A Numerical Quadrature Approach to Option Valuation in Water Markets

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Abstract

The standard Black-Scholes approach to option valuation becomes cumbersome and may fail to yield a solution when applied to non-standard options such as those emerging in water markets. An alternative tool, numerical quadrature, avoids some restrictive assumptions of the Black-Scholes framework and can more easily price options with complex structures.

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OPTION VALUATION IN WATER MARKETS

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I. Introduction

Allocating water resources among competing uses presents a host of practical problems and myriad opportunities for research by economists and other analysts of public policy. Integral to societal activities from food production to recreation, water is distinctive among natural resources due to the tangled web of legal and institutional systems at the local, state, national, and international levels that governs its use. With quality and quantity of water supply emerging as pressing concerns in many areas of the U.S. and the world, efficient allocation mechanisms are becoming more crucial.

The need for improved means of allocating water has caused gradual revision of laws and policies governing transfer of water and water rights. For example, the state legislature in Texas has allowed the State Water Development Board to form water banks, in an effort to encourage water transfers (Thompson, 1995). Similarly, California has sought to clarify water property rights to facilitate short and long-term transfers (Archibald and Renwick, 1998). Still, permanent transfers of water rights face numerous obstacles: establishing/adjudicating property rights, assessing and mitigating the effects of the transfer on downstream or third-party users, and evaluating the current value of the future stream of water. Most of these factors increase the transaction costs of permanent rights sales, suggesting that such transfers are not an effective means for addressing short or medium-term water shortages.

Changes in laws and institutions combined with the high transaction costs associated with permanent transfers of water rights have provided the opportunity and incentive to utilize market principles to allocate water among its users. Proponents of spot markets for water argue that market mechanisms can be used if water property rights are not adequately defined, and are especially attractive if the transaction costs of trading permanent water rights are prohibitive. (See, for example, Howitt, 1998.) Frederick (1998) argues that markets in many cases increase the value of water by shifting it from low-value agricultural uses to hydropower or urban consumption. For economists, markets for water hold natural appeal: ideally, as supply and demand fluctuate relative to each other, prices will provide signals to market participants and ensure efficient water allocation.

While development of spot markets and facilitation of permanent transfers of water rights have eased barriers to efficient water allocation, these two mechanisms do not adequately accommodate medium-term planning needs of many water users. For example, neither spot markets nor permanent transfers provide assistance to an urban water planner faced with a two-year horizon. Further, as Howitt (1998) notes, spot markets appear to disproportionately burden buyers with risk while permanent transfers are riskiest for sellers. Other market arrangements could spread risk more equally between the two parties. Uncertainty about future water supplies and prices, the need to time water availability to meet demand or production constraints, and the irreversibility of investments in water imply that a third market construct may be appropriate: option contracts for water.

II. Option Contracts for Stocks and Water

Options are a type of derivative, a financial instrument whose value is based on another underlying asset. A stock option, for instance, derives its value from the price of a specific stock. The two basic kinds of options are calls and puts. A call option gives the holder the right but not the obligation to purchase the underlying asset by a certain date for a specific price. Similarly, a put option gives the holder the right but not the obligation to *sell* the underlying asset by a certain date for a specific price. The date specified in the option contract is called the expiration date or exercise date. The specified price is known as the strike price or exercise price. A European style option can be exercised only on its expiration date; an American option can be exercised any time on or before its expiration.

Standard Options

Perhaps the most basic option contract is the European call. On the option's expiration date its holder must decide whether to exercise her right to purchase the underlying asset. If she exercises, she receives a payoff equal to the difference between the current price of the asset and the price specified in the option contract. If she does not exercise her option her payoff is zero. More formally, I represent the payoff from the option as

$$\max(S_T - K, 0), \qquad [1]$$

where S_T is the price of the underlying asset at the expiration date (T), and K is the strike price. Then the option's holder exercises when $S_T \geq K$.

