The Confidence Limits of a Geometric Brownian Motion

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Abstract

This paper investigates whether the assumption of Brownian motion often used to describe commodity price movements is satisfied. Using historical data from 17 commodity futures contracts specific tests of fractional and ordinary Brownian motion are conducted. The analyses are conducted under the null hypothesis of ordinary Brownian motion against the alternative of persistent or ergodic fractional Brownian motion. Tests for fractional Brownian motion are based on a variance ratio test. However, standard errors based on Monte Carlo simulations are quite high, meaning that the acceptance region for the null hypothesis is large. The results indicate that for the most part, the null hypothesis of ordinary Brownian motion cannot be rejected for 14 of 17 series. The three series that did not satisfy the tests were rejected because they violated the stationarity property of the random walk hypothesis.

JEL Classification G0;

Key Words : Random Walk, Fractional Brownian Motion, Futures Prices

1.0 Introduction

This paper investigates the existence of a geometric Brownian motion (gBm) in 17 agricultural commodity price time series by using 90%, 95% and 99% confidence intervals about the (so-called) Hurst coefficient H from 20,000 replications of a Monte Carlo simulation. The overall objective is to determine if commodity futures prices satisfy the geometric Brownian motion assumptions of linear diffusions and variance. Using a variety of techniques, recent investigations are mixed on the existence of random walks in financial and commodity price series (see Comte and Renault (1996), Hommes (2001), Greene and Fielitz (1977), Booth, Kaen, and Koveos (1981, 1982a, b) Peters (1996). Helms, Kaen and Rosenman (1984), Barkoulas and Baum (1996), Barkoulas, Labys and Onochie (1997), Corazza, Malliaris, and Nardelli, (1997), Peters (1996), Cromwell, Labys and Kouassi (2000), Gao and Wang (1999)). However, recent research using Lo's (1991) modification for correlated bias fails to reject the null hypothesis of no fractal structure in futures prices (Crato and Ray 2000). In much of the finance literature the Hurst coefficient is obtained from the R-S procedure as described in an economic context by Mandelbrot (1972), Mandelbrot and Wallis (1969), Mandelbrot and Van Ness (1968), Schroeder (1991), Peters (1996), Lo (1991), and Helms, Kaen and Rosenman (1984) among others. Another common approach in the literature is to use one of several autoregressive models to test for the H coefficient of stochastic volatility including ARCH, GARCH and more recently Fractionally Integrated GARCH (Jin and Frechette, 2004; Wei and Leuthold 1998 and citations therein). In this paper we provide the means to estimate H directly from a Brownian motion in a manner similar to that discussed in Lo and Mckinnon. For comparison purposes we provide measures of H from conventional R-S analysis but do not use them in the analysis. However, the principles involved are very much consistent with R-S. For example if the variance ratio test indicates $H = .5$, then agreement would confirm an ordinary Brownian motion. If $H \neq .5$, then the random walk follows a fractional Brownian motion.

This paper is only concerned with the random walk described by a gBm. The data are examined by a test for fractal structure (a fractional Brownian motion, fBm) against the null of no fractal structure (a geometric Brownian motion, gBm), with the latter based on the Hurst coefficient, H, which is estimated directly from variance-ratios across multiple time steps.

The paper is motivated by the observation that a geometric Brownian motion in futures prices (or other financial assets) has, for the most part, been treated as an assumption rather than a hypothesis. By this it is meant that the tests are based on in-sample properties and are generally devoid of a null. By null, it is meant that tests for random walks look only within the sampling domain, and ignore some very general properties of a random walk. For example in many studies, researchers might find a value of $H \neq .5$ and conclude that the time series is persistent, ergodic or mean reverting, or in other words has some longer term memory. But these studies fail to consider the natural distribution of a random walk, and more important fail to realize that the value H in a sample can differ from 0.5 without violating a gBm. In other words, in the absence of a true null, i.e. the observed behavior and distribution of H from a known gBm, it is impossible to determine whether a value $\hat{H}$ estimated from a sample is consistent with a gBm. This assumption has not only led to closed form solutions for pricing traded and non-traded derivatives (e.g. Black and Scholes, 1973 , Black, 1976 and Merton 1973, Boyle and Wang 1999, Cox, Ingersoll and Ross, 1985, Garman 1977, or Rubinstein 1979), but has also provided a simple mechanism for generating derivative prices using Monte Carlo methods (e,g, Boyle, Broadie and Glasserman , 1997). So critical is the Brownian motion assumption that treating it as a null hypothesis, and rejecting the null, has wide spread theoretical and practical consequences for 1) overall market efficiency which is diminished with a fBm (Rogers 1997), 2) the pricing of derivatives on agricultural futures (Cutland, Kopp and Willinger (1995) and Sottinem (2001)), and c) the effectiveness of hedging agricultural commodities with futures contracts.

To overcome this problem, we provide upper and lower confidence limits at the 90%, 95% and 99% levels for Hurst coefficient. We obtain these limits using Monte Carlo simulations of a known gBm. We show that the distribution of H is related by a power law to the sample size (N) and time-step (k). Consequently we provide, in Appendix A upper and lower confidence limits for a range of N and k. Elsewhere in the paper we provide the formulas from which the confidence intervals were obtained.

We propose with this approach a slight rethink of how random walks are evaluated. Our approach relies on the confidence intervals, which is to say that if an estimated value $\hat{H}$ falls within the upper and lower confidence limits the best we can say is that in any sequence of a pure random walk of length N and time-step k, 90% or 95% or 99% of the time a ‘true’ value will fall within the limits. We cannot say that the sample ‘is’ a gBm but rather that it is ‘consistent’ with what would be found in a gBm 90% or 95% or 99% of the time. The key finding is that the standard error of the acceptance region of the respective confidence interval for H=0.5 is quite large, increasing with the time step k, and decreasing with the sample size N. In the absence of null against which to measure a gBm many of the conclusions reported in the literature that the times series has memory may not be correct. As a case in point Jin and Frechette (2004) find H values between .50 and .60 and conclude strong persistency. Over this range, the evidence of the current paper shows that the null hypothesis of H different than .5 cannot be so easily rejected.

