# Optimal Regulation of Eutrophying Lakes, Fjords and Rivers in the Presence of Threshold Effects

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**Abstract**

In a large number of practical environmental regulation problems, the damage done by pollutants depends on stocks and/or flows of pollutants exceeding certain thresholds. A typical example is eutrophication which occur when stocks of nutrients in a lake exceeds a certain threshold. The present paper presents a model of eutrophication that accounts for such thresholds. The paper does so by applying a novel technique in optimal control theory that allows for the analysis of systems where state-variables bounce back and forth over thresholds that take the form of functions of time and state-variables.



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## Introduction

Cultural eutrophication is a process where the ecosystem in a lake, fjord or river (hereafter referred to as a lake) changes due to increased reception of nutrients such as nitrogen or phosphate from man made sources[1]. High levels of nutrients leads to excess growth of certain algae such as phyto-plankton, while larger plants (macrophytes) diminish in numbers or may even become extinct. The eutrophication process also has consequences for the fauna. For instance oligothropic lakes in North-America and northern Europe are usually dominated by salmonids while eutrophic lakes in the same area will often be dominated by other species such as roach. An eutrophication process will often involve that some species replace others as dominating. At high levels of euthropication, lakes will tend to evolve into marshlands.[2]

The degree of euthropy in a lake which is not influenced by human activity will usually be characterised by some kind of steady state[3]. The natural level of nutrient deposition corresponds to a certain type of ecosystem. A particular lake may tolerate increased loading of nutrients without impact on the ecosystem. It is only when loading increase above a certain threshold that euthropication starts. The process is usually reversible in the sense that if the loading of nutrients decrease below the threshold, the process is reversed and the dynamics of the ecosystem reverts to the state before the euthropication process begins. The threshold depends on natural characteristics to the lake such as flushing rate and mean depth. These thresholds have been mapped by Vollenveider (1975).

The traditional approach to modelling thresholds in resource economics is to model thresholds as constraints. The modeller defines a function of time $\overline{S}(t)$ and a function of time and/or state-variables $S(t, x(t), u(t))$, and require that any optimal path obeys the constraint $\overline{S}(t) \leq S(t, x(t), u(t)) \;\forall\; t$. For a good treatment of optimal control theory with pure state constraints and mixed constraints see Seierstad and Sydsæter (1987), chapters 5 and 6. For a lucid exposition of this approach in a resource economic context see Perrings and Pearce (1994).[4] This approach is unsatisfactory for two reasons. First, in many applications a dynamic system may start from an initial state where the threshold is already violated. Eutrophication is often an example of this. Usually policy measures to deal with an eutrophication process will not even be discussed for a particular lake until the process is well on its way and

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[1]Se Mason (1996) for an very readable account of the biology of freshwater pollution that is simple enough to be understandable to economists while advanced enough to be useful for model-building purposes.

[2]A lake may be naturally eutrophic when there is a biological equilibrium with high levels of nutrients deposited in the lake. The eutrophication process here modelled is one where a naturally oligothropic lake is subjected to increased depositions of nutrients due to activities of man.

[3]This process may also occur naturally.

[4]Perrings and Pearce also discuss introducing penalty functions that apply when thresholds are broken, but stops short of developing such penalty functions explicitely. As will be clear from the treatment below, explicit development of penalty functions is uneccessary.

significant damage has been done. It is usual for articles on policy in environmental and resource economics to assume that nature starts out undamaged and the question is how to exploit the resource optimally[5]. When the natural resource is already damaged at the time of the implementation of policy, one may need to consider problems where there are thresholds and the system starts from the "wrong" side. Prominent examples of this include Lake Washington in north-western USA and Lough Neagh in Northern Ireland. In both cases eutrophication had reached serious levels before purification measures were implemented, or indeed even considered. Second, even if a system initially is on the right side of the threshold, it may be optimal to break such a threshold, maybe just for a time. As is shown below, it may well be the case that although one breaks through a threshold, it may take time for significant damages to occur and that remedial policies may be implemented after the threshold has been broken. It may also be the case that it is optimal that the threshold at some point in time is violated perpetually. If this is the case, then one needs to consider optimal policies before and after the time such an violation occurs.

In a model of climatic change Farzin (1996) analyse thresholds in a way that to some extent incorporates these features. However, Farzin fail to recognise that breaching of thresholds logically implies that there must be discrete jumps in the adjoint variables[6] at the time the threshold is violated. Thus Farzin's analysis, though providing good intuition and definitely a step in the right direction, must be considered flawed. The present article applies a novel modification to standard optimal control theory that allows the explicit analysis of violations of thresholds in a stringent manner.

Eutrophication is the common term for a large class of physical and biological processes, caused by a variety of reasons. Therefore, a general model of eutrophication that incorporates all types of processes and causes is not practical. Models of eutrophication must be developed almost on a case by case basis. The present paper presents a model of eutrophication that concentrates on a basic feature of eutrophication processes: The existence of threshold effects where the dynamics of the system is fundamentally altered when the level of nutrients exceed certain thresholds. Extensions of the method here developed may be applied to more case specific models of eutrophication.
A particular aspect of the regulation of eutrophying lakes is that nutrients are often deposited from diffuse or nonpoint sources. This is suppressed here, since it is well covered in the literature. Several implications for optimal regulation of nonpoint-source pollutants are discussed in Russell & Shogren (1993) and Romstad, Simonsen and Vatn (1996).