An option contract must have both a buyer and a seller. The buyer of a call obtains the right to purchase the underlying asset but the seller of the call is *obligated* to provide the asset if the call is exercised. Thus, while the holder of the call retains flexibility, the seller has none. In order to obtain the option then, the buyer must incur an up-front cost, the option price. Determining this value of the option, how much it is worth to the buyer, is straightforward for a European call. The Black-Scholes approach yields a tidy analytical solution for simple options like the European call and put.

Exotic Options

Many options have payoff structures that are more complicated than standard calls or puts; these are called exotic options. The payoff from a lookback option, for example, depends on the maximum (or minimum) price achieved by the underlying asset during the lifetime of the option. A few types of exotic options such as European style compound options can be price analytically using extensions or the Black-Scholes pricing formulas. Most, however, require numerical methods such as binomial trees and numerical quadrature.

Option Contracts for Water

Evidence from the western United States shows that option contracts for water can be even more exotic than many exotic options considered in finance. For example, one water district negotiated a contract with an agricultural operation allowing the district to exercise a call option up to eight times in a fifteen-year period. This is a highly non-standard option; there is no simple pricing formula to calculate its value. Standard European call options involve a simple decision whether or not to exercise on the expiration date. In contrast, there are more than 65,000 ways to exercise the water district's call option eight or fewer times in fifteen years. Clearly this complicated payoff structure increases the difficulty of putting a value on the option contract.

As a starting point for my research I have generalized the "eight times in a fifteen year period" option contract to a stylized "*m* out of *n* years" option. As its name implies this option can be exercised no more than m times out of n periods, with $n > m > 0$. For convenience I denote this as an m/n option. The m/n option contract must specify in advance the strike price and exercise date for each period.

My preliminary efforts focus on computing the value of a European-style 2/3 call option with constant strike price. I describe the option explicitly in Section V. The problem is complicated because the decision whether to exercise the option in period 1 depends on the expected value of the option in periods 2 and 3. Similarly, the value of the option in period 2 depends on its value in period 3, which depends on how many times the call has already been exercised. The problem is interesting because most option valuation methods used in finance are ill-equipped to account for these intertemporal links.

III. The Black-Scholes Option Valuation Framework

On December 9, 1997, Robert C. Merton and Myron S. Scholes received the Alfred Nobel Memorial Prize in Economic Sciences for their roles in developing the conceptual and mathematical frameworks of option pricing theory. The seminal articles by Black and Scholes (1973) and Merton (1973) sparked a torrent of academic and applied research on financial derivatives, which in turn generated modifications, extensions, and innovative applications of the original Black-Scholes model. (See Merton (1998) for a brief review of the evolution of option-pricing theory since 1973.) In the context of financial derivatives that are traded on organized exchanges, the Black-Scholes formula provides the value of an option as a function of several variables: strike price of the option, time until the option expires, price of the stock, the risk-free interest rate, and the volatility of the stock.

To derive their option-pricing formula, Black and Scholes posited several assumptions. Phrased in the context of water markets, these are (see Hull, 1997, p. 236):

A1. Price of water follows a geometric Brownian motion process.

A2. Market participants are permitted to short-sell water, with full use of proceeds.

A3. No transactions costs or taxes exist.

A4. Underlying assets, including water, are perfectly divisible.

A5. No dividends accrue during the life of the option.

A6. Riskless arbitrage opportunities are not available.

A7. Water trading is continuous.

A8. The risk-free interest rate is constant, strictly positive, and compounded continuously.

While these assumptions may be justifiable in the presence of well-established, active markets, their validity in nascent or thinly-traded water markets is doubtful. For instance, the Black-Scholes assumption that the price of water follows an Ito process contains a series of additional, implicit assumptions: changes in the price of water follow a continuous-time non-stationary Markov process, and are normally distributed. Non-stationarity requires the possibility of the expected value of price growing without bound and the variance of price to increase over time. To satisfy the conditions of a Markov property, only the current price of water can be relevant to predicting tomorrow's price. Given the seasonality of water use and supply, use of the Markov assumption in this scenario is questionable. Thus, the Black-Scholes method may not be suitable for wholesale adoption in this sphere of resource economics. This suggests an opportunity for research: easing the Ito process assumption by exploring different types of stochastic forces driving water price. A mean-reverting process, for example, may bear further investigation.

