The Multi-Product Asymptotically Ideal Model:

An Application to Agriculture

Authors: Jayson L.Lusk, Allen M. Featherstone, Thomas L. Marsh*

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Abstract: This paper examines the Multi-Product Asymptotically Ideal Production Model as an alternative to the translog and normalized quadratic functional forms using farm level data. Factors such as ease of estimation, imposition of regularity conditions, and quantitative differences in empirical estimates are compared.

*The authors are USDA National Needs Fellow & Ph.D. student, Professor, and Assistant Professor, respectively - Kansas State University, 342 Waters – Manhattan, KS 66506; email – jaylusk@agecon.ksu.edu, afeather@agecon.ksu.edu, tlmarsh@agecon.ksu.edu; phone (785) 532-4445, fax (785) 532-6925; primary contact: Jayson Lusk

Introduction

Choice of functional form in empirical studies carries important implications for economists. Often, prediction of policy alternatives are based on own-price and cross-price elasticities as well as measures of returns to scale. Empirical estimates of elasticities are known to be sensitive to the choice in functional form (see Berndt and Khaled, 1979; Chalfant, 1984; Shumway and Lim, 1993). Cost estimates are typically obtained by choosing a functional form based on theoretical notions as to the true function underlying the observed data.

To this point, options of "flexible" functional forms have been generally limited to the translog, quadratic, and generalized Leontief forms. These three functional forms can be derived from a second-order Taylor series expansion and are considered to be local approximations to the true functional form. Thus, from a theoretical standpoint, there is no apparent advantage from one form to the other insofar as estimating the true but unknown functional form. One can evaluate these forms on a basis of regularity and flexibility, i.e. the ability of a functional form to 1) satisfy restrictions imposed by economic theory and 2) represent a wide variety of functional forms. While these forms have been useful in production analysis, they are limited in their ability to satisfy global regularity and flexibility conditions. The translog, for example, is locally flexible, but is not regular. Regularity may be imposed at a particular point but only through restrictions on data and parameters. Several studies using simulated data have indicated that these forms often fail to adequately approximate some technologies (see Berndt and Khaled, 1979; Barnett and Lee, 1985; and Diewert and Wales, 1987). In addition these forms, as indicated by White (1980), are problematic in that OLS regression does not produce the coefficients of the Taylor series unless the true function is of that specified form.

An alternative functional form that has received little attention in agriculture production analysis is the Asymptotically Ideal Model (AIM). The AIM model for demand analysis was introduced by Barnett and Yue (1988) and Barnett, Geweke, and Yue (1989). Later, Barnett et al.,1991 derived the AIM production model. The AIM production model provides several advantages over the currently used “flexible” forms. According to Barnett et al. (1991), it’s properties include: 1) global flexibility and global regularity, 2) global regularity can be imposed through parameter restrictions alone and do not involve the data, 3) the seminonparametric nature of the function allows the data to determine the properties of the cost function, and 4) unlike the Fourier, a seminonparametric function introduced by Gallant (1981), overfitting of the AIM is virtually impossible. The AIM not only estimates the cost and demand functions, but also its derivatives, which is necessary when estimating elasticities. Barnett et al. (1991) proved that the AIM production satisfies the Sobolev Norm for all homogeneous of degree one functions, which eliminated the problem identified by White (1980). This indicates that the AIM model can approximate any unknown cost function arbitrarily well by increasing the order of expansion.

The purpose of this paper is to utilize the multi-product AIM production model in an effort to examine its relevance to agriculture production analysis. In particular, the AIM model will be used to examine the empirical relationship among eight inputs – seed, fertilizer, chemicals, machinery, feed, fuel, labor, and land with two outputs – livestock and crops. The objectives of this paper are to determine the expansion of the AIM model that most appropriately fits the data set of interest, calculate elasticity estimates from the AIM model, and discuss practical its relevance.

