WANTED:
No Arbitrage: Dead or Alive

A comparison of hedging with futures contracts or resource reserves for arbitrage-free resource harvest contract valuation

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Paper Presented at the American Agricultural Economics Association Annual Meeting, Tampa, Florida, August 2000

I. Introduction

Tradable permits, leases, and contracts are methods of allocating access and use of valuable natural resources. The valuation issue may be as broad as resource services, or as specific as the market value of the harvested commodity. Even from the relatively narrow perspective of market value, there are at least three points to address in valuing resources: price volatility, inventory size and the appropriate discount rate. Traditionally, users have utilized net present value (NPV) methods of valuation to determine the investment value of resource harvest. NPV methods are insufficient in at least three accounts: First, they project a deterministic price path. This assumption over-values the project and results in harvesting too much, too soon (Arrow and Fisher, 1974). Second, by over-valuing the investment, they allow for arbitrage opportunities. That is, risk-free profits are possible. Third, the discount rate is subjective. Value discrepancies are often accounted for by subjectively altering the discount rate to account for price volatility. High discount rates place a low value on the future, and encourage early harvest. Conversely, low discount rates promote slower, more conservative use.

Methods which account for risk and uncertainty overcome the valuation problems associated with NPV (e.g. Arrow and Fisher, Dixit and Pindyck,). By applying modern financial portfolio theory to the resource investment, stochastic parameters may be integrated into the valuation problem. Also, an objective discount rate may be obtained by establishing a self-financing, risk-mitigating portfolio of risk-free bonds and another asset. This "other asset" is typically the underlying asset (in the case of financial stocks). However, futures may also be used as the hedging mechanism (Brennan and Schwartz). Most recently, Burnes, Thomann and Waymire (1999) have shown that it is possible to construct a portfolio that uses the undeveloped or standing asset (i.e. trees) as the hedging mechanism. The choices between not hedging, hedging with futures or hedging with resource reserves impacts the contract value.

The objective of this project is to evaluate the use of resource reserves versus futures contracts as hedging mechanisms for uncertain renewable resource harvest investment under no-arbitrage. This will be accomplished by first considering the value of a timber lease under stochastic prices using futures contracts as the hedging mechanism. Second, the value of the same timber lease is determined using the standing resource (a resource reserve) as the hedging mechanism. With each method, it will be possible to assess the role of discount rates, costs, and price volatility.

Section two provides a literature review of the motivating papers for this study, section three develops the methods used to form the analysis and results. Specifically, arbitrage is discussed, hedging portfolios are modeled, and numerical examples of means and variances under three different expectation measures are provided. Section four shares the results, and section five offers conclusions and policy implications of this work.

II Literature Review

Two pieces of literature form the basis for the development and extension of the models used in this paper. and Schwartz (1985) develop a method of using futures as the portfolio element in the hedge portfolio for natural resource investment. This is because there is often not a market in the underlying resource/asset, but active futures markets exist. Second, Burnes, Thomann and Waymire (1999) apply option methods to a natural resource harvest contract (a timber contract) given stochastic prices. Using option valuation methods, they develop a hedge portfolio based on a resource reserve.

Brennan and Schwartz (1985) discuss the case of natural resource investment under price uncertainty using futures to replicate cash flows in the hedge-portfolio. Their motivation, however, is to determine an optimal harvesting policy given stochastic prices that are hedged using the methods of no arbitrage. They develop a continuous model to determine an optimal operating policy given stochastic output prices. Another contribution of their paper is to explore the notion of "convenience yield" with respect to the futures contract holder. Brennan and Schwartz validate the use of futures to replicate cash flows, however they do not determine an explicit valuation solution for the natural resource.

Burnes, Thomann and Waymire (1999) model an arbitrage free value of a timber lease given a known inventory but stochastic prices. The authors also use a contingent claims approach. However, the hedging portfolio is comprised of a risk-free asset and a standing resource reserve of the harvestable resource. The authors explicitly value the harvest contract and perform sensitivity analyses with respect to harvesting costs, discount rates, volatility and contract duration. The authors work in a one-dimensional continuous case.

The impact of the alternative portfolio choices on the contract value in the Brennan and Schwartz and Burnes et al papers is the key issue to be addressed in this paper. That is, how does using forwards versus a resource reserve as a hedging mechanism affect the value of a renewable resource contract.

