

Parameter estimates and elasticities for the normalized quadratic cost function with curvature imposed for a system of eight inputs and two outputs are presented in table 1. After imposing curvature all restrictions are satisfied. Except for the own price elasticities for labor and machinery all own price elasticities are inelastic.

The price elasticties and the bootstrapped confidence intervals for the elasticity estimates from the Bayesian approach are presented in table 2. The confidence intervals were constructed after the burn in period. All own price elasticities are inelastic expect for the elasticities for the labor and land input.

Parameter estimates from the Bayesian approach are presented in table 3. Of the output parameters only $\gamma 22$ is statistically significant. Of the own price input parameters only $\beta 22$, $\beta 33$ and $\beta 77$ are significant at the 1 percent level.

6. Conclusions:

A Markov Chain Monte Carlo Simulation was used to impose curvature restrictions on a translog cost function. A normalized quadratic cost function was also estimated and curvature restrictions were imposed using the Cholesky factorization method. Under both approaches curvature restrictions were met after imposing curvature. The own-price elasticity estimates were smaller for the normalized quadratic cost function. Except for two all other own price elasticities were inelastic under both the approaches. All cross price elasticities were inelastic for the translog cost function approach while for the normalized quadratic cost function expect for two, all other cross price elasticities were also inelastic. Of the 55 parameters estimated using the Bayesian approach 26 were significant at the 1 percent level.

Table 1: Price Elasticities at Mean for the Normalized Quadratic Cost Function with curvature imposed.

	SEED	FERT	CHEM	FEED	FUEL	WAGE	RENT	MACH
SEED	-0.224118	0.01932235	-0.010715	-0.093368	-0.091567	-0.2570446	0.3411748	0.2424159
FERT	0.0488438	-0.1969666	0.1081472	0.2450214	-0.101205	-0.2255655	0.2241054	-0.102380
CHEM	-0.010730	0.2067133	-0.320122	-0.230893	0.0059759	0.1670780	-0.039108	0.2210881
FEED	-0.022380	0.1120929	-0.055262	-0.248595	0.0626132	0.1234789	-0.383366	0.4114198
FUEL	-0.061153	-0.1290037	0.0039852	0.1744573	-0.275801	-0.5216861	0.5571043	0.2520977
WAGE	-0.219786	-0.3681126	0.1426508	0.4404802	-0.667912	-1.374704	1.337485	0.7099008
RENT	0.0703081	0.08814477	-0.008047	-0.329597	0.1719030	0.3223481	-0.860611	0.5455527
MACH	0.0474346	-0.03823534	0.0431978	0.3358620	0.0738620	0.1624575	0.5180150	-1.142594

Table 2 Elasticity Estimates for the Translog Model with Bootstrapped 90% Percentile Confidence Intervals by Bayesian Method.

	Price Elasticities							
	SEED	**FERT**	**CHEM**	**FEED**	**FUEL**	**WAGE**	**RENT**	**MACH**
SEED	-0.956627	0.129004	0.079331	0.166465	0.060682	0.273167	-0.024394	0.264637
FERT	0.127824	-0.589615	0.018666	0.221481	0.043515	0.015552	-0.15690	-0.165327
CHEM	0.079907	0.018975	-0.744174	0.105662	-0.082558	-0.007192	0.403942	0.235596
FEED	0.171593	0.230410	0.108131	-0.905899	0.147298	0.330716	-0.227245	0.166589
FUEL	0.060648	0.043892	-0.081916	0.142816	-0.680574	0.017341	0.284463	0.214841
WAGE	0.275672	0.015840	-0.007206	0.323776	0.017510	-1.135494	0.670095	-0.146072
RENT	-0.023908	-0.015520	0.393047	-0.216068	0.278959	0.650792	-1.059589	-0.030924
MACH	0.264891	0.167013	0.234120	0.161766	0.215168	-0.144883	-0.031582	-0.869994