[5]It is strange that it should be so, since it flies in the face of the observed facts. Environmental problems have historically seldom been responded to *ex ante*. One speculation is that the rationality assumptions inherent in economics makes it psychologically hard for a trained economist to create a model where the possibility of environmental damages has not even been considered prior to their occurence.
[6]In the literature also referred to as co-state variables.

## 1. A Model of Eutrophication caused by Fertilising in Agriculture.

Let $y$ be the loading of nutrients in a particular lake and let $x$ be a suitable index of the degree of euthropy. One way of defining such an index is visibility as defined by the use of a Secchi-disk[7]. Low values of $x$ corresponds to high levels of eutrophication while high values of $x$ corresponds to low levels of euthrophication. There are several advantages to working with such an indicator function rather than an explicit model of the biology involved. One advantage is that biologists seem perfectly happy to use such indicator functions, in particular measured visibility, when evaluating the degree of eutrophication in an ecosystem. Another reason is that visibility is a variable which is straight forward to translate into economic valuation of the disutility derived from eutrophication.[8]

Consider a lake where the euthropication process is described by the following differential equation when $y < \bar{y}$:

$$\dot{x} = -\gamma x + \alpha \tag{1}$$

The steady state degree of eutrophication is given by $x_{ss} = \alpha/\gamma$. If the level of nutrients is above $\bar{y}$, the dynamics of euthropication is assumed to be given by:

$$\dot{x} = -\gamma x \tag{2}$$

The lake will then approach the level of eutrophication associated with the value $x = 0$. A variable $z$ is defined to indicate whether the lake is subject to an euthropication process or not and this variable can only take two values, 0 and $\alpha$. Thus $\dot{x} = -\gamma x + z$. The interpretation of (1) and (2) is that for sufficiently low nutrient levels the ecosystem is able to clean itself through biological processes. Excess biomass is consumed and micro organisms, plants and animals found in eutrophic waterways are replaced by species more suited for an oligothropic environment. However if the nutrient level is too high, the biomass in the lake accumulates and visibility decrease. Thus $x$ is to be interpreted as an indicator where a simple proxy variable describes a much more complicated underlying process. Note that the interpretation of $x$ as visibility, as measured e.g. with a Secchi-disk, implies an extreme eutrophication process when $z = 0$. In many cases it would be reasonable to let the variable $z$ take on some value between 0 and $\alpha$ when $y > \bar{y}$, implying that

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[7]See Mason (1996), for the details on how visibility measurements is translated into measurements of the degree of eutrophication.

[8]See Sandstrøm (1996) for an application of using visibility measured with a Secchi-disk in a contingent valuation study.

the eutrophying lake will approach a new steady state with murkier water, but not completely transformed into a marsh. The change in loading of nutrients originating from human activity into the lake is assumed to be governed by the following differential equation:

$$\dot{y} = \delta u - \beta y \tag{3}$$

Here $u$ is the amount of nutrients from which deposition into the lake is derived and $\delta \in (0,1)$ is the fraction of nutrients that is deposited into the lake. $\beta \in (0,1)$ is the proportion of nutrients that is washed out of the lake. Note that the lake is assumed to have two different ways, with quite different physical interpretations, of reversing eutrophication. First, a fraction of the nutrients is continuously washed out of the lake. Second, the if nutrient levels are sufficiently low, the ecosystem increase visibility via a mechanism where stocks of nutrient intensive species decrease or even disappear and are replaced by less demanding species.

It is assumed that there exists a regulator that is concerned about the state of the lake and that the regulator's preferences are Aristotelian in the sense that any deviance from the lakes "natural state" $\alpha / \gamma$ is considered bad. A parameterisation of the disutility from the degree of euthropication is:

$$u(x) = \frac{A}{2}\left(\frac{\alpha}{\gamma} - x\right)^2 \tag{4}$$

Here, only values of $x$ in the range 0 to $\alpha/\gamma$ is considered[9]. Thus more of $x$ implies higher visibility and is *ceteris paribus* desirable except if $x = \alpha/\gamma$. Deposition of nutrients often cause disutility for other reasons than the euthrophication of lakes, as can be testified by anyone located too close to a farm that use a mix of chicken- and pig manure for fertiliser or by unlucky enough to live close to facilities releasing untreated sewage. The disutility from $u$ caused by other effects than euthropication is assumed to be given by $\varphi u$, $\varphi > 0$. Note that this disutility depends $u$ and not on $\dot{y}$, reflecting a possible interpretation of the disutility being derived from odorous fertilising. Reduction in the depositions of nutrients is costly and the cost is assumed to be given by $\frac{c}{2}\left(u^0 - u\right)^2$. $u^0$ is the amount of nutrients that would be used in the absence of regulation and $c > 0$ is a parameter. The description above leads naturally to the following model:

$$\min_u \int_0^\infty \left(\tfrac{A}{2}\left(\tfrac{\alpha}{\gamma} - x\right)^2 + \tfrac{c}{2}\left(u^0 - u\right)^2 + \varphi u\right) e^{-rt} dt \tag{5}$$

subject to:

[9]Values of $x$ larger than $\alpha/\gamma$ in a sense implies that there is too little life in the lake.

$$\begin{aligned}
&\dot{x} = -\gamma x + z && x(0) = x^0 && x(\tau_+) - x(\tau_-) = 0 \\
&\dot{y} = \delta u - \beta y && y(0) = y^0 && y(\tau_+) - y(\tau_-) = 0 \\
&\dot{z} = 0 && z(0) = z^0 && z(\tau_+) - z(\tau_-) = \alpha - 2z(\tau_-)
\end{aligned} \tag{6}$$