Aside from the question of whether the Black-Scholes assumptions hold up in water markets, their approach of deriving and solving partial differential equations may simply be inefficient for valuing option contracts for water. For even a moderately complex option, writing down a mathematical expression for the payoff can be challenge enough. Deriving and solving the associated differential equation may well prove impossible. Valuation techniques that do not rely on partial differential equations warrant investigation and may prove to be appropriate for the complicated options that arise in water markets.

IV. Numerical Quadrature

Initial formulations of option valuation in water markets naturally mimic the conceptual and mathematical framework developed by Black, Scholes, and Merton (see Howitt, 1998 and Watters, 1995). As I argue above, while the assumptions of Black-Scholes option pricing formulas may be justifiable in the presence of well-established, active markets, their validity in nascent or thinly-traded water markets is doubtful. Furthermore, the Black-Scholes approach loses elegance and usefulness when applied to non-standard options. In this section I explain an alternative method for valuing options: numerical quadrature. Numerical quadrature uses computational methods based more directly on the option's payoff structure and the stochastic process of the underlying asset's price.

Numerical quadrature removes differential equations from the process of valuing an option. Furthermore, this method enables me to account for exotic option structures more easily than the

Black-Scholes model and is a quicker, more robust algorithm than iterative methods for solving nonlinear equations. If I can write down the mathematical expression for the option's payoff, numerical quadrature can value it.[1] Discerning an option's payoff can be decidedly simpler than deriving the differential equation that determines its value.

The procedure begins with articulating the option's payoff structure and specifying a stochastic process for the price of the underlying asset. Then I re-cast the problem as an integral. The quadrature technique values the option by estimating the solution to the integral. Nelken (1996) supplies a concise explanation of numerical quadrature and applies the method to complex chooser and compound options. To demonstrate how numerical quadrature can be used to value options I once again turn to the standard European call of Section II.[2]

Recall the payoff structure of the call option, represented by equation [1]. This suggests the following formula for the value of the call:

$$f = e^{-rT} E\left[\max\left(S_{T-K,0}\right)\right], \qquad [2]$$

where E denotes the expectation operator and T is the time until the option expires. Assuming a lognormal random walk for the price of the underlying, I can rewrite the spot price at time T as

$$S_T = S_0 e^{\mu T + \sigma z \sqrt{T}}, \qquad [3]$$

where $\mu = \sigma^2 / 2$.

In general I can use integral notation to express the expected value of a function A as the integral of the product of each value of A times the probability of achieving that value of A:

$$E[A] = \int A p(A) dA. \qquad [4]$$

Since the probability distribution, *p(A)*, assigns a likelihood to every possible value of A, the integral is computed over all values of A. In practice I choose a large but finite range of integration.

Combining equations [2], [3], and [4], I re-write the option valuation problem as an integral:

$$f = e^{-rT} \int_{-\infty}^{\infty} \left[\max\left(S_0 e^{\mu T + \sigma z \sqrt{T}} - K, 0\right)\right] n(z) dz, \qquad [5]$$

and n(z) is the standard normal distribution function. To simplify equation [5] set the two elements within the max operator equal to each other; solve for z. This shows that the option's payoff, and therefore the value of the integrand, is 0 for some values of z. Specifically, the value of the option is 0 for z that are less than

[1] This claim is subject, of course, to the constraints of computer resources but is true in principle.
[2] In this example I draw from Nelken (1996), page 131.

$$L = \frac{\ln\left(K/S_0\right) - \mu T}{\sigma\sqrt{T}} \ . \qquad [6]$$

Thus the integral that will be subjected to the numerical quadrature procedure is:

$$f = e^{-rT} \int_L^{\infty} \left(S_0 e^{\mu T + \sigma z \sqrt{T}} - K\right) n(z)\, dz \ . \qquad [7]$$

