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A QUADRATIC INVERSE DEMAND SYSTEM

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ABSTRACT: WE DERIVE A (QUADRATIC) INVERSE DEMAND SYSTEM (IQUAIDS), THAT GENERALISES THE (ALMOST IDEAL) INVERSE DEMAND SYSTEM. WE DISCUSS HOW THE ASSUMPTIONS OF SEPARABILITY AND CONCAVITY CAN BE SIMULTANEOUSLY ACCOMODATED WITHIN THE MODEL, THUS GIVING A (LOCALLY) SEPARABLE AND CONCAVE QUDRATIC DEMAND SYSTEM. AN EMPIRICAL APPLICATION IS PROVIDED FOR ILLUSTRATIVE PURPOSES.

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Introduction

In empirical demand analysis, it may be more appropriate to have quantities exogenous, and prices adjusting to clear the market.: it is the case of agricultural markets, under supply management programs. Furthermore, in the time-horizon of most time-series data in applied work (monthly or quarterly), supply can be reasonably treated as fixed in the short run: this is obviously true for perishable products.

Inverse demand systems have been proposed in empirical work: Barten and Bettendorf (1989) derived a Rotterdam-type inverse demand system; indirect translog demand systems date since Christensen et al. (1975); Moschini and Vissa (1992) and Eales and Unnevehr (1994) proposed a linear inverse demand system that resembles the more common (direct) AIDS.

Recently, there have been questions that a parsimonious representation of preferences may be able to fit well actual data: in direct demand systems, further terms in income are required, for some goods, to provide a better picture of reality. Banks et al. (1997) have proposed a Quadratic Almost Ideal Demand System (QUAIDS): it has been derived as a generalisation of the PIGLOG preferences, starting from a (general) representation of the indirect utility function. In the case of inverse demand systems, we may have the analogous situation; thus we may need to augment common inverse demand systems to account for further non-linearities. It is an important remark, especially if we use the estimated model for simulation and/or forecasting: the quadratic specification allows for more flexibility, and the more we move from in-sample values, the more the gain in flexibility may reduce the bias.

Further problems in empirical demand analysis may result from the large number of parameters that have to be estimated under flexible specifications: this number increases quadratically as the number of goods increases. In order to reduce the parameter space, we may resort to assumptions on the structure of consumers' preferences. In direct demand systems, it has been proposed to maintain weak separability of the direct utility function (Blackorby et al., 1978), because it may result in a gain in degrees of freedom and, if applied to a complete demand system, it would provide unconditional elasticities (Moschini et al., 1994).

A common empirical problem may also result from attempting to maintain the standard properties of demand theory. In fact, while symmetry, homogeneity and adding-up are easily built into the most popular demand systems, imposing the curvature condition (i.e. negative semidefiniteness of the Slutsky/Antonelli matrix) may create serious econometric problems in terms of convergence of the objective function (Moschini, 1998). A solution is that of reducing the rank of the matrix: the resulting model is semiflexible, according to Diewert and Wales (1988), and allows a further reduction in the parameter space, although it restricts substitution possibilities among goods.

In this paper, we derive a (quadratic) inverse demand system that generalises the (almost ideal) inverse demand system of Moschini and Vissa (1992) and Eales and Unnevehr (1994): this generalisation nests the linear specification. We also extend previous work on separability and derive the proper parametric restrictions within the inverse specification. Finally, we provide an illustrative example in which we estimate a concave, separable and semiflexible inverse demand system.

A quadratic inverse demand system

The parallel between cost function and distance function is well known (Blackorby et al., 1978): thus standard functional forms applied to the cost function can be extended to the distance function.