The paper proceeds as follows. The next section introduces the concepts of Brownian and fractional Brownian motion and variance ratios. Then, statistical models are developed and applied to 950 daily observations of futures prices for 17 commodities traded on the Chicago Mercantile Exchange, Chicago Board of Trade, and Winnipeg Commodities Exchange. Finally, the results are discussed and the paper is concluded.

2.0 Variance ratios and Fractional Brownian Motion

In the classical models of random walk, it is assumed that the percentage change in the futures price over a discrete interval of time is governed by

(1) $dX = \alpha X dt + X\sigma_X dZ$

where $dZ = \varepsilon\sqrt{t}$ is a Gauss - Wiener process, X is the futures price, α is the instantaneous change in futures prices and σ is the variance of the percentage change in futures prices. In contrast, a fractional Brownian motion is specified by

(2) $dX = \alpha X dt + X\sigma_X dW$

where $dW = \varepsilon\sqrt{t^{2H}}$. In (1) and (2) the term ε can be interpreted as a random shock over the prescribed time interval. However, the respective Wiener processes possess markedly different properties. The Wiener process dZ is self-similar in time, whereas dW is self-affine. While conceptually similar, self-similarity and self-affinity differ in the following way (see Mandelbrot, 1977 or Feder, 1988): Suppose that an initial sequence or set $\{X_1, X_2, X_3\}$ can be transformed to the set $\{r_1X_1, r_2X_2, r_3X_3\}$, then the transformation is said to be self similar if $r_1=r_2=r_3$ and self-affine otherwise. If the variance obeys the power law $VAR = \sigma^2 t^{2H}$, it is self-affine over the entire range of H, but is self-similar only for H=.5. Therefore, and generally speaking, self-similarity is a special case of self-affinity[1,2].

In (2), the Wiener process is described in terms of a power law. The parameter H reflects the fractal dimension of the stochastic process and can take on any value between 0 and 1. H is analogous to the Hurst (1951) coefficient in standard R-S analysis. A pure random walk has H = .5 and a biased random walk has H ≠ .5. For H > .5 the system is said to be persistent and is characterized by a long-term memory. In general, an event at some point t is positively correlated with observed events at some future period, t + Δt. In contrast, a short-memory process occurs when H < .5 . The system is anti-persistent, or ergodic, and reverses itself frequently. Because of these reversals, it is characterized by negative correlation. That is, an event at some moment in time t (say an increase in futures price) will cause a reversal at some point in the future at t + Δt.

We are concerned with the properties of $dX = X(t_2) - X(t_1)$ with expected value of zero and variance $\sigma^2(t_2 - t_1)^{2H}$. A fractal Brownian motion has a Gaussian distribution of the form

$$(3)\qquad \Pr(dX < x) = \frac{1}{\sqrt{2\pi}\sigma(t_2 - t_1)^H}\int_{-\infty}^{x}\exp(-1/2(\frac{\mu}{\sigma(t_2 - t_1)^H})^2)d\mu$$

If H = .5 then a fractional Brownian motion is the same as standard Brownian motion as used in equation (1). Likewise the variance of a fractal Brownian motion,

$$(4)\qquad E[X(t_2) - X(t_1)]^2 = \sigma^2(t_2 - t_1)^{2H} \quad ,$$

reduces to that of standard Brownian motion when H = .5,

$$(5)\qquad E[X(t_2) - X(t_1)]^2 = \sigma^2(t_2 - t_1).$$

The critical difference between (4) and (5) is that the variance property for a standard Brownian motion increases linearly in time, whereas the variance in fractal Brownian motion is increasing in H at an increasing rate. In part, the difference between variance measured by ordinary and fractional Brownian motion is due to correlation and covariance between time increments. This covariance is given by (see Crownover (1995) or Igloi and Terdik (1999));

$$(6)\qquad E\{[X(t) - X(0)]\,[X(t + \Delta t) - X(t)]\} = .5\,\sigma^2\,[(t + \Delta t)^{2H} - t^{2H} - \Delta t^{2H}]$$

By setting H = .5 the right hand side of (6) collapses to zero and the independent increments assumption is satisfied. For any H ≠ .5, it is not satisfied. As H approaches zero the limit of covariance approaches $-.5\sigma^2$, and variance falls. As H approaches +1, covariance approaches $\sigma^2(t\ \Delta t) > 0$ and variance increases. For H < .5 the covariance term decreases with increasing time steps. Hence the term 'short memory'. In contrast, the term 'long memory' comes from the results that covariance increases with increased time steps when $H > .5$.

3.0 Determining Fractals and Stationary Increments in a Time Series

Lo and Mackinnon (1999) used the variance property of Brownian motion to test for random walks. The essence of their argument is that the variance of any step or lag k ($1 < k \leq T$) must be a linear multiple of the variance of a single step or lag. For example, the variance of price changes over a 2 day period will be twice the variance of the change in 1 day or the variance over 20 days will be twenty times the variance of 1 day. Hence

(7) $\sigma^2_k = k\sigma^2_1$

and

(8) $\text{Variance Ratio} = \sigma^2_k / k\sigma^2_1 = 1$

The result suggests a specific test for a random walk. First, calculate the percentage change in prices for each of $\ln(X_{t+1}) - \ln(X_t)$, allowing for overlapping prices. Second calculate the variance, $VAR(\ln(X_{t+k}) - \ln(X_t))$, for each k step including $k = 1$. Third, divide the calculated variance for each $k \geq 1$ by the variance for $k = 1$. This gives k ratios of the form σ^2_k/σ^2_1.