Although there are several quadrature approaches, each involves choosing a finite range of integration and then evaluating the integrand at specific values within the integration range. Of the available techniques, I have chosen to use left end-point quadrature (see Nelken, 1996). This method first breaks the integration region into a pre-specified number of slices, then evaluates the integrand at the left border of each slice to determine the slice's height. The sum of the slices' areas approximates the area beneath the curve. To evaluate the area under a function *g(x)* between *a* and *b*,

$$I = \int_a^b g(x)\, dx \ ,$$

I first divide the range of integration into slices. For *n* evaluations of the function the step size is $h = (b - a)/n$. Then the following expression approximates *I*:

$$I \approx h \sum_{i=0}^{n-1} g(a + ih) \ .$$

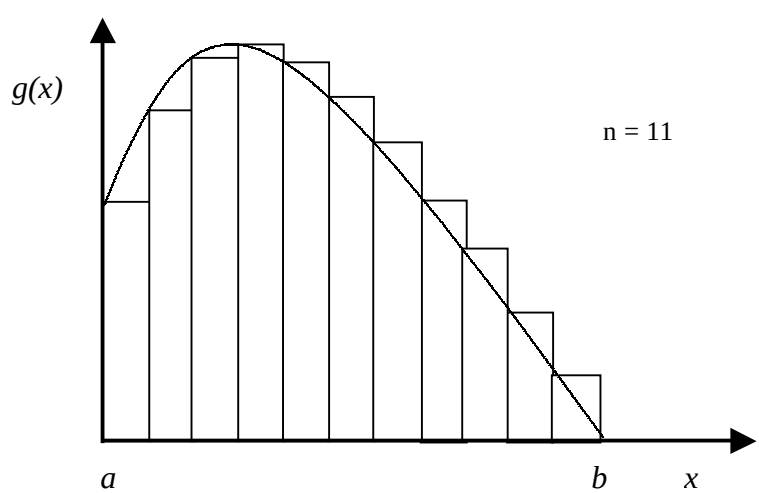


Figure 3: Illustration of Quadrature Technique

The rectangles only approximate the area under the curve: the error is evident in the space between the top of the rectangles and the function itself. I can reduce this error by increasing *n*,

the number of slices. Notice also that the distance between the top of the rectangles and the function being evaluated may vary as *x* changes. Thus the graph above alludes to a primary weakness of the left end-point quadrature method: I must specify the constant step size *h* ahead of time. If the method allowed, I could improve accuracy by decreasing the step size where the function is changing rapidly. In this instance I would decrease step size as *x* gets larger. Some of the other quadrature techniques incorporate an adaptive control component that ensures the step size will be reduced in the regions where the integrand changes quickly (Nelken, 1996).

V. Numerical Quadrature Applied to Water Options

Although the results below are for a 1 /2 call option, my next step is to value a simplified version of the m-out-of-n years call option I introduced in Section II. This call can be exercised in any two out of three years and gives the holder the right to purchase one acre foot of water at a specified price. The call can be exercised only on a certain date each year, making it a European-style contract. The diagram below depicts the structure of this 2/3 call option.

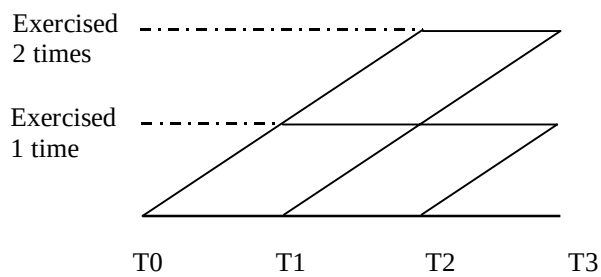


Figure 4: Exercise Paths for a 2/3 Option

The contract is signed at T_0, represented by node A in the diagram. One year later, at T_1, the call may be exercised for the first time. Node C is the case where the option is exercised in T_1; at node B the option is not exercised this period. Again the following year, T_2, the option holder must decide to exercise or keep the option. By the time T_3 arrives the option may already have been exercised twice, precluding a third use of the option. This scenario is node I in the diagram. In all, this option allows seven possible exercise paths over the three years, each with a distinct payoff. Clearly each decision point depends on both the current price of water and the expected path of water price during subsequent years in the life of the option.