Here $\tau$ is defined by any value of $t$ that solves the equation $y(\tau)$ - $\bar{y}$ = 0. Note that the formulation of the shock is such that whenever $y$ increase to levels above $\bar{y}$, $z$ = 0 and whenever $y$ decrease to levels below $\bar{y}$, $z = \alpha$. Thus $z(0) = 0$ implies that the system is starting from a point where a euthrophication process has already started. The Hamiltonian corresponding to (5) is given by:

$$H = -\left(\tfrac{A}{2}\left(\tfrac{\alpha}{\gamma} - x\right)^2 + \tfrac{c}{2}\left(u^0 - u\right)^2 + \varphi u\right)e^{-rt} + p_x(-\gamma x + z) + p_y(\delta u - \beta y) \tag{7}$$

$p_i$ is the adjoin variable to the state variable $i$. $p_z$ does not enter the Hamiltonian since $\dot{z}$ = 0. The standard sufficient conditions for optimality applies and are given by the equations of motion in (5) and the following equations:

$$u = \arg\max H = u^0 - \frac{\varphi}{c} + \frac{\delta}{c} p_y e^{rt} \tag{8}$$

$$\begin{aligned}
&\dot{p}_y = -\partial H / \partial y = \beta p_y && \lim_{t\to\infty} p_y(t) = 0 \\
&\dot{p}_x = -\partial H / \partial x = -A(\alpha/\gamma - x)e^{-rt} + p_x\gamma && \lim_{t\to\infty} p_x(t) = 0 \\
&\dot{p}_z = -\partial H / \partial z = -p_x && \lim_{t\to\infty} p_z(t) = 0
\end{aligned} \tag{9}$$

These conditions are the necessary conditions for the present type infinite horizon optimal control theory. (9) gives the differential equations determining the adjoint variables and the tranversality conditions[10]. When graphical analysis is presented it is convenient to work with the current value formulation of (9). In particular note that if $\lambda_y = p_y e^{rt}$, then:

$$\dot{\lambda}_y = r\lambda + \beta\lambda \tag{10}$$

An additional condition is required, since the jumps in $z$ whenever $y = \bar{y}$ implies that the adjoint functions will also jump whenever $y = \bar{y}$. The equation determining the size of the jump is discussed in the appendix where it is shown that the jumps in the adjoint function are governed by:

[10]The terminal conditions are applicable since the objective function obviously converge as time goes to infinity. In section 1.3 a different transversality condition is used.

$$
\begin{aligned}
\mu_y^{-1}\left(p_y(\tau_-)-p(\tau_+)\right)e^{r\tau} &= \left(-\frac{c}{2}\left(\frac{\varphi}{c}-\frac{\delta}{c}C_{t<\tau}e^{(\beta+r)\tau}\right)^2+\frac{c}{2}\left(\frac{\varphi}{c}-\frac{\delta}{c}C_{t>\tau}e^{(\beta+r)\tau}\right)^2\right) \\
&-\varphi\left(\frac{\delta}{c}\left(C_{t<\tau}e^{(\beta+r)\tau}-C_{t>\tau}e^{(\beta+r)\tau}\right)\right)+p_x(\tau_+)\left(z(\tau_-)-z(\tau_+)\right)e^{r\tau}
\end{aligned}
\tag{11}
$$

From $\dot{p}_y = \beta p_y$ we find that $p_y$ must be on the form:

$$p_y(t) = C_n e^{\beta t} \tag{12}$$

$C_n$ is a sequence of constants, related by the jump in $p_y$ whenever $y = \bar{y}$. Thus prior to the event $y = \bar{y}$ occurring the first time $p_y(t) = C_0 e^{\beta t}$. After the first time $y = \bar{y}$, $p_y(t) = C_1 e^{\beta t}$ and so on. Thus the control where the threshold is broken $n$ times must be on the following form:

$$
\begin{aligned}
u(t|t<\tau_1) &= u^0 - \frac{\varphi}{c} + \frac{\delta}{c}C_0 e^{(\beta+r)t} \\
u(t|t\in(\tau_1,\tau_2)) &= u^0 - \frac{\varphi}{c} + \frac{\delta}{c}C_1 e^{(\beta+r)t} \\
&\vdots \\
u(t|t\in(\tau_{n-1},\tau_n)) &= u^0 - \frac{\varphi}{c} + \frac{\delta}{c}C_{n-1} e^{(\beta+r)t}
\end{aligned}
\tag{13}
$$

## 2. Control Scenarios

In this section the model described above is used to characterise three different control scenarios for a lake.

1) Eutrophication in a initially eutrophying lake is reversed once and for all. Section 1.1.
2) A initially non-eutrophying lake is allowed to eutrophy completely. Section 1.2.
3) Policy drives nutrient levels to oscillate around the threshold so that the lake will approach a new steady state with “some” eutrophication. Section 1.3.