Properties of the Asymptotically Ideal Production Model

Barnett et al. (1991) derive the single output AIM model by exploiting the Muntz-Szatz expansion. They simplify the expansion by imposing linear homogeneity in input prices. Koop and Carey (1994) expanded the AIM model into a multi-product framework. In the most simple version, a constant return to scale cost function exists with a price aggregator P(p) and an output aggregator Y(y) such the cost function takes the form:

$$C(y, p) = Y(y)\,P(p), \qquad (1)$$

The above function is globally flexible and can approximate any cost function such as the translog, quadratic, or generalized Leontief. The function however, is not globally regular without parametric restrictions. One can restrict the coefficients of Y(y) and P(p) to be non-negative, thus forcing the factor demands to be downward sloping. Therefore, the cost function can be forced to be globally regular. Without these restrictions, the function can still be locally regular, and there is evidence that the AIM performs well even without the inequality constraints. These restrictions will be discussed later in detail. Non-constant returns to scale can be incorporated by allowing the cost function to take the form:

$$C(y, p) = Y(y)^{\theta}\,P(p), \qquad (2)$$

Following Koop and Carey (1994), the Muntz-Szatz expansion can be carried out on both the Y(y) and P(p) terms. The multi-product AIM can be denoted AIM(n_y, n_p), where n_y represents the order of expansion of the output aggregator and n_p represents the order of expansion of the input price aggregator. The expansion is identical for both aggregators. In general notation, let N denote the number of outputs and k the order of expansion. The set is defined as:

$$A_k = \{(i_1, i_2, \ldots, i_{2^n}) : i_1, i_2, \ldots, i_{2^n} \in i_1 \le i_2 \le i_3 \ldots \le i_{2^n}\}$$

The expansion for the output aggregator of order k is defined as:

$$Y(y) = \sum_{z \in A_k} a_z \prod_{j=1}^{2^k} y_{i_j}^{2^{-k}}$$

For the sake of discussion, consider 3 outputs. The expansion for Y(y) takes the following forms:

Y(y) for AIM(1,*)

$$Y(y) = a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_1^{1/2} y_2^{1/2} + a_5 y_1^{1/2} y_3^{1/2} + a_6 y_2^{1/2} y_3^{1/2}$$

Y(y) for AIM (2,*)

$$Y(y) = a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_1^{1/2} y_2^{1/2} + a_5 y_1^{1/2} y_3^{1/2} + a_6 y_2^{1/2} y_3^{1/2}$$
$$+ a_7 y_1^{3/4} y_2^{1/4} + a_8 y_1^{3/4} y_3^{1/4} + a_9 y_2^{3/4} y_3^{1/4} + a_{10} y_1^{1/4} y_2^{3/4} + a_{11} y_1^{1/4} y_3^{3/4}$$
$$+ a_{12} y_2^{1/4} y_3^{3/4} + a_{13} y_1^{1/2} y_2^{1/4} y_3^{1/4} + a_{14} y_1^{1/4} y_2^{1/2} y_3^{1/4} + a_{15} y_1^{1/4} y_2^{1/4} y_3^{1/2}$$

It is clear that the AIM model expands rapidly as the order of expansion increases. The expansion for the price aggregator P(p) takes the same form. Let (γ_1, γ_2,…, γ_n) represent the coefficients for the expansion of the price aggregator P(p). In the single output case, it can be easily seen that the AIM(1) is equivalent to the generalized Leontief. In the three output and three input case above the number of parameters to estimate increases dramatically. The AIM(1,1) produces as few as 12 parameters, but the AIM(3,3) produces as many as 90 parameters.

Conditional factor demands can be derived from the cost function by using Shepard's Lemma. Thus, for a given level of expansion, the derivative of the cost function with respect to an input price yields the conditional factor demand for that input. Consider for example the AIM (*,1) case with three inputs.

Conditional factor demands for AIM(*,1) are:

$$x_1 = \partial C(y,p)/\partial p_1 = Y(y)(g_1 + \tfrac{1}{2} g_4 p_1^{-1/2} p_2^{1/2} + \tfrac{1}{2} g_5 p_1^{-1/2} p_2^{1/2})$$

$$x_2 = \partial C(y,p)/\partial p_2 = Y(y)(g_2 + \tfrac{1}{2} g_4 p_1^{1/2} p_2^{-1/2} + \tfrac{1}{2} g_6 p_2^{-1/2} p_3^{1/2})$$

$$x_3 = \partial C(y,p)/\partial p_3 = Y(y)(g_3 + \tfrac{1}{2} g_5 p_1^{1/2} p_3^{-1/2} + \tfrac{1}{2} g_6 p_2^{1/2} p_3^{-1/2})$$