III Methods

The "usual" valuation approach, is to look at the probabilities of various outcomes and then take an expectation of the price at a future time period. Once this expectation is formed, one must choose the appropriate discount rate with which to bring this expected value into current terms. The first issue to be addressed is that simply taking an expected value using the price process over-values the investment and leads to arbitrage opportunities. Second, simply choosing a discount rate is not straightforward. Doing so leaves one open to disagreement regarding the appropriate rate given risk, and also provides an opportunity for arbitrage. Finally, in the case where the valuation decision includes non-market entities, such as trees, one needs to consider an appropriate expectation mechanism and discount rate.

To address these points, the notion of a hedging portfolio which relies on an underlying asset and a risk-free bond is developed. These tools are necessary to address the existence and implication of arbitrage. Arbitrage means that risk-free profits exist. A product owner and seller will not want to undervalue her product relative to the market. If she did, people could buy her product and return it to another vendor and receive more back than what she paid for it. This is a simplified example, but the idea on which this section will expand. The motivation for using the underlying asset *and* bonds to develop pricing expectations is that we use all of the tools of the market to abdicate the existence of arbitrage from this investment opportunity. Another benefit of using the arbitrage-free approach is that it objectively determines the risk-free interest rate as the appropriate discount rate.

With the hedge portfolios constructed, we have a risk-neutral probability measure, called Q, under which we can make our expectations of future prices. Q represents a martingale, which is a change of measure from the expectations under the actual price process called P. The measure Q is further modified to accommodate the reserve case. Means and variances of expectations under the various hedging scenarios are considered starting from the traditional valuation methods, then extending to the case of futures.

III.1. The Hedging Portfolios

The hedging mechanisms are derived to provide an understanding of how the market tools of assets, futures and bonds are used to derive risk-neutral, arbitrage free expectations. The contract writer determines the value of the contract, and then uses that money to buy a designated amount of stocks and bonds to effectively hedge loss exposures. The proportion of the portfolio held in stocks and bonds may be traded continuously as asset prices change. The outcome of this portfolio is such that when the contract is exercised, the writer has enough to buy the asset, and pay off any borrowings (bonds), and the payoff to the writer is exactly zero.

Three portfolio scenarios are considered. The first scenario is the traditional Black-Scholes binomial option pricing formula. In this case, the hedge portfolio is comprised of the underlying financial stock and risk-free bonds. The second scenario utilizes futures to formulate the hedge portfolio. This method is useful for resources, such as timber, oil, and agricultural products for which a "stock" market does not exist. The significance of trading futures to obtain a risk-neutral portfolio is that we still end up with the traditional Black-Scholes formula. Finally, in the third case, the hedge portfolio using resource reserves is derived.

III.1.a Hedging with Stock Options

In finance, an option is the right to buy (a call) or sell (a put) a stock for a pre-specified price in the future. An American option is the right to "exercise" (buy or sell) *by* a certain date. A European option is the right to exercise *on* a certain date. Let's consider the case of a non-dividend paying stock. Then the value of the European option equals the value of the American option. In case I, the question is, how much should I pay today, for the right to buy a stock in the future. That is, what is the value of the portfolio: $V_{0s} = \varphi S_0 + \Psi B_0$. In words, the value at time zero of the hedge portfolio using stocks (V_{0s}) , is a proportion of the underlying stock (S) valued at time 0, and risk-free bonds (B) valued at time zero. In the next time period, the value of the stock can either go up by a unit u or down by a unit d. That is:

$$V_1 = \begin{cases} \varphi S_0 u + \Psi B_0 r = V_1(u) \\ \varphi S_0 d + \Psi B_0 r = V_1(d) \end{cases} \quad \textbf{(1)}$$

Notice that in either case, the risk free bond moves with certainty by an increment r, the risk free discount rate. Now subtract $V_1(d)$ from $V_1(u)$ and solve algebraically for φ. Equating the outcomes makes the investor indifferent between up and down movements: the portfolio is worth the same in either case.

$$\varphi = \frac{V_1(u) - V_1(d)}{S_0(u-d)} \quad \textbf{(2)}$$

and $\Psi = \frac{1}{rB_0}\left[\frac{V_1(d)u - V_1(u)d}{u-d}\right]$. **(3)**

Now substitute these values into V_{0s} .

$$V_{0s} = V_1(u)\left(\frac{r-d}{u-d}\right) + V_1(d)\left(\frac{u-r}{u-d}\right) \text{ (after canceling).} \quad \textbf{(4)}$$

That is, the value of V_0s is a function of the value of the portfolio in the up case (V_1(u)), times the risk-neutral probability of the up case, plus the value of the portfolio in the down case (V_1(d)), times the risk-neutral probability of the down case. These probability weightings are martingale risk-neutral probability measures. Notice that they are only functions of magnitude of the changes in value. Given the Markov solutions underlying the initial discrete models, the magnitude of value changes are constant over time.