### 2.1. Climbing over the Edge

The first control scenario to be considered is the case where the threshold is breached at most once and that the breach is such that nutrient levels are initially too high. It is thus assumed that $y(0) > \bar{y}$ and that $z(0) = 0$. This implies that the eutrophication process has already begun and that the regulator considers policies for reversing the process. (11) reduces into:

$$p_y(\tau_-) = -\tfrac{c}{2} p_y(\tau_-)^2 e^{-r\tau}\mu_y - p_x(\tau_+)\alpha\mu_y \qquad (14)$$

The equations in (8), (9), (14), and the equations of motion in (6) define the solution to the problem. If the threshold is breached once, then by hypothesis there exists a $\tau$ such that $y(t) > \bar{y} \; \forall \; t \in [0 , \tau)$ and $y(t) < \bar{y} \; \forall \; t \in (\tau , \infty)$. Intuition tells us that if $y(t) < \bar{y}$ and will remain so until the end of the planning period, then $C_{t>\tau} = 0$, since there is no disutility from *y per se* . If an optimal path leads us to some value $y(t) = \bar{y}$ from a starting point $y^0 > \bar{y}$, $C_{t<\tau}$ must be a negative constant, since $y$ is bad. This reasoning combined with (8) yields some intuition to the qualitative character of an optimal path. We must have that:

$$\begin{aligned} u^*(t|t<\tau) &= u^0 - \frac{\varphi}{c} + \frac{\delta}{c} C_{t<\tau} e^{(\beta+r)t} \\ u^*(t|t>\tau) &= u^0 - \frac{\varphi}{c} \end{aligned} \qquad (15)$$

Interpreting (15) is straightforward. Until the lake is clean enough to reverse the eutrophication process, $u$ must be reduced from it's initial level $u^0$. As time goes $u$ will be continuously reduced, thereby making $\dot{y}$ more and more negative. This continuos reduction in $u^*$ is reversed at the time $\tau$ when the stock of $y$ is reduced enough so that the lake is able to cleanse it self. Then $u$ is allowed to jump up to the level $u^0 - \varphi/c$. In short, at time $\tau$, deposition of nutrients cease to be a stock/flow problem, and reverts to being just a flow problem where only the direct disutility from $u$ matters. The condition of the lake will deteriorate for a while even if policies for continuos reduction of $u$ are in place. It is only when $y$ has been reduced to below $\bar{y}$ that starts to improve. For such an optimal path, the closed form integral solution to $x(t)$ will take the following form:

$$\begin{aligned} x(t|t \le \tau) &= x^0 e^{-\gamma t} \\ x(t|t \ge \tau) &= \left( x^0 - \frac{\alpha}{\gamma} \right) e^{\gamma(\tau - t)} + \frac{\alpha}{\gamma} \end{aligned} \qquad (16)$$

Note that $x(t)$ is continuos, but not differentiable at $t = \tau$. Optimal paths of $x$ and $u$ are illustrated in figure 1.

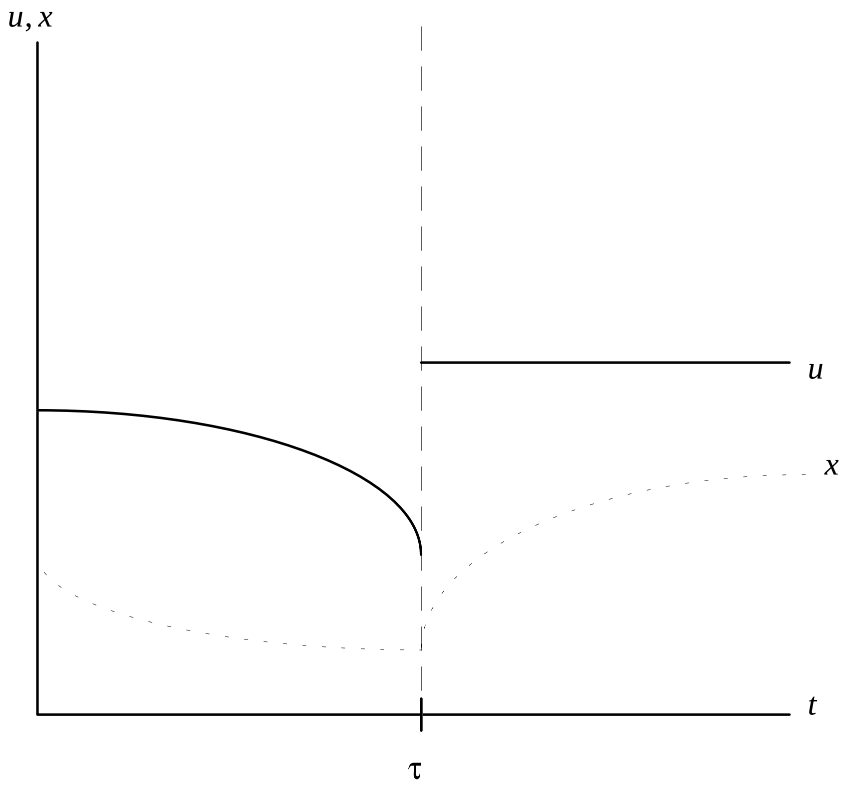


Figure 1.

Note from Figure 1 that $u$ becomes smaller and smaller as $t \to \tau$. This is caused by two effects. One partial effect is discounting, which implies that when a threshold is to be breached at a given time, it makes economic sense to postpone costs until a later date. The second effect, compounded by the discounting, is that when $y$ is high, large amounts of nutrients are washed out of the ecosystem through natural processes, and it makes little economic sense to assume costs for large reductions in $u$ when the ecosystem is rapidly cleaning itself anyway. Although the intuition the policy is clear, intuition does not suffice for the implementation of practical policy. For this we need to know $p_x(\tau_+)$, $p_y(\tau_-)$ and $\tau$. In particular, it must be proven that a negative $p_y(\tau_-)$ may be found for the solution to make economic sense. Using the transversality condition $\lim_{t\to\infty} p_x(t) = 0$, and noting that $p_x$ does not jump since $x$ neither jumps nor enters the equation determining $\tau$, it is straightforward to calculate that:

$$p_x(t|t>\tau) = \frac{\left(\frac{\alpha}{\gamma} - x(\tau)\right)}{r+2\gamma} e^{-\gamma(t-\tau)-rt} > 0 \,\forall\, x(\tau) < \frac{\alpha}{\gamma} \qquad (17)$$

Note that the assumption that $y = \overline{y}$ only once implies that $\dot{y}(\tau_+) < 0$ since otherwise $y$ would immediately cross the threshold again. Then we have that

$\dot{y}(\tau_+) < 0 \Rightarrow \dot{y}(\tau_-) < 0 \Rightarrow \mu_y > 0$. From (14) it is then straight forward to see that if there exists a value of $p_y(\tau_-)$ that solves (14) it must be negative.