90% Confidence Interval
Upper Critical Value

	SEED	**FERT**	**CHEM**	**FEED**	**FUEL**	**WAGE**	**RENT**	**MACH**
SEED	-0.873034	0.234028	0.169424	0.236886	0.161836	0.363469	-0.044980	0.315040
FERT	0.233444	-0.486349	0.048240	0.253641	0.172457	0.171226	0.247076	0.185062
CHEM	0.171334	0.048073	-0.685772	0.229992	0.033320	0.043450	0.537439	0.268365
FEED	0.242064	0.260243	0.225158	-0.801435	0.208938	0.335444	0.167871	0.243274
FUEL	0.160097	0.169242	0.033006	0.205441	-0.662257	0.171024	0.286307	0.266982
WAGE	0.378119	0.175654	0.044322	0.337895	0.178244	-0.789256	0.764069	-0.053672
RENT	-0.044425	0.236511	0.515565	0.161908	0.280454	0.724267	-0.970022	0.271734
MACH	0.305260	0.188346	0.256982	0.238910	0.262254	-0.050691	0.271889	-0.790628

Lower Critical Value

	SEED	**FERT**	**CHEM**	**FEED**	**FUEL**	**WAGE**	**RENT**	**MACH**
SEED	-1.151637	0.130613	0.026019	0.103490	0.063144	0.259840	-0.221659	0.205197
FERT	0.128977	-0.661072	-0.160225	-0.015629	-0.031951	-0.048408	-0.022567	-0.041567
CHEM	0.026338	-0.162282	-0.845017	0.105547	-0.056209	-0.072312	0.240630	0.145352
FEED	0.102352	-0.015432	0.104761	-1.137286	0.051420	0.058307	-0.020313	0.074611
FUEL	0.063269	-0.032158	-0.055832	0.049255	-0.864193	0.015377	0.087518	0.151681
WAGE	0.268511	-0.049455	-0.074891	0.059364	0.016195	-1.197084	0.371130	-0.337549
RENT	-0.212764	-0.021693	0.229834	-0.197365	0.085133	0.349765	-1.463842	0.047275
MACH	0.207045	-0.040506	0.144113	0.070699	0.148341	-0.327453	0.047067	-1.031926

Table 3: Parameter Estimates from the Bayesian Approach

	Parameter	Standard Error
α0	-39.0723*	0.517638
α1	0.073781*	0.014301
α2	0.133218*	0.023053
α3	0.202415*	0.02332
α4	0.051473	0.041825
α5	0.126489*	0.017575
α6	0.177866*	0.015272
α7	0.129551*	0.03366
α8	0.020451	0.043209
α9	-0.36586*	0.170293
β11	-0.01393	0.010498
β12	0.007832*	0.003641
β13	-0.00523	0.005229
β14	0.004792	0.004361
β15	-0.00263	0.004015
β16	0.023855*	0.003826
β17	-0.03045*	0.006026
β22	0.036498*	0.006318
β23	-0.02347*	0.007731
β24	0.004091	0.012694
β25	-0.01013	0.008841
β26	-0.0075	0.008907
β27	-0.00494	0.010225
β33	0.014957*	0.006481
β34	0.003579	0.004464
β35	-0.01783	0.003302
β36	-0.01724	0.004316
β37	0.036297	0.011686
β44	-0.01419	0.014033
β45	0.003108	0.005771
β46	0.009176	0.010852
β47	-0.0181	0.014619
β55	0.013297	0.009003
β56	-0.00592	0.006777
β57	0.008789	0.007771
β66	-0.01291	0.014879
β67	0.051968*	0.014617
β77	-0.04709*	0.01883
β18	0.002277	0.001332
β19	-0.00319	0.001622
β28	0.000506	0.001223
β29	-0.00259*	0.00095
β38	0.001992	0.00115

Table 3: Parameter Estimates from the Bayesian Approach (con't)

	Parameter	Standard Error
$\beta39$	-0.0031*	0.001085
$\beta48$	-0.01065	0.005757
$\beta49$	0.015919*	0.007717
$\beta58$	-0.0003	0.000995
$\beta59$	0.001496	0.001255
$\beta68$	0.00328	0.003643
$\beta69$	0.004557*	0.002131
$\beta78$	-0.00641*	0.002279
$\beta79$	-0.00876*	0.003187
$\gamma11$	0.001685	0.004459
$\gamma22$	0.062805*	0.027085
$\gamma12$	-0.00434	0.006479

*** indicates significance at the 1 percent level**

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