Start from a distance function of the form:

(1) $$\ln D(u,\boldsymbol{q}) = \ln a(\boldsymbol{q}) - \left[\frac{u \cdot b(\boldsymbol{q})}{1 - u\lambda(\boldsymbol{q})}\right]$$

where u indicates utility, ln$a(\boldsymbol{q})$, $b(\boldsymbol{q})$ and $\lambda(\boldsymbol{q})$ are functions of the vector of quantities $\boldsymbol{q}$; the minus sign states that the distance function is decreasing in u. In order to guarantee some properties for the distance function $D(u,\boldsymbol{q})$ (i.e. homogeneity of degree 1 in $\boldsymbol{q}$), we require $a(\boldsymbol{q})$ to be homogeneous of degree 1 in $\boldsymbol{q}$, and $b(\boldsymbol{q})$ and $\lambda(\boldsymbol{q})$ homogeneous of degree 0 in $\boldsymbol{q}$.

By applying the derivative property, $\partial \ln D(u,\boldsymbol{q})/\partial \ln q_i = w_i(u,\boldsymbol{q}) \equiv (p_i q_i)/m$, where p indicates prices and m indicates income (expenditure), we get compensated quantity-dependent demand functions: given the adopted functional form for the distance function, we get:

(2) $$w_i = \frac{\partial \ln a(\boldsymbol{q})}{\partial \ln q_i} - \frac{\partial \ln b(\boldsymbol{q})}{\partial \ln q_i}\frac{u \cdot b(\boldsymbol{q})}{1 - u \cdot \lambda(\boldsymbol{q})} - \frac{u^2 \cdot b(\boldsymbol{q})}{[1 - u \cdot \lambda(\boldsymbol{q})]^2}\frac{\partial \lambda(\boldsymbol{q})}{\partial \ln q_i}$$

By exploiting the fact that at D=1 the distance function is an implicit representation of the direct utility function, that is $U(\boldsymbol{q}) = \ln a(\boldsymbol{q}) / [\lambda(\boldsymbol{q}) \cdot \ln a(\boldsymbol{q}) + b(\boldsymbol{q})]$, we get uncompensated inverse demand functions:

(3) $$w_i = \frac{\partial \ln a(\boldsymbol{q})}{\partial \ln q_i} - \frac{\partial \ln b(\boldsymbol{q})}{\partial \ln q_i} \ln a(\boldsymbol{q}) - \frac{\partial \lambda(\boldsymbol{q})}{\partial \ln q_i} \frac{1}{b(\boldsymbol{q})} [\ln a(\boldsymbol{q})]^2$$

To get a parametric specification of a quadratic inverse demand system, we set $\ln a(\boldsymbol{q})$ as a translog quantity aggregator function:

(4) $$\ln a(\boldsymbol{q}) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln q_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln q_i \ln q_j$$

$b(\boldsymbol{q})$ as a Cobb-Douglas quantity aggregator function:

(5) $$b(\boldsymbol{q}) = \prod_{i=1}^{n} q_i^{\beta_i}$$

and $\lambda(\boldsymbol{q})$ as a linear quantity aggregator function:

(6) $$\lambda(\boldsymbol{q}) = \sum_{i=1}^{n} \lambda_i \ln q_i$$

Hence, the parametric specification is:

(7) $$w_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \ln q_j - \beta_i \ln a(\boldsymbol{q}) - \lambda_i \frac{1}{b(\boldsymbol{q})} [\ln a(\boldsymbol{q})]^2$$

where, given the homogeneity property of the distance function and symmetry property of demand functions, the following set of restrictions applies:

(8) $$\sum_i \alpha_i = 1 \quad \sum_i \beta_i = 0 \quad \sum_i \gamma_{ij} = 0 \quad \sum_j \gamma_{ij} = 0 \quad \sum_i \lambda_i = 0$$
$$\gamma_{ij} = \gamma_{ji}$$

We term this model as the Inverse Quadratic AIDS (IQUAIDS), although we recognise the improper use of the term "almost ideal", since this model does not share the same aggregation properties of the AIDS-QUAIDS specifications. The linear specification of Moschini and Vissa (1992) and Eales and Unnevehr (1994) is nested into the quadratic model: it is retrieved by setting $\lambda_i = 0 \ \forall i$.