An explicit solution may be found, but it is algebraically cumbersome[11]. In stead a phase diagram of $\lambda$ and $y$ is examined. The threshold is illustrated by the line marked $\bar{y}$. Obviously $\lambda \le 0 \Rightarrow \dot{\lambda} \le 0$, so no arrows indicating the movement in $\lambda$ points upwards. We have to draw two lines for $\dot{y} \le 0$. One for $\dot{y}(t|t > \tau)=0$ and one for $\dot{y}(t|t < \tau) = 0$.

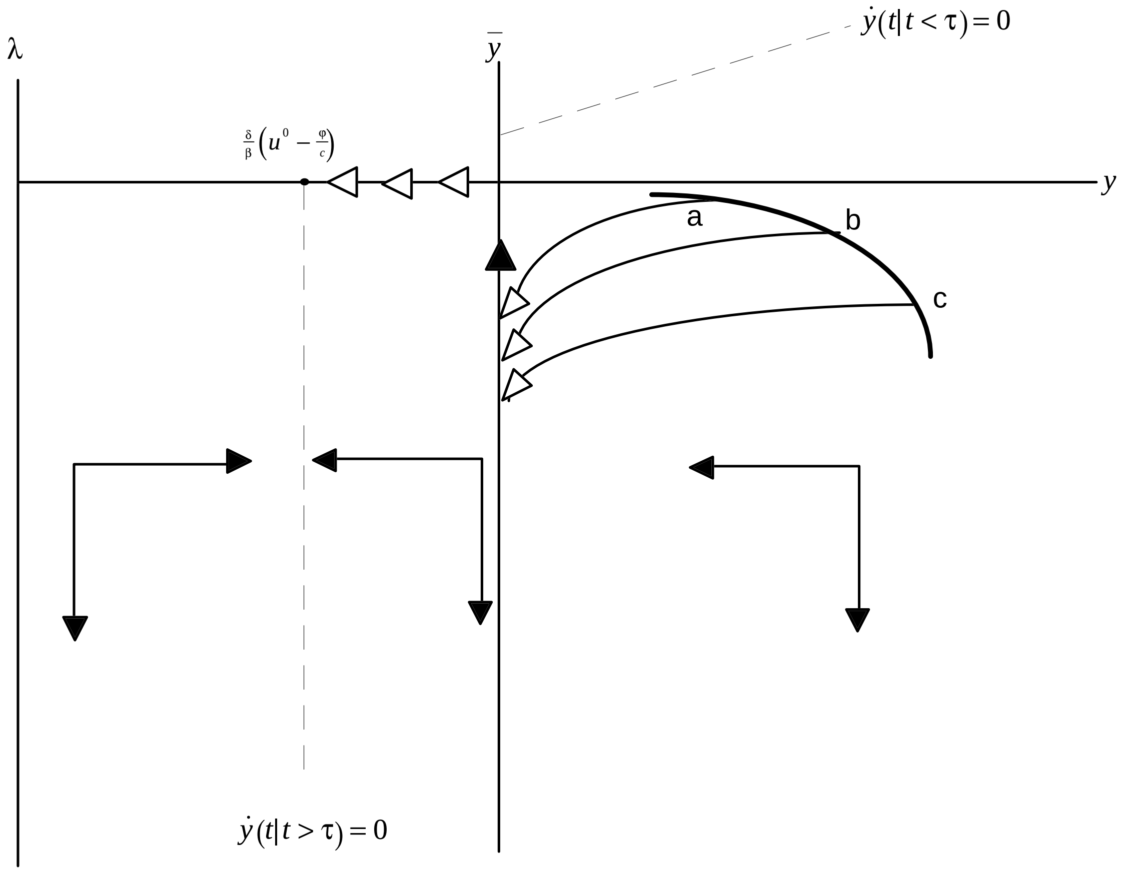


Figure 2

Here a, b and c are initial states of the system with $y$ exceeding $\bar{y}$. The line connecting these points show the relationship between different values of $y(0)$ and $\lambda_y(0)$. Starting from any of these point it can be seen that $y$ is continuously reduced, while $\lambda_y$ jumps when $y = \bar{y}$. Note that the size of the jump depends on the initial value of $y(0)$. The larger $y(0)$, the more negative is $\lambda_y$ and therefore also $C_{t<\tau}$. After the jump, $y$ approaches a steady state $y_{ss}$= $\frac{1}{\beta}\left(\delta\left(u^0 - \varphi/c\right)\right)$. Implicit in the assumption that the threshold is crossed only once is that $\dot{y}(\tau_+)$< 0. is that the policy $u(t\,|\,t < \tau)$=$u(t\,|\,t > \tau) = u^0 - \varphi/c$ will eventually reduce the nutrient level to a steady state below $\bar{y}$. Thus the

[11]In a real world application, numerical methods for solving xx are however easy to apply.

control scenario here described is only applies only to lakes that would have been cleaned up anyway by only taking the disutility from $\varphi u$ into account.