The results allow for a number of tests. The Lo and Mackinnon (1999) approach is to treat each of the k ratios as a separate hypotheses. That is

(9) $$Ho: \frac{\sigma_k^2}{\sigma_1^2} - k = 0 \qquad \forall k .$$

Lo and Mackinnon (1999) provide a formula for calculating the asymptotic variance of the ratio and provide a standardized test for the null hypotheses. In the alternative, an equivalent test would be to regress

(10) $$\sigma_k^2 / \sigma_1^2 = \hat{a} + \hat{b}k + \varepsilon$$

and set $Ho: \hat{b} = 1$. Failure to reject H_0 would indicate that variance increases linearly in time as required by the random walk hypothesis.

In the context of fractional Brownian motion the above model may not be specific enough since its variance is given by $\sigma^2 T^{2H}$. To test for a biased random walk follow the steps described above for calculating variance ratios. To test for fractional Brownian motion we need an estimate of the H coefficient. The following regression can be used to estimate the value for H.

(11) $$\ln(\sigma_k^2 / \sigma_1^2) = \alpha_0 + \alpha_1 \ln(k) + \varepsilon$$

with Ho: $\alpha_0 = 0$ and Ho: $\alpha_1 = 1$. In (11) the value of H can be calculated from $\alpha_1 = 2H$ or $H=\alpha_1 /2$. If $\alpha_1 =1$ then H = .5 and there is no evidence of fractal structure. If $\alpha_1 >1$ then H > .5 and this would indicate long-term memory and positive autocorrelation. If $\alpha_1<1$ then H < .5, memory is short and the system is ergodic or mean reverting.

Even though fractional Brownian motion does not satisfy the property of independent increments, it still must satisfy the Gaussian assumption of stationary increments. In general, stationary increments imply that the first difference of the returns series are independent for any choice of t. That is, the differenced series

$$\{x_2 - x_1, x_3 - x_2, x_4 - x_3 \ldots x_t - x_{t-1}\}$$

are independent. Furthermore the process is stationary across any time step. This means that for a time step k (where k can equal days or weeks, etc.)

(12) $$E[x_k - x_1] = E(x_k - x_{k-1} + x_{k-1} - x_{k-2} + \ldots + x_2 - x_1) = k\mu$$

where $\mu = E[x_t - x_{t-1}]$ across all t. This definition of stationarity states that the k-step difference between any two observations is a linear function of the mean 1-step difference. Hence

(13) $$E\left[\frac{x_{t+k} - x_t}{x_{t+1} - x_t}\right] = k \,.$$

This leads to a simple test for stationarity by estimating the following regression,

(14) $$\left[\frac{x_{t+k} - x_t}{x_{t+1} - x_t}\right] = \beta_0 + \beta_1 k + e \,.$$

Under the null hypotheses H_0: $\beta_1 = 1$, the linearity assumption, and hence a finding of a stationary process, will be rejected if Ho is rejected. The alternative hypothesis H_A: $\beta_1 \neq 1$ implies that the increments are non-stationary.

We can use the stationarity and independence assumptions to test for fractional versus ordinary Brownian motion in the following way. Under the null hypothesis of a stationary time series a specification test that rejects the null automatically eliminates the time series as being either ordinary or fractional Brownian motion, let alone a random walk. Failure to reject the null hypothesis is sufficient to conclude a random walk, but is not sufficient on its own to declare a Brownian motion. In order for the time series to be declared a Brownian motion a stationary time series must also satisfy the null hypothesis Ho: H=.5. Failure to reject the null implies an ordinary Brownian motion. Rejecting the null implies $H \neq .5$ and a persistent or antipersistent fractional Brownian motion would be concluded for H>.5 and H<.5 respectively. In other words while a test of stationarity is not sufficient to conclude a random walk, rejection of the null Ho: H=.5 is sufficient to reject the stationarity hypothesis.

5.0 Methods

Whether or not one rejects or fails to reject the null depends on the sampling properties of the underlying distribution. However, the sampling properties of the underlying distribution depend also on the size of the sample and the periodicity of the steps being considered. That is, does a representation of an AR(1) process measured by the standard unit root tests meet the condition of the null or should some other AR(k) process be used. This is a rather critical step. Although the unit root test might be a strong indicator of whether a process follows a random walk, the true measure of a random walk is that it must hold for all k (i.e. steps). Furthermore, one cannot ignore that any measure of $\hat{H}$ estimated from a sample represents the mean of a sampling distribution, and in the absence of knowledge about the 'true' distribution of $\hat{H}$, its standard error is also measured by the sample. In reality the measure $\hat{H}$ can, by chance alone, be less than or greater than 0.5 and in the absence of a null, it is difficult to determine whether the estimated value represents the true value.

There is, of course, no 'true' value. The best that can be provided by statistical methods is a representation of the probability of the distribution about H for some sample size N when it is known, for sure, that H was obtained from a Brownian motion of sample N. That is by comparing the sample $\hat{H}$ to the probability limits or confidence intervals for values of H at the 95% or 99% that were obtained from a known Brownian motion, one can then test the null.

The confidence intervals about H were obtained using Monte Carlo simulations of a known Brownian motion. Brownian paths of size N=2,150 were generated with zero drift and with volatilities 0.10, 0.15….0.60. Each of the sample paths were simulated 20,000 times. Overlapping samples of sizes N=200,400 … 2,000 were obtained for steps k=1,2,3…150. For each combination of k and N a value $H_{N,k}$ and $\sigma^H_{N,k}$ were calculated. For example $\sigma^H_{1000,50}$ represents the population or true standard deviation of $H_{1000,50}$ for a sample size N and steps k=50. For a unit root equivalent test $\sigma^H_{1000,1}$ is the k=1 step standard deviation for $H_{1000,1}$. For a pure geometric Brownian motion then $E\left[\frac{\sigma^H_{1000,50}}{\sigma^H_{1000,1}}\right] = 50$.