Assumptions and Data

Any model of a complex problem utilizes simplifying assumptions. My water option valuation program is no exception to the rule, and I attempt to be explicit about my assumptions from the outset. They are:

- Strike price is constant.
- No dividend is issued.

- The stochastic process of water price is a lognormal random walk.
- Risk-neutral probabilities are appropriate.
- Risk-free discount rate is appropriate and constant.

The first two assumptions are easily overcome, my program allows flexibility for a positive dividend rate and a different strike price for each period. The last three assumptions, however, are more substantive and I plan to relax each of them.

As initial inputs for this option valuation problem my Gauss program uses data from Watters (1995). Specifically:

Spot price of water: $122
Strike price: $172
Expiration date(s): 1 year, 2 years
Volatility: 0.11
Discount rate: 0.0588

Clearly these assumptions and data leave room for improvement. The assumption of lognormal random walk for the stochastic process of water price is a strong one I plan to relax in the near future. Using the interest rate supplied by Watters is also problematic since it likely does not reflect the project-specific risk inherent in the option contract.

Computational Methodology

As usual the value of the call is a function of the spot price of water (w_t) in each of the three periods, the interest rate (r), the strike price in each period (k_t), and the exercise deadlines. Let E again denote the expectation operator. Then the value of the call (C) is:

$$C = e^{-r} E [\max \{(w_1-k_1), e^{-r}(w_2-k_2), e^{-2r}(w_3-k_3), ((w_1-k_1) + e^{-r}(w_2-k_2)), ((w_1-k_1) + e^{-2r}(w_3-k_3)), (e^{-r}(w_2-k_2) + e^{-2r}(w_3-k_3)), 0\}]. \qquad [8]$$

Since the discount rate and strike prices are pre-determined, the random elements of equation [8] are simply the prices of water: w_1, w_2, and w_3. I have assumed a lognormal random walk for the price of water in accordance with standard option pricing theory. The stochastic process for water price follows:

$$w_t = w_0 e^{(r-\sigma^2/2)t+\sigma z\sqrt{t}}. \qquad [9]$$

Here, w_0 is the spot price of water when the contract is signed, σ measures the volatility of water prices, and z is a normally distributed random variable with mean 0 and standard deviation 1. Substituting from [9] for each occurrence of w_t in [8] leads me to the expression that re-casts the problem as an integral, the value of the 2/3 call option.

Numerical quadrature and dynamic programming are the two primary techniques I use to value this 2/3 call option. The quadrature component calculates the probability distribution of water

prices for each time period while the dynamic programming code determines the optimal decision path. The probability distributions for water price in T_2 and T_3 are conditional on the price realized in the previous period.

VI. Preliminary Program Results: A 1 / 2 Option

Although valuing a 2/3 option is my current goal, I began by writing a Gauss program to value a 1 / 2 European style call. Using the numbers suggested by Watters (1995) and listed above, the call's value is $1.45. I am not surprised that this number is low. Because the strike price is much higher than the initial price of water it is unlikely that the option holder will exercise the option during its three-year lifespan. Thus, the option buyer will insist upon a low price for the contract.

The graphs on the two following pages summarize the comparative statics of the 1 / 2 option price. Figure 5 shows that, holding all else constant, increasing the strike price decreases the value of the option. Increasing the initial price of water has an equal and opposite effect. This reflects the basic relationship between spot price, strike price, and option payoff. Figure 6 reveals that raising the volatility or discount rate while leaving all other parameters unchanged causes option price to rise.