III.1.b Case Two: Hedging With Futures

In the case of futures, we are trading the right to buy a specified quantity at a specified price. Unlike options, no money is traded at time zero. Rather, the asset must be purchased at the pre-determined future time, for a price k. The question becomes, how much should we agree to pay, k, in the future.

At time (t) =0, the value of the futures portfolio is: $V_{f0} = \varphi Stocks + \Psi Bonds$.

However, since no money is traded at time zero, the value of the investment is zero, and so the value of V_{f0} is also zero. Again, the bonds are risk-free government bonds. Since we are agreeing on a price k to pay in the future *regardless* of future asset prices, the payoff after one time step is:

$$V_{f1} = \begin{cases} \varphi S_0 u - k + \Psi B_1 = V_{f1}(u) \\ \varphi S_0 d - k + \Psi B_1 = V_{f1}(d) \end{cases} \qquad \textbf{(5)}$$

Again, solve for φ and Ψ algebraically by equating the outcomes, up and down.

$$\varphi = 1, \qquad \textbf{(6)}$$

and $\Psi = \frac{uS_0 - k - uS_0}{rB_0} = -\frac{k}{rB_0}$ **(7)**

Now insert these values into the initial V_{f0} equation and by setting V_{f0} equal to zero, solve for k. $k = rS_0$. That is, we come out with the traditional Hotelling (1937) outcome, that prices (rents) rise at the rate of discount. We should be willing to pay k, in the future (tomorrow), where we consider prices today, S_0 and the risk-free discount rate. (For multiple timesteps this becomes the familiar e^{rt}).

The risk-free interest rate is the appropriate discount rate given the hedge portfolio created. This can be checked by showing that this result is compatible with the Black Scholes outcome in section IV.iv**.** Recall that under Black Scholes

$$V_0 = V_1(u)\frac{r-d}{u-d} + V_1(d)\frac{u-r}{u-d}$$

We can show that the same outcome is accessible using futures. Given the solution that $k=rS_0$, substitute into the V_{f1} up and down outcomes to get:

$$V_{f1} = \begin{cases} \varphi(uS_0 - rS_0) + rB\Psi = V_1(u) \\ \varphi(dS_0 - rS_0) + rB\Psi = V_1(d) \end{cases} \quad \textbf{(8)}$$

Solving for φ we obtain: $\varphi = \dfrac{V_1(u) - V_1(d)}{(u-d)S_0}$ **(9)**

and solving for Ψ results in:

$$\Psi = \frac{1}{rB_0}\left[V(u)\left(\frac{r-d}{u-d}\right) + V(d)\left(\frac{u-r}{u-f}\right)\right]. \quad \textbf{(10)}$$

Consider again the portfolio at time zero: $V_0 = \varphi S_0 + \Psi B_0$. But as mentioned, in the case of futures, no asset is purchased at time zero, so S_0=0 and we are left with

$$V_0 = \psi 0 + \frac{1}{rB_0}\left[V(u)\left(\frac{r-d}{u-d}\right) + V(d)\left(\frac{u-r}{u-d}\right)\right]B_0 = \frac{1}{r}\left[V(u)\left(\frac{r-d}{u-d}\right) + V(d)\left(\frac{u-r}{u-d}\right)\right],$$

(11)

which is exactly equal to the Black Scholes value for an option on an asset.