Use of Monte Carlo techniques, at least within the context of this paper, have not been widely used in academic research. Similar ideas however have been used in several applications. For example, Fama and French (1988) use a similar approach to estimate the standard errors of first order autocorrelation coefficients. Qualitatively they are able to support the conjecture that stock price movements have stationary and random components. However, when their specific tests were assessed using standard errors from Monte Carlo simulations they found that the null hypothesis (of non-stationarity in prices) was difficult to reject. In fact they speculate that the large standard errors in a pure random walk may make such hypotheses altogether untestable (Fama and French, 1988, page 257). A wide acceptance region for unit roots in time series data has also been discussed by Kwiatkowski et al (1992) and critical values for fractional cointegration generated from Monte Carlo methods are described in Sephton (2002). Panas (2001) uses a bootstrapping method to estimate standard errors for stocks traded on the Athens Stock Exchange and is able to reject the null for 11 of 13 stocks. In an application to self-similar properties in ethernet traffic, Leland, Taqqu, Willinger and Wilson (1994) apply numerical techniques to obtain confidence intervals. However, in their model the confidence intervals were constructed around the estimate of H, whereas in the current study the confidence intervals were constructed around a fixed point of H=.5. In some disciplines of the social sciences and humanities a surrogate approach has been used. The surrogate approach repeatedly randomizes observations from a particular sample to remove all correlations. The H values are then calculated for each surrogate, and their sample standard deviations used in the statistical tests. (see West and Griffin (1998) and West, Hamilton and West (1999); See Rangarajan and Ding (2000) for a critique).

Using Monte Carlo techniques to determine the standard errors about H, it was found in the current study that the confidence intervals followed a power law that decreased as the sample size increased and increased as the step-length k increased. The exact form of this power law for 90%, 95% and 99% confidence intervals about the null H0:H=.5 was found to be

(15)
$$P_{90\%}^{U} = e^{\{-0.20681-0.07906 ln(N)+0.051308 ln(k)\}}$$
$$P_{90\%}^{L} = 1 - P_{90\%}^{U}$$

(16)
$$P_{95\%}^{U} = e^{\{-0.13087-0.09109 ln(N)+0.059433 ln(k)\}}$$
$$P_{95\%}^{L} = 1 - P_{95\%}^{U}$$

(17)
$$P_{99\%}^{U} = e^{\{0.005838-0.11251 ln(N)+0.074106 ln(k)\}}$$
$$P_{99\%}^{L} = 1 - P_{99\%}^{U}$$

for the upper and lower confidence limits respectively. Tables of approximate intervals are provided in the appendix. For example suppose an analyst was checking a time series of 1,000 daily observations for a memory of 30 days, then N=1,000 and k=30. Plugging these into the equation gives $P_{95\%}^{U} = 0.572$ and $P_{95\%}^{L} = 0.427$. Therefore if the analyst computes a value of H between 0.572 and 0.427 there can be 95% confidence that the time series follows a geometric Brownian motion.

There are several key observations arising from the Monte Carlo simulations.

1. The confidence limits are approximations. Regressions started for k=10 through 150 so approximations for lower step values are not as stable as those provided.
2. As the number of steps increase the confidence intervals widen. This simply reflects the power of the test. For an analysis based on k=10 for example there are much fewer degrees of freedom to influence the overall path and variance within the path. Compare this with k=150 which has many more opportunities for the random walk to wander off. This equals not only the randomness that would be observed for k=10, but also for k=25, 50 and so on.
3. As the sample size increases the upper and lower limits converge closely to 0.5. In other words, the range of H values for which the null would not be rejected falls as N increases.

4. Combined, observations 2) and 3) show that one cannot take any sampled value of H at its recorded value. Nor is it meaningful in the absence of the null to compare values of H across studies without also identifying sample size and step.
5. While the null is the same for any N or k, the upper and lower bounds for acceptance are related via a power law. For example suppose that two researchers using different data sets test for a geometric Brownian motion. The first researcher used a sample size of 500 and k=75 while the second had a sample of 2,000 and used k=100. From the appendix table for 95% confidence interval the acceptance range for H is between 0.3429 and 0.6438 for the first researcher and 0.4150 and 0.5849 for the second. Thus if both obtained estimates of $\hat{h} = 0.40$ the first would fail to reject the null that $H = 0.50$ while the second would reject the null. On the other hand, if both found $\hat{H} = 0.65$ then both would reject the null and if $\hat{H} = 0.45$ neither would reject the null.
6. The confidence limits of a Brownian motion are determined independently of the drift and volatility of the underlying stochastic process. In other words if two researchers used identical N and k on different data series, or even sub samples of the same data series that differed in drift and/or volatility both would use the same confidence intervals, and cannot use differences in either drift or volatility to explain differences in their estimates of $\hat{H}_{N,k}$.

6.0 Data

Seventeen futures contracts for agricultural commodities were examined for Brownian motion. Summarized in Table 1, the data represent 950 matched daily observations from 1996 through February 7, 2001 on the nearby futures price. The futures contracts include grains and oilseeds, livestock and livestock products, and cocoa, coffee, orange juice and sugar. The contracts are traded on the Chicago Mercantile Exchange (CME), the Chicago Board of Trade (CBOT), the Coffee, Sugar and Cocoa Exchange (CSCX) and the Winnipeg Commodity Exchange (WCE). Alberta barley, rapeseed, Winnipeg oats and Winnipeg wheat are denominated in Canadian dollars while others are in $U.S.

The sample means and range are given in Table 1. In the last two columns the annualized geometric growth rate and volatility based on a 250-day trading year are presented. The results show that 13 of 17 commodities faced price declines over this period with the largest declines being on CBOT and WCE oats at -19.7% and -22.7% respectively. Feeder cattle (CME) showed the largest annual gain of approximately 10.2%/year.