In inverse demands the analogues of (uncompensated) price elasticities are (uncompensated) quantity elasticities, (also known as price flexibilities); after defining $\pi_i \equiv p_i/m$ as normalised prices, then quantity elasticities are given by $f_{ij} \equiv (\partial \pi_i / \partial q_j)(q_j / \pi_i) = \partial \ln \pi_i / \partial \ln q_j$; on the other hand, the analogues of expenditure (income) elasticities are the scale elasticities (flexibilities).

Quantity elasticities can be derived as $f_{ij} = \left(\partial w_i / \partial \ln q_j\right) / w_i - \delta_{ij}$, where δ_{ij} is the Kronecker delta (δ_{ij}=1 for $i=j$ and δ_{ij}=0 otherwise). Given our parametric specification, they are computed as:

(9) $$f_{ij} = \frac{\gamma_{ij}}{w_i} - \frac{\beta_i}{w_i}\left(\alpha_j + \sum_k \gamma_{jk} \ln q_k\right) - \frac{\lambda_i}{w_i}\frac{2\ln a(\boldsymbol{q})}{b(\boldsymbol{q})}\left(\alpha_j + \sum_k \gamma_{jk} \ln q_k\right) + \frac{\lambda_i \beta_i}{w_i}\frac{1}{b(\boldsymbol{q})}\left[\ln a(\boldsymbol{q})\right]^2 - \delta_{ij}$$

Scale elasticities, defined as $f_i \equiv \partial \ln \pi_i(\theta \mathbf{q}^*) / \partial \ln \theta$, where θ is a scale parameter such that $q \equiv \theta \mathbf{q}^*$, are retrieved by exploiting the relation $f_i = \sum_j f_{ij}$. In our quadratic specification, we have:

(10) $$f_i = -\frac{\beta_i}{w_i} - \frac{\lambda_i}{w_i}\frac{2\ln a(\boldsymbol{q})}{b(\boldsymbol{q})} + \frac{\lambda_i \beta_i}{w_i}\frac{1}{b(\boldsymbol{q})}\left[\ln a(\boldsymbol{q})\right]^2 - 1$$

The link between compensated and uncompensated flexibilities is obtained using the scale decomposition of inverse demands (Anderson, 1980), that is the analogue of the Slutsky decomposition in direct demands. A change in the quantity of a good produces a change in (normalised) prices: the total change in prices can be decomposed in a substitution effect and in a scale effect. The (Antonelli) substitution effect is the change in prices that allows to move along the initial indifference curve to induce a consumption of goods in the new proportion; the scale effect is the change in prices that, moving along the new ray from the origin, will induce to consume the new bundle on the final indifference curve. Hence the Antonelli substitution effect is defined as $a_{ij} = \partial \bar{\pi}_i(\mathbf{q}, u) / \partial q_j$, where $\bar{\pi}_i(\mathbf{q}, u)$ indicates compensated inverse demand, and the Antonelli equation becomes $a_{ij} = \partial \pi_i / \partial q_j - \pi_j (\partial \pi_i(\theta \mathbf{q}^*) / \partial \theta)$ (see Anderson, 1980). The matrix of Antonelli substitution effects $\lfloor a_{ij} \rfloor$ is just the (symmetric) matrix of the second derivatives of $D(u,\boldsymbol{q})$; concavity of $D(u,\boldsymbol{q})$ in quantities ensures that the Antonelli matrix is negative semidefinite. The Antonelli effects can be used to class goods as q-complements (if a_{ij}>0) and q-substitutes (if a_{ij}<0), according to Hicks (1956).