Once the conditional factor demands are obtained, elasticities may be derived in typical fashion. Own price elasticities for AIM(*,1) are:

$$e_{1,1} = (\partial x_1 / \partial p_1)(p_1 / x_1) = [Y(y)(-\tfrac{1}{4} g_4 p_1^{-1/2} p_2^{1/2} - \tfrac{1}{4} g_5 p_1^{-1/2} p_2^{1/2})] / x_1$$

$$e_{2,2} = (\partial x_2 / \partial p_2)(p_2 / x_2) = [Y(y)(-\tfrac{1}{4} g_4 p_1^{1/2} p_2^{-1/2} - \tfrac{1}{4} g_6 p_2^{-1/2} p_3^{1/2})] / x_2$$

$$e_{3,3} = (\partial x_3 / \partial p_3)(p_3 / x_3) = [Y(y)(-\tfrac{1}{4} g_5 p_1^{1/2} p_3^{-1/2} - \tfrac{1}{4} g_6 p_2^{1/2} p_3^{-1/2})] / x_3$$

In addition, the cross price elasticity of input 1 with respect to input 2 AIM(*,1) is estimated by:

$$e_{1,2} = (\partial x_1 / \partial p_2)(p_2 / x_1) = [Y(y)(\tfrac{1}{4} g_5 p_1^{1/2} p_2^{1/2})] / x_1$$

Conditional factor demands and elasticities for higher order expansions of the AIM model as well as alternative numbers of inputs are merely extensions of the above example (see Barnett et al. (1991) for additional expansions).

While there are many potential advantages to the AIM model, there are some drawbacks. The major disadvantages are discussed below.

1) One disadvantage is that the multi-product AIM model only asymptotically approximates the unknown cost function if the function is of the form given in (1). The function used in (1) also assumes expansion for the outputs is separate from the expansion for the inputs. Such an assumption may provide undesirable results in certain circumstances. For example, if the true, unknown cost function is not of the form given by (1), improper estimates may be obtained.

2) Another issue, which remains unresolved, is the use of parameter restrictions to impose regularity on the AIM model. Restricting (γ_1, γ_2,… , γ_n) and (α_1, α_2,… ,α_n,) to be positive in the previous models insures the AIM to quasi-concave and non-decreasing in inputs and outputs. Barnett et al. (1991) indicated that this restriction may only be used when all factors are substitutes. Using simulated data, Terrell (1995) concluded that the unconstrained AIM can approximate both

translog and CES technologies as well as providing significant advantages over the two functional forms. Terrell (1995) continued by indicating that the constrained AIM was not capable of approximating some technologies. As an alternative, however, it was noted that the unconstrained AIM rarely violated concavity. Barnett et al. (1991) indicate, “results suggest that imposition of regularity may be less necessary in general than with other models, since we achieve regularity within the region of the data with out the need to impose any restrictions at every order of approximation that we estimate.” Thus, one is left with a decision to determine what circumstances call for parameter restrictions. It should be noted that the parameter restrictions are a sufficient and not a necessary condition for global regularity.

3) An additional difficulty with the AIM model is the choice of expansion order. There is no definite rule indicating the level of expansion necessary to conduct accurate analysis. Koop and Carey (1994) using hospital data found that long expansion of the price aggregator did not improve model performance. Using their data they determined that the AIM (3,1) was appropriate in the unrestricted model and AIM (2,1) or AIM (1,1) was appropriate for the restricted model. Jensen (1997) concluded that the flexibility of the AIM did not always increase with the order of expansion. He cautions against the use of higher order AIM models (i.e. AIM (3) or higher) when using noisy data to make predictions.

Despite the drawbacks of the AIM model, there are many properties, which are rather intriguing such as global flexibility and the ability to impose regularity. The AIM model does provide several advantages over current functional forms thus qualifying the AIM model as a potentiality advantageous alternative for use in production analysis. The goal of this paper is to examine the applicability and validity of the AIM model in larger agricultural production models.

Data and Procedures

The data for the analysis was obtained from the Kansas Farm Management Data Base. The data contained price and quantity information for several inputs and outputs from 144 Kansas farms over a 24 year time period from 1973 to 1996 resulting in 3456 observations. Two outputs, livestock and crops, along with eight inputs, seed, fertilizer, chemicals, machinery, feed, fuel, labor, and land, from the data set was used in the analysis.