III.1.c. Case III: Hedging with Resource Reserves

The same process is utilized to construct the hedge portfolio using the resource reserve. However, the portfolio is different. $V_{R0} = \Psi(P_0 - c) + \Phi B_0$ (c = the cost of harvest). In this case we use current prices of timber minus harvesting costs as the value of the reserve. That is current timber price minus harvesting cost equals the value of the standing resource. The proportion of φ and Ψ in this case is:

$$\varphi = \frac{V_1(u) - V_1(d)}{P_0(u-d)} \quad \textbf{(12)}$$

and $$\Psi = \frac{1}{rB_0}\left[V_1(u) - \frac{V_1(u) - V_1(d)}{u-d}\left(u - \frac{c}{P_0}\right)\right] \quad \textbf{(13)}$$

And the value of the portfolio (V_{0R}) is :

$$V_{0R} = V_1(u)\left(\frac{r-d}{u-d} + \frac{c}{P_0}\left(\frac{1-r}{u-d}\right)\right) + V1(d)\left(\frac{u-r}{u-d} + \frac{c}{P0}\left(\frac{r-1}{u-d}\right)\right) \quad \textbf{(14)}$$

Under the reserve method there exists a higher probability of an upward movement ($\frac{c}{P_0}\left(\frac{1-r}{u-d}\right) > 0$), and a lower probability of a downward movement ($\frac{c}{P_0}\left(\frac{r-1}{u-d}\right) < 0$) in prices. This implies that the value of the contract using reserves as a hedging mechanism will be greater than the value of the contract using futures as a hedging mechanism. Note that the magnitude of up and down movements in both hedging cases are the same.

The measure Q is based on the relationship between u, r and d which form the probabilities of up and down movements of asset prices when hedging is performed with the asset or with futures. When the resource reserve is used as the hedging mechanism, the measure Qc relies on u, r, d, c and S_0, where c enters as a conversion factor that links the standing resource to the market.

III.2. Expectations Under Uncertainty

This section develops the rationale and for developing an arbitrage free contract price in the case of pricing futures, using futures to hedge, and using resource reserves as a hedging mechanism. Means and variances of expectations based on the various hedging scenarios are considered starting from the traditional valuation methods, then extending to the case of futures.

Futures pricing is used as a motivation for finding an arbitrage free price. Futures prices are considered under the probability measures P, Q, and Qc. This allows us to derive the intuition for the options applications based on the same probability distributions as the more complicated options cases.

Under each distribution, P, Q, and Qc, prices are assumed to be lognormally distributed. Consider an asset with value S_0, which at each time step under the measure P moves up with probability p and magnitude u and down with probability 1-p and magnitude d (see Figure 1). Now I want to know how much I should be willing to pay for that asset next period.

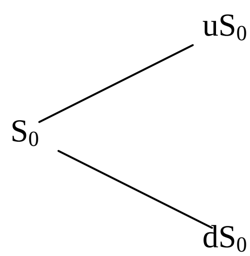

Figure 1: A binomial tree depicting asset price evolution after one time period

S_0 = \$50/mbf (thousand board feet)

u = 1.4

d = .8

p = .6

r = 1.06

c = \$10/mbf

S_0 equals the value of the asset today, u is the magnitude of the upward movement, d is the magnitude of the downward movement, p is the probability of the upward movement ((1-p) is the probability of the downward movement), r is the risk-free interest rate and c is the marginal cost of harvesting 1000 board feet of timber.

$$\mathrm{E}_P\left(\frac{S_1}{S_0}\right)=\frac{puS_0}{S_0}+\frac{(1-p)dS_0}{S_0}=\frac{.6*(1.4*50)}{50}+\frac{.4*(.8*50)}{50}=1.16 \qquad \textbf{(15)}$$

That is the expected value under P of (S_1/S_0) = 1.16. 1.16 implies a 16% rate of return, and that one should be willing to pay $58 next period to obtain this asset. However, as asserted at the beginning of the paper, this price is too high and arbitrage is possible. The agreement is that the buyer will pay the seller $58 in the next period. Arbitrage is possible because the seller may borrow $50 today at the risk free interest rate of 6% to buy the asset, and in the next period pay back 1.06*50, or $53 and receive $58 for a risk-free $5 profit.

What we need is a way to calculate the expectation that achieves a rate of return equal to the risk-free interest rate, or 6%. The good news is that we have calculated an arbitrage free expectation measure. In section III.1.a,b, the probabilities were applied to the value of the contract in the up and down case. The value of the contract explicitly depends on the value of the underlying or S. Therefore, it is appropriate to apply these probabilities to S in this example as well, since S is the source of volatility. It was determined that the probability of an upward movement equals q=$\frac{r-d}{u-d}$, and that the probability of a downward movement is 1-q=$\frac{u-r}{u-d}$. These measures represent changes of measure under the measure q and are referred to as Q-Martingales.