It is more convenient to express the Antonelli equation in elasticities terms as $f_{ij}^c = f_{ij} - w_j f_i$ where f_{ij}^c is the compensated quantity-elasticity; starting from this relation it is possible to derive the Antonelli substitution effect in our quadratic specification as:

$$a_{ij} = \frac{m}{q_i q_j}\left[\gamma_{ij} + w_i w_j - \delta_{ij} w_i - (\beta_i - \lambda_i)\beta_j \ln a(\boldsymbol{q}) + (\lambda_i \beta_i - \lambda_i \beta_i w_i - \lambda_j \beta_i - 2\lambda_i \beta_j)\frac{[\ln a(\boldsymbol{q})]^2}{b(\boldsymbol{q})} - 2\lambda_i \lambda_i \frac{[\ln a(\boldsymbol{q})]^3}{[b(\boldsymbol{q})]^2}\right] \tag{11}$$

While symmetry, adding-up and homogeneity can be imposed globally by parametric restrictions, negativity involves inequality constraints; furthermore, since the Antonelli substitution terms involve shares, prices and income, there do not exist parameter values ensuring that negativity will be satisfied globally. Therefore negativity can be checked after estimation for some values of the variables, or, otherwise, it can be imposed only locally, at a given point.

If we scale quantities to equal unity at a point (e.g., the mean of the sample), the expressions for elasticities and the Antonelli substitution effects become simpler: in fact, at this point $\ln a(\boldsymbol{q}) = \alpha_o$ and $b(\boldsymbol{q}) = 1$. Given that the parameter α_0 cannot be estimated, it is fixed before estimation. However the chosen value does not affect estimation results: hence, we can set α_0=0.

Elasticity formulas at the scaling point become:

$$f_{ij} = \frac{\gamma_{ij}}{w_i} - \frac{\beta_i w_j}{w_i} - \delta_{ij} \tag{12}$$

$$f_i = -\frac{\beta_i}{w_i} - 1 \tag{13}$$

while the Antonelli terms are computed as:

$$a_{ij} = \bar{m}\left[\gamma_{ij} + w_i w_j - \delta_{ij} w_i\right] \tag{14}$$

It is interesting to note that, at a scaling point, elasticity formulas and Antonelli substitution terms for the quadratic specification boil down to those of the linear specification (at the scaling point). The proposed IQUAIDS specification is of course flexible, but it does not reach the so called "minimal property" (Barnett and Lee, 1985), because it has *(n-1)* additional parameters.

Separability in inverse demands

The notion of direct weak separability (DWS) essentially relates to the possibility of partitioning goods in the direct utility function *U(.)*. More formally, if the set of indices of the *n* goods is *N={1,...,n}*, DWS implies that these goods can be ordered in S separable groups according to a mutually exclusive and exhaustive partition $\Psi=\{N_1,N_2,...,N_S\}$ of the set *N*. For example, any utility function *U(.)* is directly separable in the partition Ψ if it can be written as:

(15) $$U(\boldsymbol{q}) = U^0(u^1(\boldsymbol{q}^1),...,\ u^S(\boldsymbol{q}^S))$$

where $u^s(.)$, $s=1,\ldots,S$ are sub-utility functions that depend on a subset $\boldsymbol{q}^s$ of goods (Blackorby et al., 1978).

The separable structure of *U(.)* implies restrictions on the substitutability of goods belonging to different groups. In particular, the marginal rate of substitution between two goods belonging to the same group is independent of all goods that are not in that group. Moschini et al. (1994) have derived necessary and sufficient conditions for DWS in direct (quantity-dependent) demand functions. Kim (1997) has derived restrictions from homothetic DWS in inverse demands, in terms of the Antonelli elasticities of complementarity, which are defined as $\rho_{ik} \equiv f_{ik}^c / w_k$.

Theorem 1. *The utility function U(q) is weakly separable in the partition* Ψ *if and only if*

(16) $$\rho_{ik} - \rho_{jk} = f_j - f_i \qquad i, j \in N_g \ \ and \ \ k \in N_h$$

Proof. To show that DWS implies the above set of restrictions, we start from the Hotelling-Wold identity (Anderson, 1980; Weymark, 1980) that defines inverse demands as:

(17) $$p_i = \frac{\frac{\partial U}{\partial q_i}}{\sum_t \frac{\partial U}{\partial q_t} q_t} m$$