The AIM models were estimated using full information maximum likelihood regression procedures in SAS. Coop and Carey (1994), in the only paper to date utilizing the multi-product AIM production function, estimated only the cost function – no factor demand equations were estimated. Since elasticity results are the goal of this paper, it is vital to estimate factor demand equations to obtain reliable elasticity estimates since elastities are calculated from derivatives of the input demand functions. In the case of the single output AIM model, Barnett et al. (1991) estimated all n input/output equations (factor demands divided by output), and no cost function. However, in a multiple output case, it is infeasible to normalize the factor demands by the inputs. Therefore, the cost function as well as all eight factor demands were estimated here. The addition of the cost function to the factor demands provides efficiency to the model without singularity.

As indicated previously, choice of the level of expansion for the AIM model can be somewhat difficult. Due to the results of Koop and Carey (1994) and Jensen (1997), the expansion of the model will be limited to the 1st order expansion for the inputs and 3rd order expansion for the outputs. In this particular case, the 2nd order expansion of the eight inputs results in 335 coefficients – which adds computational complexity to the model, thus for present purposes, it will be excluded from this paper. Six combinations of the expansion are estimated here. The AIM(1,1), AIM(2,1), and AIM(3,1) are estimated twice, once unrestricted, and once with all coefficients restricted to be greater than or equal to zero to impose regularity conditions. Following Koop and Carey (1994), log liklihood ratio tests will be used to identify the appropriate model. Since the AIM(1,1) and the AIM(2,1) models are a restricted version of the AIM(3,1), the likelihood ratio test can be performed to selected the appropriate model.

The hypothesis is tested that the coefficients on the additional expansion terms are zero. The following test was used where j is the number of restrictions (degrees of freedom):

$$l = -2\left[\ln L_{restricted} - \ln L_{unrestricted}\right] \sim c_j^{2}$$

The normalized quadratic functional form is also used to estimate the system as a basis for comparison.

Results

The AIM(1,1) contained 39 parameters, the AIM(2,1) contained 41 parameters, and the AIM(3,1) contained 44 parameters. The log-likelihood values are shown in Table 1. The log-likelihood ratio test indicated that the AIM(2,1) was the appropriate model for both the restricted and unrestricted versions of the AIM. The likelihood function for the normalized quadratic is also listed in Table 1. The log-likelihood value for the normalized quadratic cost function was –162466, which was somewhat similar to the AIM models. Since the normalized quadratic is not a restricted version of the AIM, the log-likelihood ratio test may not be used to indicate which model more appropriately fits the data. Out of sample prediction may be useful in determining the preferred functional form. Potentially, a J or P test may also be used to measure model performance.

Compensated elasticity estimates calculated at the means for the AIM(2,1) unrestricted and unrestricted models are presented in Tables 2 and 3 respectively. In the unrestricted version of the AIM, two of the inputs are not consistent with rational economic behavior, seed and labor have positive own price elasticities. Several of the inputs including fertilizer and seed, seed and chemicals, and machinery and labor have positive signs indicating the goods are compliments.

Conversely, inputs such as seed and feed, land and labor, and land and machinery are substitutes. When the estimates are restricted to be greater than zero, all own price elasticities are negative as indicated in Table 3. All inputs are inelastic with feed being the most elastic and land being the most inelastic. All inputs are either independent of one another, or they are substitutes. For example, seed and labor are compliments, while a change in the price of fertilizer does not change the quantity demanded of seed. The restriction forces negativity of input demands, however many of the cross-price effects are zero, which is produces an undesirable result. Additional work may lead to more appropriate measures of imposing curvature. Once again, out of sample forecasting may further indicate that this restriction is inappropriate.

For comparison, elasticity results for the normalized quadratic are shown in Table 4. Homogeneity and symmetry were imposed on the normalized quadratic, but curvature was not imposed. As in the unrestricted AIM model, the own price elasticity for seed was positive. The own price elasticities in the normalized quadratic are more similar to the restricted version of the AIM, at least in magnitude. When comparing cross-price effects between the two models, some consistencies were found between the unrestricted AIM and the normalized quadratic. For example, seed and labor and seed and land were found to be compliments in both models. In addition, it can be seen that homogeneity holds by noting that the sum of each row in the elasticity table sums to zero.

There were advantages and disadvantages to both the AIM and the normalized quadratic. The normalize quadratic was computationally easier to implement than the AIM, however, regularity conditions were more readily adapted into the AIM. In theory, the AIM should more closely approximate the true functional form, if in fact the true cost function is in the form of (1). The question still persists as to which functional from is preferred. The efficiency of out of sample forecasts may shed light on the subject and may give a clear indicator of which functional form is a better predictor. In general, elasticity results are not robust to changes in functional form and imposition of theoretical restrictions.

