Curing Commodity Kurtosis?*

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Abstract

Recent research has determined that commodity prices often exhibit distributional characteristics inconsistent with normality or log-normality. We utilize discrete mixtures of log-normals in a GARCH framework to model corn, wheat, and soybean prices. Options premiums are simulated and compared to actual premiums and premiums generated under standard Black-Scholes assumptions.

1 Introduction

Modeling the price behavior of financial instruments has been the impetus for much of the development of modern time-series econometrics. Early on, these endeavours were viewed by many as alchemy: a quest to find some lucrative, hidden structure buried deep beneath the erratic movements of prices. As the Efficient Markets and Rational Expectations paradigms overtook economics, the study of price processes became less concerned with price changes themselves, and more concerned with the stochastic processes that govern them.(Fama (1965) and Mandelbrot (1963))

Though options contracts had existed as far back as the 18th century, no rational pricing mechanism for them existed until Black and Scholes' (1973) pricing model was developed. This advancement greatly expanded the variety of derivatives available and the number of underlying assets upon which options were available grew as well.

A sizable literature exists, however, on the shortcomings of the B-S model. The model is primarily criticized for the restrictive assumptions about the price process of the underlying assets. These critcisms focus on two themes: that price realizations are iid and log-normally distributed. Mandelbrot(1963) was the first to observe that "random variables with an infinite population variance are are indispensible for a

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workable description of of price changes." He further observed the thick tails of asset price distributions, as well as second moment serial correlation. Numerous authors (add cites here) have studied these phenomena in the intervening years, and have found these characteristics to be present in nearly every speculative price series studied. Later econometric advances, especially ARCH and GARCH models of Engle (1982) and Bollerslev's (1986) GARCH model provided a parsimonious yet flexible technique for modelling processes with time-varying variances. This variation in the conditional distribution of returns did much to explain the kurtosis observed in unconditional distributions. (Millhøj (1985) and Bollerslev (1986)) However, excess kurtosis as well as skewness have also been found in the conditional distributions. (Cornew, Town and Crowsen (1984) Hudson, Leuthold and Sarassoro (1987) and Hsieh (1989)) These findings have prompted, among other approaches, the use of conditionally t-distributed errors, as in Bollerslev (1987) and Myers and Hanson (1993), and mixtures of discrete distributions. (Hsieh (1989)) The TGARCH specification incorporates the additional kurtosis, whereas the mixture distributions allow both excess kurtosis and skewness. This study will incorporate a mixture of discrete normal distributions into a GARCH framework, and assess the performance of the model by its ability to forecast observed options prices on futures contracts.

This paper has seven sections. The second section will the use of mixture distributions and GARCH models with emphasis added to the issues relevant here. The third section will discuss the estimation of the futures price model. The fourth section reviews the theory of options pricing, while the fifth section details the procedures used to estimate the options prices. The sixth section reports the results of the study and the seventh concludes.

2 Mixture Distributions and GARCH

In addressing the issues of leptokurtosis and skewness, discrete mixtures of normal distributions are simultaneously very flexible and tractable. With arbitrarily many distributions, a mixture distribution can replicate nearly any other distribution of similar support. Further, when using mixtures of normals, the moments of the mixture distribution are easily obtained and have relatively simple closed-form solutions.

The density function of a mixture of k discrete distributions is

$$f_k^*(x;\Theta) = \sum_{j=1}^{J} \lambda_j f_j(x;\theta_j) \qquad 0 \geq \lambda_j \geq 1\,\forall j \quad \sum_j \lambda_j = 1 \tag{1}$$

where λ_j is the weight of the the jth distribution, $f_j(x)$. This p.d.f. implies that the variance for the distribution has the form The log-likelihood function associated with a mixture distribution is then

$$LLF(x;\Theta) = \sum_{i=1}^{N} ln \sum_{j=1}^{J} \lambda_j f_j(x_i;\theta_j) \tag{2}$$

The use of Auto-Regressive Conditional Heteroskedasticity (ARCH) and Generalized ARCH (GARCH) models was introduced by Engel(1982) and Bollerslev(1986), respectively, to reflect the common observation that volatility tends to be highly correlated. The GARCH(p,q) process is directly analogous to specifying the variance of a process using an ARMA(p,q) model.