### 2.2. Going over the Edge

If initial levels of nutrient depositions are below $\bar{y}$, it may be thought that under certain conditions it is optimal to let the lake eutrophy completely even if nutrient depositions are initially lower than $\bar{y}$, but that one postpones the eutrophication process by delaying the time when the threshold is breached. Thus one enjoys the benefits of a pristine lake until $y$ reaches $\bar{y}$ a little longer, and then let the lake eutrophy forever. The necessary conditions in equations (8), (9) and the equations of motion in (6) still hold. $p_x(t)$ is also different since in the present scenario the (undiscounted) disutility from eutrophication increase continuously after the time the threshold is violated. $p_x(t)$ will in this still be non-negative for all $t > \tau$. Assuming that $x(0) = \alpha/\gamma$ simplifies the exposition somewhat since this implies that $x(\tau) = \alpha/\gamma$ and then $p_x(\tau_+) = 0$.
The jump equation in (14) reduces to:

$$p_y(\tau_-) = -\tfrac{c}{2} p_y(\tau_-)^2 e^{-r\tau} \mu_y \tag{18}$$

Since $\dot{y}(\tau_-) > 0$, $\mu_y < 0$. From (18) one can then see $p_y(\tau_-)$ must be non-negative in order to solve (18), and this does not make economic sense. Thus the scenario sketched here will never be optimal if one starts out with a initially pristine lake.

### 2.3. Living on the Edge

The previous sections have analysed two possible paths, both such that the threshold was breached but once. The case where the threshold is breached more than once still needs to be considered. Consider in particular the case where $y(0) > \bar{y}$ and $\delta(u^0 - \varphi/c) - \beta\bar{y} > 0$. The solution described in section 1.2. will not work since $y$ after being reduced to a level below $\bar{y}$, will bounce right back again. It may still be optimal to steer $y$ towards $\bar{y}$. After reaching $\bar{y}$, the optimal path of $y$ will oscillate around the threshold. In the case where $y(0) > \bar{y}$ such oscillation will occur if $c$ is sufficiently low relative to $\delta$ and $\delta(u^0 - \varphi/c) - \beta\bar{y} > 0$, This oscillation is caused by jumps in the adjoint variable $p_y(t)$ that occur when the threshold is broken. In principle such a model may be solved. We need to define two functions $C(s; y(s) < \bar{y})$ and $C(s; y(s) > \bar{y})$. The domain of these functions is the points in time where the inequalities in the definition of the function holds. The adjoint function then becomes $C(s; y(s) < \bar{y})e^{-\beta s}$ and $C(s; y(s) < \bar{y})e^{-\beta s}$. These functions need to obey the equation (A.8) in the appendix that the function defining the jumps in the $p_y$ and in addition they must have the following properties:

$$\liminf_{t\to\infty} C(t; y < \bar{y})e^{\beta t} = \liminf_{t\to\infty} C(t; y > \bar{y})e^{\beta t} = 0 \qquad (19)$$

The good news is that this leads to an optimal control $u$. The bad news is that such a control is discontinuous at every point in time. As soon as the threshold is broken and $y$ is e.g. reduced infinitesimally below $\bar{y}$, $u$ is increased so that $\dot{y}$ changes sign and $y$ is increased to a level infinitesimally above $\bar{y}$ and so on. An illustration is given if Figure 3 below. As in Figure 2 there are two lines that illustrate $\dot{y}$ = 0. One for $t < \tau$ and one for $t > \tau$. Note that the arrows indicating movement in variables indicates movement in variables *after* the threshold has been violated. $y$ moves along its path until it reaches $\bar{y}$. Then switching starts to occur. $y$ will (for all practical purposes) remain at $\bar{y}$, but $\lambda_y$ behaves oddly. The movement in $\lambda_y$ has two components. One continuos movement that is negative. This movement is illustrated by the white arrows. There is also the jump movements in positive direction.