Conclusions

The Asymptotically Ideal Model has been introduced as an alternative functional form into production literature. The AIM model provides several advantages over currently used flexible functional forms – such as the ability to be globally flexible and regular. Despite the fact that the AIM model has the potential has the ability to be globally regular and globally flexible, it has received little attention in agriculture production literature.

The AIM model was used to estimate a cost function consisting of two inputs – crops and livestock, as well as eight factor demand equations for seed, fertilizer, chemicals, feed, fuel, labor, land, and machinery. Log likelihood tests indicated that the appropriate expansion was the AIM(2,1) model. Elasticity results for the restricted and unrestricted AIM and the normalized quadratic differed substantially. A consistent, yet undesirable finding was that the own price elasticity for seed was found to be positive in both the unrestricted AIM and the normalized quadratic models. In addition, when the AIM model was restricted, all own-price elasticities were negative, however, all cross-price elastcities were either substitutes or zero, which is an unappealing result of the restricted AIM. Restricting the AIM in this manner may be too restrictive. A next step in the research is to examine which of the three models performs best. To determine which of the three models would be appropriate to use, out-of-sample forecasting or non-nested tests may be used.

Table 1 – Log Likelihood Values for the AIM Model

Expansion	Unrestricted	Restricted
AIM(1,1)	-201970	-202470
AIM(2,1)	-201955	-202462
AIM(3,1)	-201953	-202469
Quadratic	-162466	

Table 2 – Elasticity Estimates for the Unrestricted AIM(2,1) Model

	Seed	Fertilizer	Chemical	Feed	Fuel	Labor	Land	Machine
Seed	0.531	-0.364	-0.913	1.083	-0.056	-1.042	-1.276	2.037
Fertilizer	-0.193	-0.481	0.095	1.162	-0.132	-0.246	0.028	-0.233
Chemical	-0.999	0.195	-1.124	0.525	-0.142	0.456	-0.151	1.189
Feed	0.281	0.568	0.125	-2.920	0.343	0.0490	0.382	1.190
Fuel	-0.035	-0.159	-0.083	0.844	-0.176	-0.870	-0.207	0.690
Labor	-0.859	-0.380	0.343	0.155	-1.119	0.719	2.746	-1.554
Land	-0.240	0.010	-0.026	0.276	-0.061	0.626	-0.643	0.058
Machine	0.378	-0.081	0.202	0.850	0.200	-0.350	0.057	-1.255

Table 3 – Elasticity Estimates for the Restricted AIM(2,1) Model

	Seed	Fertilizer	Chemical	Feed	Fuel	Labor	Land	Machine
Seed	-0.296	0	0	0	0.046	0.049	0	0.200
Fertilizer	0	-0.158	0	0.158	0	0	0	0
Chemical	0	0	-0.492	0	0	0.395	0	0.097
Feed	0	0.077	0	-0.609	0.049	0	0	0.484
Fuel	0.0296	0	0	0.119	-0.153	0	0	0
Labor	0.040	0	0.297	0	0	-0.521	0.181	0
Land	0	0	0	0	0	0.041	-0.041	0
Machine	0.037	0	0.016	0.345	0	0	0	-0.399

Table 4 – Elasticity Estimates for the Normalized Quadratic

	Seed	Fertilizer	Chemical	Feed	Fuel	Labor	Land	Machine
Seed	0.568	-0.121	0.035	-0.130	0.164	-0.747	-0.390	0.621
Fertilizer	-0.062	-0.107	0.027	0.361	-0.085	-0.343	0.074	0.136
Chemical	0.038	0.057	-0.327	-0.251	0.008	0.351	0.294	-0.170
Feed	-0.030	0.165	-0.055	-0.120	0.076	0.014	-0.560	0.510
Fuel	0.109	-0.109	0.005	0.214	-0.174	-0.827	0.323	0.459
Labor	-0.641	-0.569	0.278	0.052	-1.066	-0.154	2.825	-0.726
Land	-0.075	0.028	0.052	-0.456	0.093	0.636	-0.413	0.136
Machine	0.115	0.049	-0.029	0.398	0.128	-0.157	0.130	-0.634

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