$$\begin{aligned} y_t &= X_t B + \epsilon_t & (3) \\ \epsilon_t &= \sqrt{h_t}\eta_t & (4) \\ h_t &= \kappa + \sum_{p=1}^{P} \alpha_j h_{t-j} \sum_{q=1}^{Q} \delta_i \epsilon_{t-i}^2 \end{aligned}$$

where η_t is an iid distribution of unit variance. In this framework, $E_{t-1}(\epsilon_t^2) = h_t$. Bollerslev showed that the unconditional variance of the GARCH process is $\kappa/(1 - \sum \alpha - \sum \delta)$. In the analysis of time-series models, the use of GARCH(1,1) models has dominated other CH specifications, having proved itself to be a remarkably flexible and robust technique.(Myers, Hanson (1993), [insert others])

3 Estimation of Futures Prices

In order to explore the effects of differing parameterizations of the underlying price process on options pricing, estimations of the hypothesized model must first be obtained. The synthesis of the mixture distribution and the GARCH(1,1) specification proceeds quite naturally from the underlying components. In this paper, futures prices are assumed to follow the process $E_{t-1}(P_t) = P_{t-1}$, i.e. that they are martingales.[1] Note, however, that due to the very nature of the mixtures of normals distribution, this assumption is made only for the construction and estimation of the model, it is not possible to allow an intercept in the model and identify it and the means of the distributions simultaneously,[2] and therefore, this model may in actual practice produce non-martingale behavior.

The role of seasonality in commodity markets is well-known (Fackler (1986), [cite others]) , and their use in modeling second-moment behavior, especially for option pricing is likewise well-known. [cites] As in Fackler(1986), among others, a Fourier expansion is used to estimate the seasonal effects of variance. The form is

$$s_t = \sum_{m=1}^{M} \phi_m sin(2m\pi t) + \psi_m cos(2m\pi t) \qquad 0 \geq t \geq 1 \tag{5}$$

where t denotes the time of year of the observation and M denotes the 'order of seasonality'. The use of the Fourier form produces a simple and smooth approximation of the seasonal variance effects.

In order to estimate the parameters of the futures price process, a rolling sequence of July soybean futures prices was employed. A parameter was added to the model to account for the changes in variance when the underlying contract switches expiration dates.[3]

Again, as per common practice, the model is estimated in the log of the futures prices.[4] The model to be estimated has become

$$\Delta ln\, P_t = \epsilon_t \tag{6}$$

$$\epsilon_t = \sqrt{h_t}\eta_t \tag{7}$$

$$h_t = \kappa + \sum_{j=1}^{p} \alpha_j h_{t-j} \sum_{i=1}^{q} \delta_i \epsilon_{t-i}^2 + \gamma Z_t \tag{8}$$

[1] See Campbell, Lo and MacKinlay (1997), Ch. 2 for a fuller explanation of martingales and their relation to Efficient Markets.

[2] The specification of a drift term also has no baring on the pricing of options, see Campbell, Lo, MacKinlay, (1997).

[3] Only one additional parameter was added to capture these effects. This implicitly assumes that the effect of the change in expiration is constant across years. Some have suggested that a better parameterization might be to have an individual dummy for each change of expiration, however, for this model, repeated attempts to incorporate this method failed to converge.

[4] Note that this implies that the model here is conditionally distributed as a log mixture of normals, as opposed to a mixture of lognormals.

Where now η is a unit-variance mixture of normals distribution, and γZ_t incorporates the seasonality and switching parameters.

As an explicit distribution of the errors is asserted, maximum likelihood estimation is the natural choice for obtaining parameters for the model. MLE estimation of both GARCH and mixture distributions is well-documented. One difficulty encountered in combining the processes is that in order for h_t to be the expectation of the variance on date t, the mixture density must be constrained to be of unit variance.

$$1 = Var(x) = \sum_{j=1}^{J} \lambda_j (\sigma_j^2 + \mu_j^2) - \left[\sum_{k=1}^{K} \lambda_k \mu_k \right]^2 \tag{9}$$

The unconditional variances of the discrete densities comprising the mixture cannot be simply scaled by h_t, instead, the price changes must be scaled by h_t before the log-likelihood function of the mixture distribution is calculated. The unconditional log-likelihood function is

$$LLF(x;\Theta) = \sum_{i=1}^{N} \frac{1}{h_i} ln \left(\sum_{j=1}^{m} \lambda_j f_j \left\{ \frac{x_i}{h_i}; \mu_j, \sigma_j \right\} \right) \tag{10}$$

where $f(x,\theta)$ is the normal density and h_1, the expected variance in the first period, was estimated as a parameter, as the conditioning information, ϵ_0^2 and h_0 doesn't exist.