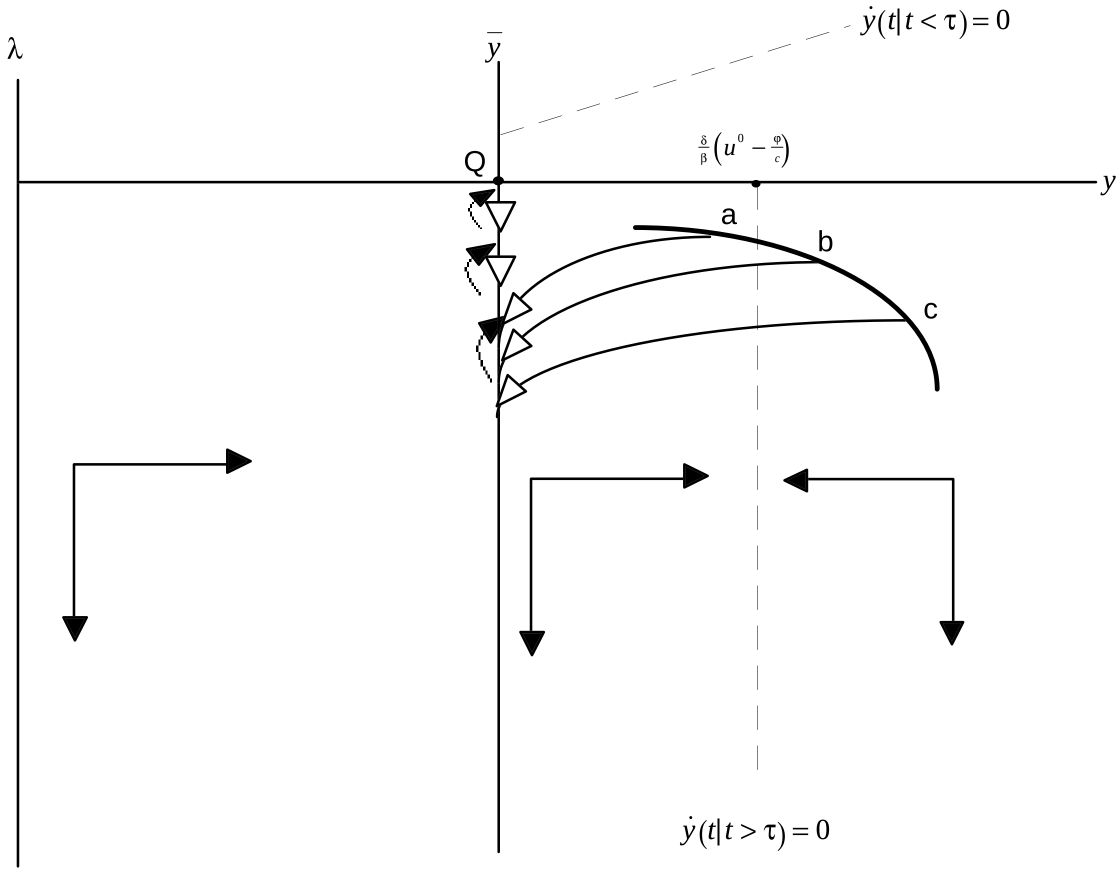


Figure 3.

Having examined the development of $y(t)$, it is possible to *approximate* the path of $x(t)$ from the time $y(t)$ is first reduced to a level below $\bar{y}$. A good approximation is the following differential equation:

$$\dot{x} \approx -\gamma x + \tfrac{\alpha}{k} \quad x(\tau_1) = x(x_0), \quad k \in (0,1) \qquad (20)$$

$\tau_1$ indicates time $y(t)$ is first reduced to a level below $\bar{y}$, and the notation $x(x_0)$ indicates that the level of eutrophication at the time $\tau_1$ is a function of how far the eutrophication process had progressed at $t = 0$. $k$ is a constant that depends on the parameters in the model. If $c$ is high then $k$ is low and if $A$ is high, then $k$ is high and so on. The solution to (20) is given by:

$$x(t) = \frac{\alpha}{k\gamma} - \left(\frac{\alpha}{k\gamma} - x(x_0)\right)e^{-\gamma(t-\tau)} \tag{21}$$

Comparing (21) with (16) we see that $x(t)$ in this case approach a steady state with a higher level of eutrophication and that the rate that $x(t)$ approach this steady state is slower. The intuition about this result makes economic sense to a certain extent. The regulator does not like neither eutrophication nor the costs of reducing $u$. By switching back and forth between a state of increased eutrophication and a state where the degree of eutrophication is reduced, the regulator strikes a balance between the joys of nature and the cost of reducing deposition of nutrients. It should be noted that $k$ is a function that depends on optimal values of $u(t\,|\,y > \bar{y})$ and $u(t\,|\,y < \bar{y})$.

A problem with the result briefly sketched above is that it is hard to imagine a control that discontinuously jumps at every point in time. Indeed, it is not mathematically well defined. In order to resolve this problem one must add more structure to the model. At least two possible extensions are possible.

- *Switching costs.* It may be argued that discrete jumps in control variables imply costs, and that these costs should be incorporated into the model.
- *Periodicity in the dynamics.* In many cases there are intrinsic fluctuations in the unregulated nutrient depositions, $u^0$. In agriculture fertilising is concentrated to particular seasons. In sewage treatment, the released amount of sewage has a twenty-four hour cyclical component. In both cases the natural cycles in nutrient depositions could be utilised to optimise switching policies.

Unfortunately both these approaches are analytically untractable. Further exploration must await development of numerical methods able to handle mixed boundary differential equation problems with discontinuities.[12]

**2.4. Choosing control scenario**

Three control scenarios have been sketched. One additional control scenario may also be taken into consideration. When $x(0) < \alpha/\gamma$, one must also evaluate controls that ignore eutrophication altogether in the sense that $p_y(t) = 0$ for all $t$. Alas the necessary conditions stipulated by the maximum principle and the jump conditions will rarely single out a single control scenario as the optimal one. Sufficiency theorems for the kind of problems discussed here are

[12]Such methods are currently under development by the authour.

extremely complicated, but the following procedure will work with most problems found in resource economics. 1) Use the necessary conditions to identify candidates for optimal control scenarios. 2) Evaluate the objective function for all scenarios found in 1). Choose the control scenario with the highest corresponding value to the objective function.

## Summary and Conclusions

The present article has analysed the optimal regulation of eutrophying lakes, rivers and fjords in the presence of threshold effects. It has been shown that threshold effects implies discontinuities in the control variable. Conditions for when eutrophication is a problem solved once and for all are presented. It has been argued that under certain condition threshold effects imply that optimal policy is everywhere discontinuous, and that this leads to some particular difficulties in determining optimal policies.

## Appendix - Derivation and use of the jump equation

There is a small, but distinguished literature on discontinuities in optimal control theory. Early economic applications include Arrow and Kurz (1970) and Vind (1967). A introduction to the topic is given in Seierstad and Sydsæter (1987), pages 194-209. This appendix develops a equation that determines the jump in the adjoint function when the state-variables jump, and then derives the specific jumps for the model presented in this article. The approach used here is more general than other approaches found in the literature. The formulation is due to Professor Atle Seierstad at the Dept. of Economics, University of Oslo.