On average, volatility exceeded 30% per year. The most volatile commodity was pork bellies (CME) at 55.2% followed by coffee (CSCX) at 52.8%, lean hogs (CME) at 42.6% and wheat (CBOT) at 40.1%. Sugar (CSCX) was the least volatile at only 8.5% and Alberta barley (WCE) had the second lowest volatility at 19.4%.

Table 1: Sample Statistics for Futures Price Series

contract	Exchange	Mean	Variance	Standard Dev.	Maximum	Minimum	Geometric Mean	volatility
Alberta Barley price	WCE	137.60	476.14	21.82	196.80	108.50	-0.108	0.194
coffee price	CSCX	129.15	996.54	31.56	261.00	81.35	-0.028	0.528
cocoa price	CSCX	1337.50	62310.14	249.62	1762.00	763.00	-0.116	0.274
corn price	CBOT	272.33	5928.11	76.99	548.00	178.50	-0.146	0.373
Feeder Cattle price	CME	71.50	65.10	8.07	86.88	47.65	0.102	0.208
Fluid Milk price	CME	13.51	4.75	2.18	21.70	9.47	-0.074	0.342
Lean Hogs price	CME	60.25	221.06	14.87	90.12	25.22	-0.050	0.426
live cattle price	CME	65.61	10.49	3.24	73.63	54.80	0.023	0.211
oats price	CBOT	148.85	1755.47	41.89	286.00	99.00	-0.197	0.377
orange juice price	CSCX	97.57	292.75	17.11	138.00	66.80	-0.126	0.395

Pork Bellies price	CME	63.98	249.857	15.80	104.475	32.75	0.069	0.552
Rapeseed canola price	WCE	376.71	3563.54	59.69	490.20	251.30	-0.137	0.213
Soybeans price	CBOT	637.02	15992.26	126.46	894.25	410.00	-0.095	0.277
Sugar price	CSCX	21.78	1.66	1.29	23.09	16.55	-0.070	0.085
wheat price	CBOT	345.37	9079.49	95.28	716.50	224.00	-0.156	0.401
Winnipeg oats price	WCE	121.87	1906.97	43.66	243.00	83.00	-0.227	0.297
Winnipeg Wheat price	WCE	165.08	998.46	31.59	293.40	121.70	-0.128	0.235
Average							-0.067	0.297

7.0 Results

The parametric tests of stationarity and H are based on a lag structure with k = 150 days (following Peter's (1996) suggestion). Hence, all estimation was done for k = 150 days.

There are very few studies that have taken an interest in measures of variance about H, but those that do find similar dispersion. Bassingwaighte and Raymond find for a series of 512 points a 95% confidence interval about H from .2 to .9 a dispersion (which is wider than those found in the current study)that was confirmed in a later study by Cannon et al, which also showed that the standard error about a point estimate of H was sensitive to sample size. [3]

The estimates of $\hat{H}$ are presented in Table 3. In Table 3, column 2 provides the estimate of H used in the hypothesis test and columns 3 through 5 show the four 180-day sub periods. Column 6 provides an estimate of H using R-S analysis as a point of comparison. Since no value of $\hat{H}$ falls outside of the asymptotic 95% confidence limit there is no instance where the estimated value of H is statistically different from .5. The 90%, 95% and 99% confidence intervals for these estimates are presented in Table 2

Table 2: Confidence Intervals

	90%		**95%**		**99%**	
Sample Size (N)	**Upper**	**Lower**	**Upper**	**Lower**	**Upper**	**Lower**
940	0.6121	0.3880	0.6334	0.3666	0.6749	0.3250
760	0.6224	0.3776	0.6458	0.3542	0.6913	0.3087
580	0.6359	0.3641	0.6618	0.3381	0.7126	0.2875
400	0.6548	0.3452	0.6846	0.3154	0.7431	0.2569
220	0.6865	0.3135	0.7229	0.2770	0.7948	0.2052

The results in Table 3 show the sensitivity to sample size. In Table 3 the superscripts a, b and c represent rejection of H=0.5 under the 90%, 95% and 99% confidence intervals. The interpretation of these confidence intervals is as follows. The distribution of H was drawn from Monte Carlo simulations of a known geometric Brownian motion. With 20,000 replications the simulated value for H for a sample of 400 fell between 0.6548 and 0.3452 90% of the time, 0.6846 and 0.3154 95% of the time and 0.7431 and 0.2569 99% of the time. Thus, if an estimated value of H from a sample of 400 fell between these ranges we can stipulate within the confidence limits that it is consistent with, or within the range of, values generated from a geometric Brownian motion.

Table 1: Estimated Values of Hurst Coefficient from Equation (16) where H=α_1/2 for Days in Sample and from R-S Calculations from Equation (27). Null Hypothesis Ho= .5

Days/Contract	940	760	580	400	220	R-S Hurst
Alberta Barley price	0.414	0.431	0.451	0.428	0.333	0.489
coffee price	0.402	0.441	0.448	0.376	0.065 [a,b,c]	0.467
cocoa price	0.465	0.431	0.280 [a,b,c]	0.291	0.117 [a,b,c]	0.446
corn price	0.348 [a,b]	0.363 [a]	0.362 [a]	0.363	0.254 [a,b]	0.503
Feeder Cattle price	0.401	0.407	0.360 [a]	0.099 [a,b,c]	0.059 [a,b,c]	0.461
Fluid Milk price	0.481	0.489	0.473	0.523	0.381	0.540
Lean Hogs price	0.438	0.388	0.342 [a]	0.203 [a,b,c]	0.156 [a,b,c]	0.486
live cattle price	0.272 [a,b,c]	0.269 [a,b,c]	0.243 [a,b,c]	0.256 [a,b,c]	0.088 [a,b,c]	0.460
oats price	0.348 [a,b]	0.321 [a,b]	0.323 [a,b]	0.356	0.194 [a,b,c]	0.436
orange juice price	0.458	0.479	0.514	0.248 [a,b,c]	0.141 [a,b,c]	0.425
Pork Bellies price	0.381 [a]	0.356 [a]	0.380	0.253 [a,b,c]	0.206 [a,b]	0.519
Rapeseed canola price	0.396	0.336 [a]	0.352 [a]	0.251 [a,b,c]	0.267 [a,b]	0.494
Soybeans price	0.332 [a,b]	0.331 [a,b]	0.324 [a,b]	0.269 [a,b]	0.226 [a,b]	0.515
Sugar price	0.543	0.285 [a,b,c]	0.261 [a,b,c]	0.200 [a,b,c]	0.029 [a,b]	0.519
wheat price	0.231 [a,b,c]	0.221 [a,b,c]	0.171 [a,b,c]	0.168 [a,b,c]	0.123 [a,b,c]	0.483
Winnipeg oats price	0.481	0.477	0.459	0.480	0.200 [a,b,c]	0.566
Winnipeg Wheat price	0.341 [a,b]	0.349 [a]	0.345 [a]	0.334 [a]	0.265 [a,b]	0.499