Since DWS implies that $\partial U/\partial q_i = \frac{\partial U}{\partial u^s}\frac{\partial u^s}{\partial q_i}$, the price ratio $p_i / p_j \equiv \frac{\partial u^s / \partial q_i}{\partial u^s / \partial q_j}$ for $(i,j)\in N_s$ becomes independent of quantities of goods outside that group, that is:

(18) $$\frac{\partial (p_i / p_j)}{\partial q_k} = 0 \quad \Rightarrow \quad \frac{\partial p_i}{\partial q_k}\frac{1}{p_i} - \frac{\partial p_j}{\partial q_k}\frac{1}{p_j} = 0 \qquad i, j \in N_g \ \ and \ \ k \in N_h$$

that can be expressed in elasticity terms, using the notion of quantity-elasticity (flexibility) f_{ik}, as:

$$(19)\qquad f_{ik} - f_{jk} = 0 \qquad i,j \in N_g \quad and \quad k \in N_h$$

which is equivalent to the restrictions in (16).

To prove necessity, we consider that the set of restrictions in (16) implies:

$$(20)\qquad \frac{\partial\left(\dfrac{dU/\partial q_i}{\sum_t (dU/\partial q_t) q_t}\right)}{\partial q_k}\frac{q_k}{\pi_i} = \frac{\partial\left(\dfrac{dU/\partial q_j}{\sum_t (dU/\partial q_t) q_t}\right)}{\partial q_k}\frac{q_k}{\pi_j} \qquad i,j \in N_g \quad and \quad k \in N_h$$

where $\pi_i \equiv p_i/m$ are normalised prices. Manipulation of (20) leads to the following equality:

$$(21)\qquad \frac{1}{\pi_i}\left[\frac{\partial^2 U}{\partial q_i \partial q_k} - \frac{\partial^2 U}{\partial q_j \partial q_k}\frac{\pi_i}{\pi_j}\right] = 0$$

which is equivalent to the restriction:

$$(22)\qquad \frac{\partial\left(\dfrac{dU/\partial q_i}{dU/\partial q_j}\right)}{\partial q_k} = 0$$

that implies DWS, once we recognise that the tangency condition implies $\frac{\pi_i}{\pi_j} = \frac{\partial U/\partial q_i}{\partial U/\partial q_j}$.

In inverse demands, it may be more interesting to analyse the role of (weak) separability of the indirect utility functions (IWS). Assuming that goods can be partitioned in the indirect utility function according to the partition Ψ, then $V(\pi)$ can be written as:

$$(23)\qquad V(\pi) = V^0(v^1(\pi^1),\ldots, v^S(\pi^S))$$

where $v^s(.)$, $s=1,\ldots,S$ are sub-utility functions that depend on a subset π^s of normalised prices.

Given the duality between cost function and distance function, and between direct and indirect utility functions, we may extend results of implicit separability obtained by Blackorby et al. (1991) to the indirect utility function. Starting from theorem 3 in Blackorby et al. (1991), we may follow the approach in Moschini et al. (1994) and write the following necessary and sufficient conditions for IWS:

$$(24)\qquad \frac{\partial \bar{\pi}_i(\boldsymbol{q},u)}{\partial q_k} = \Gamma(\boldsymbol{q},u)\frac{\partial \Xi(\boldsymbol{q},u)}{\partial q_i}\frac{\partial \Xi(\boldsymbol{q},u)}{\partial q_k}$$

$$(25)\qquad \frac{\partial \bar{\pi}_i(\boldsymbol{q},u)}{\partial u} = \Gamma(\boldsymbol{q},u)\frac{\partial \Xi(\boldsymbol{q},u)}{\partial q_i}\frac{\partial \Xi(\boldsymbol{q},u)}{\partial u}$$

where with $\bar{\pi}_i(\boldsymbol{q},u)$ we indicate compensated inverse demands of normalised prices, while $\Gamma(\boldsymbol{q},u)$ and $\Xi(\boldsymbol{q},u)$ are some appropriately defined functions. Following Anderson (1980), compensated inverse demands give the levels of normalised prices that induce consumers to choose a consumption bundle that is along a ray passing through $\boldsymbol{q}$ and that gives a utility u. On the other hand, uncompensated inverse demands $\pi_i(\boldsymbol{q})$ give the level of normalised prices that induce consumers to choose the consumption bundle $\boldsymbol{q}$. Compensated and uncompensated normalised prices are equal once quantities are properly scaled: therefore, we define the bundle $\boldsymbol{q}^* = \theta\boldsymbol{q}$ as exactly the bundle along a ray through the origin that gives utility u: then we may write the equality:

(26) $$\bar{\pi}_i(\boldsymbol{q},u) = \pi_i(\boldsymbol{q}^*) \quad \Rightarrow \quad \bar{\pi}_i(\boldsymbol{q},u) = \pi_i(\theta\,\boldsymbol{q}) \equiv \hat{\pi}_i(\boldsymbol{q},\theta)$$

Then, by the definition of the distance function as the amount by which $\boldsymbol{q}$ must be divided in order to bring it on the indifference curve u, we may define $1/\theta=D(\boldsymbol{q},u)$ and substitute it in (26); then, using (25), the restrictions in (24) can be written as:

(27) $$\frac{\partial \bar{\pi}_i(\boldsymbol{q},u)}{\partial q_k} = \Theta_k(\boldsymbol{q},u)\frac{\partial \hat{\pi}_i(\boldsymbol{q},\theta)}{\partial \theta}$$

where $\Theta_k(\boldsymbol{q},u)$ is a function that depends on the good q_k we are considering.

Thus, taking two goods $(i,j)\in N_g$ and a good $(k)\in N_h$, by using equation (27), necessary and sufficient conditions for IWS can be restated in terms of the Antonelli elasticities of complementarity as:

(28) $$\frac{\rho_{ik}}{\rho_{jk}} = \frac{f_i}{f_j} \qquad i,\, j \in N_g \;\; and \;\; k \in N_h$$

Homothetic DWS (i.e. separability of the sub-utility functions) implies that the distance function is implicitly separable in the same partition of the direct utility function. Under homothetic separability, we have that the above set of restrictions becomes $\rho_{ik} = \rho_{jk}$ (see also Kim, 1997); this is a consequence of the fact that along a ray from the origin the marginal rate of substitution is constant. Hence, under homotheticity of the sub-utility functions (for both the direct and indirect utility), we have that DWS and IWS provide the same restrictions.

In order to count the number of restrictions that are implied by our separability assumptions, we may resort to the formula proposed by Moschini et al. (1994: their equation (10), page 63) for DWS in direct demand systems; the formula applies also to either DWS or IWS in inverse demands. Furthermore, the *(i,j,k)* non–redundant combinations implied by the separable structure can be retrieved using the scheme proposed by Nayga and Capps (1994: their table 4, page 805).

Separabilty and concavity in the IQUAIDS: the case of DWS

The local separability restrictions in (16) can be expressed in terms of the parameters of any flexible functional form. In our case, if we consider the scaling point such parametric expressions become the same for both the IQUAIDS and the (linear) inverse AIDS (IAIDS), and take the form:

(29) $$\frac{\gamma_{ik} - \alpha_k \beta_i - \alpha_i \beta_k}{\gamma_{jk} - \alpha_k \beta_j - \alpha_j \beta_k} = \frac{\alpha_i}{\alpha_j} \qquad i, j \in N_g \ \text{and} \ k \in N_h$$

Also concavity of the Antonelli matrix can be imposed only locally, using the Cholesky decomposition; a necessary and sufficient condition for negative semidefiniteness of a matrix $A \equiv \lfloor a_{\mathrm{ij}} \rfloor$ is that it can be written as $A = -\Gamma'\Gamma$, where $\Gamma \equiv \lfloor \tau_{ij} \rfloor$ is an upper triangular matrix (Diewert and Wales, 1987). Ryan and Wales (1998) shows that this procedure preserves flexibility of the underlying functional form. The notion of semiflexibility pertains to the possibility of restricting the substitution matrix and reducing the parameter space, and it may be a solution when the estimation of the fully concave model gives problems of convergence. The solution originally adopted by Diewert and Wales (1988) was that of restricting the rank of the matrix $\Gamma'\Gamma$. Moschini (1998) extended it to derive a semiflexible AIDS model