When these measures are applied to the expectation of (S_1/S_0) the following is obtained:

$$E_Q\left(\frac{S_1}{S_0}\right)=\left(\frac{r-d}{u-d}\right)*\frac{uS_0}{S_0}+\left(\frac{u-r}{u-d}\right)*\frac{dS_0}{S_0}=$$
$$\left(\frac{1.06-.8}{1.4-.8}\right)*\frac{(1.4*50)}{50}+\left(\frac{1.4-1.06}{1.4-.8}\right)*\frac{(.8*50)}{50}=1.06 \qquad \textbf{(16)}$$

That is, the expected value under the measure Q of S_1/S_0 is 1.06. This means that the rate of return is 6%, exactly the risk free interest rate and that the appropriate price to contract for the asset in the next period is $53.

Finally, consider the case where the "asset" is the resource reserve. Again, to obtain an arbitrage free valuation the probability measure must change due to the cost conversion factor. This is called the expected value under the cost adjusted measure Qc. In this case, the expected value of S_1/S_0 moves up with probability $\frac{r-d}{u-d}-\frac{c}{S_0}\left(\frac{r-1}{u-d}\right)$

and down with probability $\frac{u-r}{u-d}+\frac{c}{S_0}\left(\frac{r-1}{u-d}\right)$.

$EQc(S_1/S_0)=1.048$ -- that is a 4.8 rate of return implying that $52.40 - c is an appropriate amount to contract for purchasing tomorrow.

In summary, the $E_P(S_1/S_0)>E_Q(S_1/S_0)>E_{Qc}(S_1/S_0)$. The expectation under P does not consider arbitrage opportunities and subsequently overvalues the investment opportunity. It the asset is standing, or unharvested, Q overvalues the investment opportunity since it does not account for the conversion cost factor, c.

Consider also the impact that each of these probability measures (distributions) has on the variance of the expectations. Since the actual outcome is u or d in each case, and the $E_p > E_Q$, E_{Qc}, one would expect the $Var_{Qc} > Var_Q > Var_P$. Solving the general variance formula under P, Q and Qc respectively, and inserting the chosen parameter values, we get $var_P = \sigma_P^2$ =.0144, $var_Q = \sigma_Q^2$ =.0884, and the varQc= σ_{Qc}^2 =.0982. The mean and variance results are summarized in Table 1.

Distribution	rate of return	mean (price in next period)	variance
P	1.16	58	.0144
Q	1.06	53	.9884
Q(c)	1.048	52.40-c	.0982

Table 1: Summary of Mean Variance and Rates of Return under different probability distributions.

To reiterate, prices still follow the process P in the market. Under P, prices are lognormally distributed. Prices are exogenous to the contract writer. If the writer prices the contract based on the distribution P, the contract is over-valued and arbitrage is possible. The writer constructs the distribution Q, under which prices are also lognormally distributed in order to match the contract outcome with the return available by investing in the risk-free bonds. Under this setting, arbitrage is not possible. Finally, the case in which there is not a market for the contracted asset is considered. Again, a new distribution Qc is required. Figure 2 depicts the relationship between these distributions. Note that it is from these distributions that we draw price expectations, not payoff expectations. f(S) represents a lognormal distribution.