The results are mixed and contingent on the confidence limits. Alberta Barley, Fluid Milk are consistent with a gBm at all confidence levels. Coffee and Winnipeg oats follow a random walk for samples greater than 220. Corn is consistent with a gBm at the 99% level for all N, while feeder cattle and live hogs are consistent in large samples of 940 and 760 but not consistent at the 90% level for N=580. Live cattle, and wheat prices do not appear to follow a random walk and sugar prices are consistent with a random walk only over the larger sample. Pork bellies are consistent with a random walk at the 95% and 99% levels for samples greater than 400. Orange juice futures are consistent with a random walk in large samples but not with the N=400 and N=220 sub samples. Canola is consistent with a random walk at the 90% level for N=940 and at the 95% level for N=760.

The differences between small and large samples is evident. It is much more likely to reject a random walk with small samples. For n=220, 9 series cannot be concluded as a gBM, 6 are consistent at the 99% confidence level, and only 2 are consistent with a gBm. In contrast, for N=940 10 series are consistent with a gBm, 1 is consistent at the 95% level, 4 are consistent at the 99% level and only 2, wheat and live cattle, are not consistent with a gBm.

In a qualitative sense, accepting the values as given has several implications. First, with the exception of sugar which shows a slightly persistent dynamic with H = .543, the evidence suggests that commodity futures prices are ergodic or mean-reverting. This observation is in opposition to recent concerns regarding persistent long-term memory in commodity futures contracts (Barkoulas et al. 1997, Corazza et al., 1997 or Crato and Kay 2000). The results in Table 3 provide no support for long-term memory[4].

8.0 Implications of Results

The results of this study have significant implications for the analysis of futures (and other financial) time series. The evidence of this paper is that the null hypotheses of H = .5 cannot generally be rejected at least for large samples. However, the results show that at least two price series, live cattle and wheat do not follow a gBm. The Hurst coefficients are low and this indicates strong mean reversion. More generally the majority of futures and cash prices are consistent with a random walk at least within he boundaries of a 99% confidence limit for samples of N=940, but the majority of series do not display random behavior in the smaller sample but whatever erratic price behavior is observed in he short run appears to work its way through in the longer run. Even so, when one considers that a sampling frame of 220 days is almost a year of trading this should not be trivialized. Furthermore, while short run departures due perhaps to spurious correlations or other economic impacts are not inconsistent with patterns of a random walk the results suggest a need to investigate short run price movements. More likely the results are due to the sampling frame. A combination of N=220 and k=150 does not provide a lot of degrees of freedom, so the results may also be indicative of a failure of the Hurst measure to adequately pick up or absolve correlations in the short run.

Qualitatively it is interesting to note that none of the results had h>5. Thus if there is any tendency within commodity price series to be correlated, the correlation in time is negative and not positive. In other words there is no evidence with these price series of persistent behavior that can be arbitraged in time. Instead the (qualitatively) low values indicate that there is a continual rebalancing between supply and demand   This conclusion is consistent with recent findings by Corazza et al. (1997) and Crato and Ray (2000) and is at odds with earlier findings by Helms et al. (1984) and Barkoulas et al. (1997).

In terms of futures market efficiency, it is unlikely that there are any self-similar properties that would allow a speculator to arbitrage from one period to the next. Speculative gains and losses can only be attributed to luck, rather than predictive ability. The luck arises from the fact that those series that did display some form of persistent behaviour did so by chance alone over the subset of time used. From a statistical point of view, there is no reason to expect that any gains or losses could be repeated in a different subset of time. In short, the evidence points to weak-form market efficiency in most commodity futures contracts in that successive price changes tend to be independent of each other. There is no evidence that trends in market prices can, unto themselves, be used to predict and benefit from future price changes.

From an analytical perspective this paper has provided a means to empirically test for fractional Brownian motion using variance ratios. This is a parametric approach that relies on the fractional definition of the Wiener process. In contrast, the Hurst-Mandelbrot approach is non-parametric. Given the qualitatively similar results, this is not necessarily a criticism of the Hurst-Mandelbrot approach, but an approach to measuring fractals and fractal dimension using a consistent-theoretical structure has its advantages. From a computational perspective the approach was less cumbersome than the R-S approach.

Finally, the overall intent of this paper was to determine if commodity futures prices followed a random walk process consistent with non-fractal Brownian motion. The results indicate that futures price movements are consistent with Brownian motion. One of the beneficial outcomes is that, for the most part, the assumption of Brownian motion used in the pricing of options on futures is justified. If Brownian motion is consistent with the efficient market hypothesis (an inference that is, according to Mandelbrot (1963), Mandelbrot and Taqqu (1979), Lo and Mackinnon (1999) and Corazza et al. (1997), debatable) then the results of this study indicate that markets are indeed efficient. However, while this paper provided an approach to test for fractional Brownian motion, future research should verify the results by using the technique to assess randomness in other financial time series, and should also compare and contrast the current technique with more conventional econometric approaches to measuring and testing for stationarity.