A concave and separable inverse demand system must accomodate restrictions in (29) on the terms γ_{ij} , with the imposition of concavity on the matrix $A \equiv \lfloor a_{ij} \rfloor$, that places restrictions on the matrix $\lfloor \gamma_{ij} \rfloor$; a semiflexible specification will impose further restrictions on the matrix of price coefficients $\lfloor \gamma_{ij} \rfloor$.

In order to provide an empirical example, we have applied the IQUAIDS to meat demand in Italy. The use of an inverse specification for meat demand may be justified by the relatively fixed supply in the short run, due to the time lag between investment decisions and final production. Data for the period

1960-1990 are yearly current and constant price expenditures for private consumption (ISTAT, National Institute of Statistics). Implicit prices are obtained dividing current by constant price expenditures. We have specified a conditional meat demand system for beef, pork and poultry. The scaling point is the sample mean. The model is specified in levels, with a logarithmic trend, and estimated via maximum likelihood techniques, dropping one of the equations because of singularity of the system. We have also postulated separability between beef and other meats.

The contemporaneous imposition of the standard set of restrictions (adding-up, homogeneity and symmetry) together with separability and concavity, may give some technical difficulties, because of restrictions involving the same parameters. Although a systematic procedure is not feasible, we show how to solve the problem by following a precise order in imposing the whole set of restrictions.

At first, we impose adding-up, homogeneity and symmetry on the original parameters (α'*s*, β'*s*, γ'*s* and λ'*s*), taking into account that we need to further impose negativity and separability. Since negativity involves the *rank (n-1)* matrix $\lfloor \gamma_{ij} \rfloor$, it may be convenient to impose homogeneity on a single row/column of the *(nxn)* matrix. We choose as "numeraire" a good which belongs to a group of more than one element, because only a few of these coefficients may be further restricted under DWS. In our case, homogeneity was imposed on the poultry coefficients.

The second step is to impose concavity on the Slutsky matrix, even though the characteristics of the data set may force the adoption of a more parsimonious semiflexible specification, that restricts the rank of the matrix $\Gamma\Gamma$. In our application the rank is restricted to 1.

The third step is the imposition of DWS. Clearly, these restrictions must be rewritten incorporating the new concave form of the price parameters. This operation requires two types of cautions: first, the restricted parameters, once all restrictions are imposed, must be independent of each other; second, the restricted parameters must belong to the set of parameters that can actually be estimated under the semiflexible specification. Note that this second limit imposes an upper bound to the number of DWS restrictions that can be maintained in a semiflexible specification.

In the appendix we provide an illustrative example of this 3-step procedure, which refers to our simple empirical application, while in table 1 we compare estimation results from the three possible specifications of the IQUAIDS: the unconstrained system, the (semiflexible) concave system and the (semiflexible) concave and separable system. The table allows to verify how the imposition of such restrictions affect elasticities and R^2 values.

Concluding remarks

We have proposed a new inverse demand system, the Inverse Quadratic AIDS (IQUAIDS), that nests the Inverse AIDS specification. Furthermore, we have derived a set of necessary and sufficient restrictions implied by direct weak separability and indirect weak separability within inverse demands.

Maintaining both separability and concavity in a demand system allows to ensure that the model will satisfy integrability conditions while saving degrees of freedom. We have provided an example of how direct weak separability and concavity can be managed to obtain a set of nonredundant restrictions within the IQUAIDS, applying the procedure to a 3-good system.