Consider standard optimal control problem with a present value formulation. Let $f_0(t, u, x)$ be the instantaneous utility function and let $f(t, u, x)$ be the equation of motion depending on time, $t$, the controls, $u \in \mathbb{R}^n$, and the state-variables $x \in \mathbb{R}^m$. It is well known that the differential equation determining the adjoint variables, $p$, is given by:

$$\dot{p} = -\frac{\partial H}{\partial x} \tag{A.1}$$

Here $H$ is the present value Hamiltonian defined by $H = f_0(t, u, x) - pf(t, u, x)$. Assume that at some point(s) in time $\tau$, there is a shock so that the state variables jump. This shock occurs at the time defined by the following equation $\phi(x(\tau)) = 0$. Let the jump be given by $x(\tau_+) - x(\tau_-) = K$. Here $K$ is constant. Then a modification of (A.1) is to let the adjoint function jump according to the following formula.

$$\begin{aligned} p(\tau_+) - p(\tau_-) &= -\big(H(\tau, x(\tau_-) + K, u(\tau_+), p(\tau_+)) - H(\tau, x(\tau_-), u(\tau_-), p(\tau_-))\big)(-\mu) \\ &= -\big(H(\tau_+) - H(\tau_-)\big)(-\mu) \end{aligned} \tag{A.2}$$

$\mu$ is a $m \times 1$ vector where the elements capture the marginal effect of the control, $u$, via $x(t)$ on $\tau$. This effect is defined by the following equation:

$$\phi_x'\big(x(\tau_-)\big)f\big(\tau_-, x(\tau_-), u(\tau_-)\big)\mu_j + \phi_x'\big(x(\tau_-)\big)e_j = 0 \tag{A.3}$$

Here $e_j$ is the j-th unit vector. If $x(\tau_+)$ - $x(\tau_-)$ = $g(\tau$ , $x(\tau_-))$ with $g$ differentiable with respect to $x$, then (A.2) must be extended to take into account marginal effects of $x$ on $p(\tau_+)$ that is due to the size of the shock. (A.2) may then be written:

$$p(\tau_+) - p(\tau_-) = -\big(H(\tau_+) - H(\tau_-)\big)(-\mu) - p(\tau_+)g_x'\big(\tau, x(\tau_-)\big) \tag{A.4}$$

Writing out (A.4) completely and multiplying on both sides with -1 yields the general formula.

$$\begin{aligned} p(\tau_-) - p(\tau_+) &= \Big(f\big(\tau, x(\tau_-), u(\tau_-)\big) - f\big(\tau, x(\tau_+), u(\tau_+)\big)\Big)\mu + p(\tau_+)g_x'\big(\tau, x(\tau_-)\big) \\ &+\Big(p(\tau_+)f\big(\tau, x(\tau_-), u(\tau_-)\big) - p(\tau_+)f\big(\tau, x(\tau_+), u(\tau_+)\big)\Big)\mu \end{aligned} \tag{A.5}$$

Applying formula (A.3) to the model in the main text, it is easily seen that:

$$\mu_y = -\left(\begin{bmatrix}1\\0\\0\end{bmatrix}^T \begin{bmatrix}\delta u(\tau_-) - \beta\bar{y} \\ -\gamma x(\tau_-) + z(\tau_-) \\ 0\end{bmatrix}\right)^{-1} = \frac{-1}{\delta u(\tau_-) - \beta\bar{y}} \tag{A.6}$$

From (A.5) one may then verify that:

$$\begin{aligned} \mu_y^{-1}\big(p_y(\tau_-) - p(\tau_+)\big) &= \Big(-\Big(\tfrac{A}{2}\big(\tfrac{\alpha}{\gamma} - x(\tau_-)\big)^2 - \tfrac{c}{2}\big(u^0 - u(\tau_-)\big)^2 - \varphi u(\tau_-)\Big)\Big)e^{-r\tau} \\ &-\Big(-\Big(\tfrac{A}{2}\big(\tfrac{\alpha}{\gamma} - x(\tau_+)\big)^2 - \tfrac{c}{2}\big(u^0 - u(\tau_+)\big)^2 - \varphi u(\tau_+)\Big)\Big)e^{-r\tau} \\ &+p_x(\tau_+)\Big(\big(-\gamma x(\tau_-) + z(\tau_-)\big) - \big(-\gamma x(\tau_+) + z(\tau_+)\big)\Big) \end{aligned} \tag{A.7}$$

Inserting from equation (8) determining the optimal $u$ and noting that $x(t)$ is continuos at $\tau$:

$$\mu_y^{-1}\left(p_y(\tau_-)-p(\tau_+)\right)e^{r\tau}=\left(-\frac{c}{2}\left(\frac{\varphi}{c}-\frac{\delta}{c}C_{t<\tau}e^{(\beta+r)\tau}\right)^2+\frac{c}{2}\left(\frac{\varphi}{c}-\frac{\delta}{c}C_{t>\tau}e^{(\beta+r)\tau}\right)^2\right)$$
$$-\varphi\left(\frac{\delta}{c}\left(C_{t<\tau}e^{(\beta+r)\tau}-C_{t>\tau}e^{(\beta+r)\tau}\right)\right)+p_x(\tau_+)\left(z(\tau_-)-z(\tau_+)\right)e^{r\tau} \tag{A.8}$$

(A.8) is the jump condition used in (11).

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