Table one displays the results of the futures price estimation across a variety of parameterizations. By comparing the first and second columns, we can see the effects of adding seasonality to a model utilizing two normal densities. A likelihood ratio test of significance of the four additional parameters yields a p-value of .073. The second and third columns compare the results of incorporating a third distribution in the mixture density. However, due to the lack of identification of the mean and variance of the distribution under the null hypothesis of $\lambda_3 = 0$, standard asymptotic results don't apply to hypothesis tests regarding the appropriate number of distributions in a mixture. (see Feng and McCulloch (1994) and (1996), and McLachlan (1987))

4 Pricing Options

The problem of pricing European-style options owes its first solution to Fisher Black and Myron Scholes (1973). A complete explanation is beyond the scope of this article, an interested reader should see

	MLE	$s.e.$	MLE	$s.e.$	MLE	$s.e.$	MLE	$s.e.$
κ	0.0168	0.2021	0.0143	0.1728	0.0143	0.0185	0.0129	0.0201
α	0.8976	0.0509	0.9239	0.0159	0.9238	0.0023	0.9267	0.0005
δ	0.0905	0.0681	0.0623	0.0230	0.0624	0.0007	0.0598	0.0000
μ_1	0.0290	0.0228	0.0302	0.0097	0.0321	0.0005	0.0321	0.0016
σ_1	−0.8627	0.0172	−0.8904	0.0090	−0.8907	0.0019	−0.8957	0.0000
λ_2	0.1127	0.0086	0.0807	0.0016	0.0787	0.0015	0.0809	0.0000
μ_2	−0.1070	0.1140	−0.2434	0.0009	−0.2428	0.0502	−0.2668	0.0478
σ_2	−1.7311	0.0739	−1.8146	0.0434	−1.8258	0.0069	−1.7786	0.0000
λ_3					0.0118	0.0000		
μ_3					−0.1582	0.0955		
σ_3					−0.9415	0.1276		
$switch$	2.1938	0.2061	2.1946	0.0297	2.2007	0.1719	2.4153	0.0817
$\hat{h}_1$	2.1940	0.0574	1.9059	0.0304	1.9128	0.0799	1.8609	0.0870
ϕ_1			0.0182	0.0021	0.0183	0.0006	0.0210	0.0001
ψ_1			0.0000	0.0004	0.0001	0.0005	0.0013	0.0001
ϕ_2			−0.0160	0.0013	−0.0160	0.0009	−0.0198	0.0012
ψ_2			−0.0114	0.0025	−0.0114	0.0007	−0.0149	0.0001
ϕ_3							0.0051	0.0001
ψ_3							0.0056	0.0013
ll	−1968.38		−1959.728		−1959.72		−1959.10	

Merton (1990), Ch. 8 for a fuller explanation. The modal method for options-pricing, the Black-Scholes model, obtains from primitive assumptions of the iid normality of log returns and frictionless trading. It is especially the former that is investigated here, though violations of the latter may also influence estimation of the conditional distributions of returns, as well. The Black-Scholes model obtains regardless of the risk preferences of the investor, as it is based upon the premise that the sources of risk influencing the option price may all be hedged.[5]

The market price of a European-style call option should be the discounted value of the right conferred by the contract:

$$G_t = e^{-r(T-t)}\, E_t(max[P_t - K, 0]) \tag{11}$$

where K is the strike price of the option, T is the date of maturity and r is the risk-free rate of interest. From this simple concept, the pricing formula for European options can be derived to be (Campbell, Lo,

[5]Check this, I think that there is something subtly incorrect here

MacKinlay (1997)).

$$G_t = P_t F\left(\frac{log(P_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}F\left(\frac{log(P_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \quad (12)$$

where $F(\dot{})$ is the normal culmulative distribution function. The adaptation of the options-pricing formula to the mixture of normals distribution is not unique. Ritchey (1990) showed that under risk-neutrality options prices derived from a linear combination of normal distributions are equivalent to the linear combinations of options prices derived from the Black-Scholes model.

5 Estimation of Options Prices

6 Results

Conclusion

References

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