With proper scaling, the imposition of these restrictions at the scaling point in the IQUAIDS does not imply more complexity than in the IAIDS, since all elasticity formulas for the IQUAIDS exactly duplicate those for the IAIDS at the point of interest. In principle, our method can be extended to manage other separability assumptions, such as indirect weak separability or implicit separability, and other flexible functional forms.

Furthermore, we have shown that separability and concavity can be accomodated within a semiflexible specification, although the separable structure may place a minimal requirement for the rank of the substitution matrix. The only drawback is that maintaining both separability and concavity may produce high non-linearities in the estimated model, and that its implementation may become cumbersome with many goods and a detailed structure of preferences.

Table 1: R^2 values and estimated quantity- and scale-elasticities at the mean point .

	q_{BF}	q_{PK}	q_{PO}	θ
	Unconstrained system (11 parameters)			
p_{BF}	-0.60	-0.09	-0.16	-0.85
p_{PK}	-0.24	-0.42	-0.32	-0.97
p_{PO}	-0.64	-0.47	-0.26	-1.37
R^2	0.964	0.988	-	
	Semiflexible (rank=1) and concave system (10 parameters)			
p_{BF}	-0.60	-0.09	-0.15	-0.85
p_{PK}	-0.25	-0.44	-0.28	-0.96
p_{PO}	-0.63	-0.43	-0.34	-1.40
R^2	0.964	0.987	-	
	Semiflexible (rank=1) concave and separable system (9 parameters)			
p_{BF}	-0.59	-0.13	-0.09	-0.82
p_{PK}	-0.43	-0.39	-0.34	-1.16
p_{PO}	-0.43	-0.43	-0.38	-1.23
R^2	0.964	0.986	-	
shares	0.510	0.245	0.246	

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Appendix

As a simple example, consider our 3-good system with the following separable tree:

(A.1) $$U(q) = U\left(u^a(q_1), u^b(q_2, q_3)\right)$$

where q_1 is beef. Homogeneity, symmetry and adding-up imply the following restrictions (the "numeraire" for homogeneity is q_3, which belongs to the two-good group):

(A.2) $$\begin{aligned} \gamma_{33} &= -\gamma_{13} - \gamma_{23} \\ \gamma_{13} &= -\gamma_{11} - \gamma_{12} \\ \gamma_{23} &= -\gamma_{12} - \gamma_{22} \\ \beta_1 &= -\beta_2 - \beta_3 \\ \lambda_3 &= -\lambda_1 - \lambda_{32} \\ \alpha_3 &= 1 - \alpha_1 - \alpha_2 \end{aligned}$$

while imposing full concavity on the *rank-2* matrix implies the following reparameterisation:

(A.3) $$\begin{aligned} \gamma_{11} &= -\tau_{11}^2 - \alpha_1^2 + \alpha_1 \\ \gamma_{12} &= -\tau_{11}\tau_{12} - \alpha_1\alpha_2 \\ \gamma_{22} &= -\tau_{12}^2 - \tau_{22}^2 - \alpha_2^2 + \alpha_2 \end{aligned}$$

Finally, the separability structure in (A.1) implies one additional parametric restriction:

(A.4) $$\gamma_{13} = \alpha_1\beta_3 + (\gamma_{12} - \alpha_1\beta_2)\frac{\alpha_3}{\alpha_2}$$

Given that we cannot estimate a fully concave model, we are forced to set $\tau_{22}=0$. Under this specification, we first impose the restrictions in (A.2) on the original price and income parameters, and, in a second step, we impose the Cholesky expressions on price parameters, setting $\tau_{22}=0$. Now the separability restriction in (A.4) must be rewritten to take into account that γ_{13} and α_3 are restricted by homogeneity and both γ_{11} and γ_{12} are restricted by concavity. Thus substituting the corresponding expressions and rearranging, the same restriction becomes:

(A.5) $$\tau_{12} = \frac{-\tau_{11}^2\alpha_2 + \alpha_1\alpha_2\beta_3 - \alpha_1\beta_2(1 - \alpha_1 - \alpha_2)}{\tau_{11}\left(1 - \alpha_1\right)}$$