References

Barkoulas,J., T. and C.F. Baum (1996) “Long-term Dependence in Stock Returns” Economic Letters 53:253-259.

Barkoulas, J., W.C. Labys and J. Onochie. (1997). "Fractional Dynamics in International Commodity Prices." J. Futures Markets. 17(2):161-189.

Bassingthwaighte, J. B., and G. M. Raymond. “Evaluating Rescaled Range Analysis For Time Series”. Ann. Biomed. Eng. 22:432-444, 1994.

Black, F. (1976). “The pricing of Commodity Contracts” Journal of Financial Economics 3:167-177.

Black, F. and M. Scholes (1973). “The Pricing of Options and Corporate Liabilities” Journal of Political Economy 81:637-659

Booth, G.G., F.R. Kaen, and P.F. Koveos. (1982a). "Persistent Dependence in Gold Prices." J. Finance Research (Spring):85-93.

Booth, G.G., F.R. Kaen and P.F. Koveos. (1982b). "R/S Analysis of Foreign Exchange Rates Under Two International Monetary Regimes." J. Monetary Economics. 10:407-415.

Boyle, P.P., M. Broadie, and P. Glasserman (1997) “Monte Carlo Methods in Security Pricing” Journal of Economics, Dynamics and Control 21:1267-1327.

Boyle, P.P. and T. Wang (1999) “The Valuation of New Securities in an Incomplete Market: The Catch-22 of Derivative Pricing” Working Paper, University of Waterloo

Cannon, M.J., D.B. Percival, D.C. Caccia, G.M. Raymond and J.B. Bassingthwaighte (1996) “Evaluating Scaled Windowed Variance Methods for Estimating the Hurst Coefficient in Time Series” Physica A 241:606-626.

Comte, F. and E. Renault “Long Memory Continuous Time Models” Journal of Econometrics 73(1996):101-149

Corazza, M., A.G. Malliaris, and C. Nardelli. (1997). "Searching for Fractal Structure in Agricultural Futures Markets." J. of Futures Markets. 17(4):433-473.

Cox, J.C. , J.E. Ingersoll, and S.A. Ross (1985) “An Intertemporal General Equilibrium Model of Asset Prices” Econometrica 53:363-384.

Crato, N. and B.K. Kay (2000). "Memory in Return and Volatilities of Futures Contracts." J. Futures Markets. 20(6):525.543.

Cromwell, J.B. , W.C. Labys and E. Kouassi (2000) “What Color are Commodity Prices?: A Fractal Analysis” Empirical Economics 25:563-580.

Crownover, R.M. (1995). Introduction to Fractals and Chaos. Tones and Bartlett Publishers, London, U.K.

Cutland, N.J. , P.E. Kopp and W. Willinger (1995) “Stock price returns and the Joseph Effect: A Fractional Version of the Black-Scholes Model” Progress in Probabilities 36:327-351.

Fama, E.F. and K.R. French. (1988). "Permanent and Temporary Components of Stock Prices." J. of Political Economy. 96(2):246-273.

Feder, J. (1988) Fractals Plenum Publishing Corp. NY NY

Igloi, E. and G. Terdik (1999) “Bilinear Stochastic Systems with Fractional Brownian Motion Input” Annals of Applied Probability 9(1):46-77

Gao, A.H. and G.H.K. Wang (1999) “Modeling Nonlinear Dynamics of Daily Futures Price Changes” Journal of Futures Markets 19(3):325-351.

Garman, M.B. (1977) “A General Theory of Asset Valuation under Diffusion State Processes” Working Paper, University of California, Berkeley

Greene, M.T. and B.O. Fielitz. (1997). "Long-Term Dependence in Common Stock Returns." J. Financial Economics. 4:339-349.

Helms, B.P., F.R. Kaen and R.E. Rosenman. (1984). "Memory in Commodity Futures Contracts." J. Futures Markets. 4(4):559.567.

Higuchi, T. (1990) “Approach to an Irregular time Series on the Basis of Fractal Theory” Physica D 31:277-283

Higuchi, T. (1990) “Relationship Between the Fractal Dimension and the Power Law Index for a Time Series: A Numerical Investigation” Physica D 46:254-264

Hommes, C.H. “Financial Markets as Nonlinear Adaptive Evolutionary Systems” Quantitative Finance 1:149-167

Hurst, H.E. (1951). "Long-Term Storage Capacity of Reservoirs." Transactions of the American Society of Civil Engineers. 116:770-799.

Jin, H.J. and D. L. Frechette (2004) “Fractional Integration in Agricultural Futures Price Volatilities” American Journal of Agricultural Economics 86(2):432-433

Kwiatkowski, D, P.C.B. Phillips, P Schmidt, and Y. Shin (1992) “Testing the Null Hypothesis of Stationarity against the Null Hypothesis of a Unit Root” Journal of Econometrics 54:159-178

Leland, W.E., M.S. Taqqu, W. Willinger, and D.V. Willson (1994) “On the Self-Similar Nature of Ethernet Traffic (Extended Version) IEEE/ACM Transactions on Networking 2(1):1-15.

Lo, A.W. (1991). "Long-Term Memory in Stock Market Prices." Econometrica. (59):1279-1313.

Lo, A.W. and A.C. Mackinnlay. (1999). A Non-Random Walk Down Wall Street. Princeton Press, Princeton, N.J.

Mandelbrot, B. (1963). "The Variation of Certain Speculative Prices." J. of Business. 36:394-419.

Mandelbrot, B.B. (1972). "Statistical Methodology for Non-Periodic Cycles: From Covariance to R/S Analysis." Annals of Economic and Social Measurement. 1(July):259-290.

Mandelbrot, B.B. (1977). Fractals: Form, Chance and Dimension. W.H. Freeman and Co., San Francisco.