VII. Policy Relevance

Efficient water allocation mechanisms are becoming increasingly important as population dynamics and pollution problems put pressure on quality and quantity of water supply in the U.S. and around the globe. This research explores option contracts as a means of improving water allocation, using both theory and empirical analysis. Proof that my results will have direct policy implications is given by the fact that at least two states, Arizona and Nevada, are currently considering adopting a program of options in water markets. Further, contingent contracts for water are become more common in California. Since established methods for valuing financial options are poorly equipped to assign a price to option contracts for water, the numerical quadrature components of my research may serve as a tool for water authorities, utilities, or agricultural concerns who buy or sell water options. Regardless of whether I conclude that options can play a substantial role in improving water allocation or not, my research will have direct implications for the direction of local and state efforts to solve water allocation problems. If options hold promise, decision-makers should make necessary changes in institutions and laws, then implement an options program. If options result in little benefit, the public interest may be better served by devoting time and energy to other mechanisms for increasing efficiency in water allocation, such as reducing transaction costs of water transfers.

Figure 5: Sensitivity of Call Value to Changes in Strike Price and Initial Spot Price

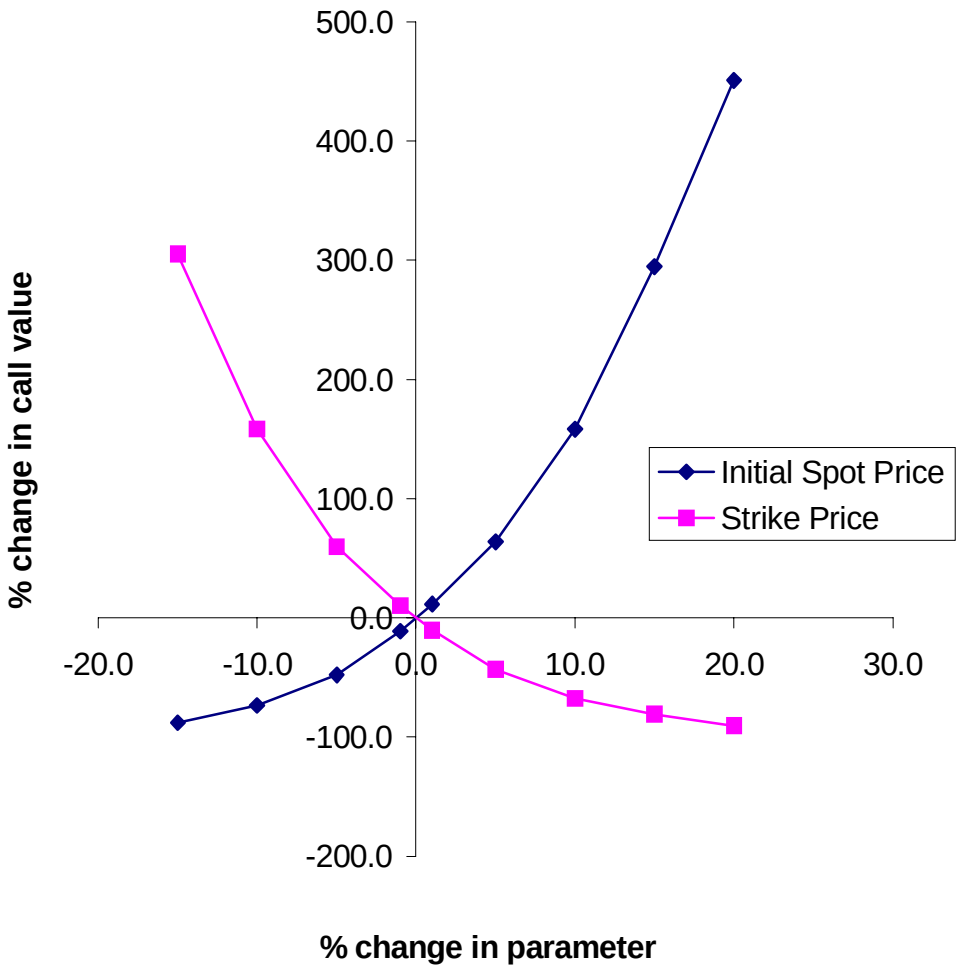
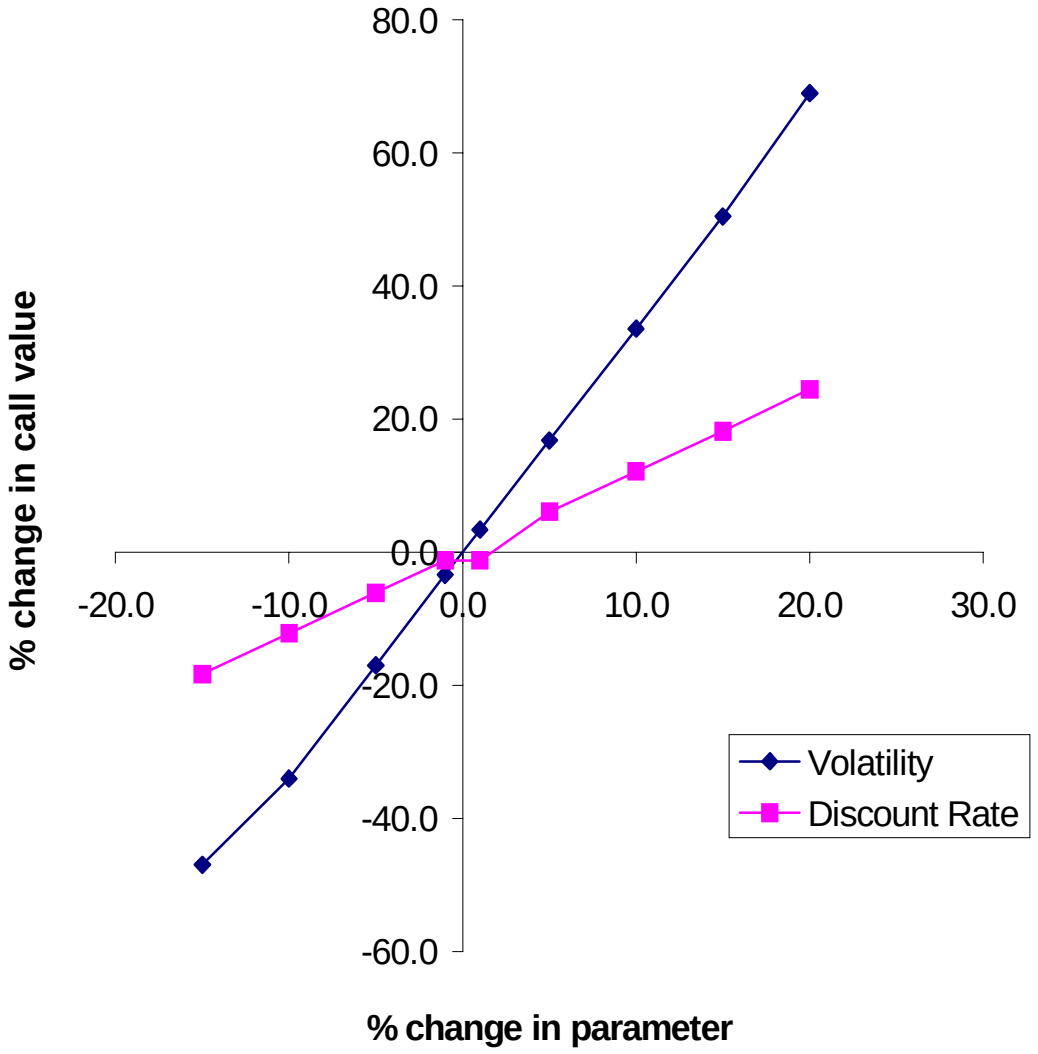


Figure 6: Sensitivity of Call Value to Changes in Volatility and Discount Rate



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