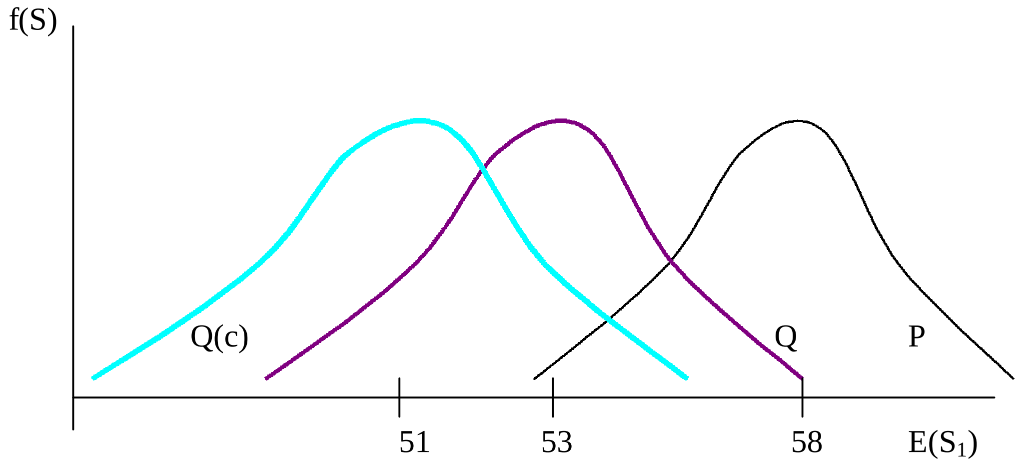

Figure 2. Lognormal Probability Distributions under P, Q and Qc

The far right distribution represents P. The mean of the distribution falls at $58. The middle distribution represents Q, and the mean falls at $53. Finally, the left distribution represents Q(c) and the mean is $51. Notice that as we move right, each distribution is wider. This is due to the increasing variance associated with the expectations under each measure.

IV. Results

Now we have seen how the hedging portfolios are formed under the various expectation measures. We have seen how the three distribution measures impact the mean and variance of price expectations. Now let's combine these results and apply the numerical examples derived in section 3.2 in the futures case to the hedging portfolios derived in section 3.1. Recall that the previous examples in section 3.2 depicted contract prices for futures. In the case of futures contracts, one decides today how much to pay for an asset at a specified time in the future. In contracts of this type, no money is exchanged today (this was invoked as the solution method for futures valuations in section 3.1.b). The contract is mandatory. That is, at the settlement date, the buyer must buy the asset for the specified price regardless of the current market price. However, it is still possible to create a risk-free portfolio by selling bonds and buying the asset.

In this section, the value of each contract is derived, and an example is worked through to clarify how the hedge portfolio is determined maintain a risk neutral position for the contract writer.

Consider the same parameter values as in section III.

S_0	=	\$50/mbf (thousand board feet)
u	=	1.4
d	=	.8
p	=	.6
r	=	1.06
c	=	\$10/mbf

In the case of expectations under P, when no hedging occurs, the price that the writer agrees on for the asset in the next period is \$58. As long as the risk-free interest rate, r is less than the $E_P(S_1/S_0)$, there is no value to forming a contract to buy the resource in the future. This is because the value of the contract today is $V_0=58/1.06=\$54.72$. The writer would have to borrow \$54.72 today to hedge the \$58 expected return in one period. That is, the writer is spending \$54.72 today on an asset that is worth \$50. Of course this does not make sense. This is the problem, the expected value under P does not account for interest rates or volatility in its expectations.

Under the measure Q,

The writer contracts to receive $53 for the asset in the next period. She borrows $50 in bonds today to buy the asset. Tomorrow, if the asset price goes up to $70, she still receives $53 and pays of the bond debt which equals $50*1.06=$53. If the asset price goes down to $40 she receives $53 and again pays off the bond debt.

Now consider the portfolio under Qc. With the cost adjustment, Stocks= φ=.8 units and bonds = Ψ=40 units. The portfolio is: V_0= .8*50 -40*1. The buyer agrees to pay $52.39 in the next period, and the writer sees 52.39-10=$42.39=1.06*40. Notice that the buyer agrees to pay based on the amount S, while the writer is hedging on S-c. The buyer is indifferent to the writer's hedge portfolio.

V Conclusions and Policy Implications

This paper has shown that even if we include uncertainty in investment calculations, we may still over-value the investment if arbitrage is possible. It is necessary to change the measure on which we make the expectations on prices in order to develop a risk-neutral, arbitrage free contract price. This is possible by creating an underlying hedge portfolio comprised of the asset and a risk-free asset. In cases where the underlying is not traded in the market, it has been shown that we can develop yet another measure to accommodate the necessary conversion costs in "bringing the asset to market". That is, we include the harvest costs of timber for example when the marketed asset is harvested trees but the contract writer is holding timber (standing trees).

These results have policy implications for resource managers interested in setting contract rates for natural resources such as trees, fish and water. First, in the cost adjusted case, we can directly see the impact that costs have on the contract price. In instances where harvester costs are uncertain, managers can better model the impact that cost variations across harvesters may have on contract values. Second, using the risk-neutral case allows managers to use an objective interest rate measure (the risk free interest rate) in discounting contract values over time. Third, these results provide a market-based foundation for integrating non-market resources into a valuation system. These results are strictly based on observable market characteristics of prices and hedging opportunities. If one wishes to value social and resource derived from different policy scenarios, these values suggest a starting value in the absence of public or environmental welfare. Certainly, there is a place for amenity and existence values, but this framework provides a method for setting a reliable and justifiable benchmark.

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