Mandelbrot, B.B. and M.S. Taqqu (1979) “Robust R/S Analysis of Long Run Serial Correlation” in Proceedings of 42nd Session ISI :69-99

Mandelbrot, B.B. and J.R. Wallis. (1969). "Robustness of the Rescaled Range R/S in the Measurement of Non-Cyclic Long-Run Statistical Dependence." Water Resources Research. 5:321-340.

Mandelbrot, B.B. and J.W. Van Ness (1968) “Fractional Brownian Motions, Fractional Noises and Applications” SIAM Review 10 (4 October):422-437

Merton, R.C. (1973) "The Theory of Rational Options Pricing" Bell Journal of Economics 4:141-183

Panas, E. (2001) "Estimating Fractal Dimension Using Stable Distributions and Exploring Long Memory Through ARFIMA Models in Athens Stock Exchange" Applied Financial Economics 11:395-402.

Peters, E. (1996). "Chaos and Order in the Capital Markets." 2nd Edition. John Wiley and Sons, New York.

Rangarajan, G. and M. Ding (2000) "Integrated Approach to the Assessment of Long Run Correlation in Times Series Data" Physical review A 61(5):4991-5001.

Rogers, L.C.G. (1997) "Arbitrage with Fractional Brownian Motion" Mathematical Finance 7 (1 January): 95-105.

Rubinstein, M. (1979) "The Pricing of Uncertain Income Streams and the pricing of Options" Bell Journal of Economics 7:407-424

Schroeder, M.N. (1991) Fractals, Chaos and Power Laws W,H. Freeman and Company, NY

Sottinen, T (2001) "Fractional Brownian Motion, Random Walks and Binary Market Models" Finance and Stochastics 5:343-355.

Sephton, P (2002) "Fractional Cointegration: Monte Carlo Estimates of Critical Values, With an Application" Applied Financial Economics 12:331-335

Wei, A. and R.M. Leuthold (1998) "Long Agricultural Futures Prices: ARCH, Long Memory or Chaos Processes" OFOR Paper # 98-03, University of Illinois, May

West, B.J. and L. Griffin (1998) "Allometric Control of Human Gait" Fractals 6(2):101-108.

West, B.J., P. Hamilton and D.J. West (1999) "Fractal Scaling and the Teen Birth Phenomenon" Fractals 7(2):113-122

Appendices

Approximate Upper and Lower 90% Confidence Intervals For Geometric Brownian Motion

	k						
N/k	10	25	50	75	100	125	150
	Upper Confidence Limit						
100	0.635883	0.666491	0.69062	0.705138	0.715623	0.723864	0.730667
200	0.601975	0.630951	0.653794	0.667538	0.677464	0.685265	0.691705
300	0.582985	0.611047	0.633169	0.646479	0.656092	0.663647	0.669884
400	0.569875	0.597306	0.618931	0.631942	0.641339	0.648724	0.654821
500	0.55991	0.586862	0.608108	0.620892	0.630124	0.63738	0.64337
600	0.551898	0.578463	0.599406	0.612006	0.621107	0.628259	0.634163
700	0.545213	0.571457	0.592146	0.604593	0.613583	0.620649	0.626482
800	0.539487	0.565456	0.585927	0.598244	0.60714	0.614131	0.619903
900	0.534487	0.560215	0.580497	0.5927	0.601513	0.608439	0.614158
1000	0.530054	0.555568	0.575682	0.587783	0.596523	0.603392	0.609063
1100	0.526075	0.551398	0.57136	0.583371	0.592046	0.598863	0.604491
1200	0.522468	0.547618	0.567443	0.579372	0.587987	0.594758	0.600347
1300	0.519173	0.544163	0.563864	0.575717	0.584278	0.591006	0.59656
1400	0.51614	0.540984	0.56057	0.572354	0.580865	0.587553	0.593075
1500	0.513332	0.538042	0.557521	0.569241	0.577705	0.584357	0.589849
1600	0.51072	0.535304	0.554684	0.566344	0.574765	0.581383	0.586847
1700	0.508278	0.532744	0.552031	0.563636	0.572017	0.578604	0.584042
1800	0.505986	0.530342	0.549543	0.561095	0.569438	0.575995	0.581408
1900	0.503828	0.52808	0.547199	0.558701	0.567009	0.573538	0.578929
2000	0.501789	0.525943	0.544984	0.55644	0.564715	0.571217	0.576586
	Lower Confidence Limit						
100	0.364117	0.333509	0.30938	0.294862	0.284377	0.276136	0.269333
200	0.398025	0.369049	0.346206	0.332462	0.322536	0.314735	0.308295
300	0.417015	0.388953	0.366831	0.353521	0.343908	0.336353	0.330116
400	0.430125	0.402694	0.381069	0.368058	0.358661	0.351276	0.345179
500	0.44009	0.413138	0.391892	0.379108	0.369876	0.36262	0.35663
600	0.448102	0.421537	0.400594	0.387994	0.378893	0.371741	0.365837
700	0.454787	0.428543	0.407854	0.395407	0.386417	0.379351	0.373518
800	0.460513	0.434544	0.414073	0.401756	0.39286	0.385869	0.380097
900	0.465513	0.439785	0.419503	0.4073	0.398487	0.391561	0.385842
1000	0.469946	0.444432	0.424318	0.412217	0.403477	0.396608	0.390937
1100	0.473925	0.448602	0.42864	0.416629	0.407954	0.401137	0.395509

1200	0.477532	0.452382	0.432557	0.420628	0.412013	0.405242	0.399653
1300	0.480827	0.455837	0.436136	0.424283	0.415722	0.408994	0.40344
1400	0.48386	0.459016	0.43943	0.427646	0.419135	0.412447	0.406925
1500	0.486668	0.461958	0.442479	0.430759	0.422295	0.415643	0.410151
1600	0.48928	0.464696	0.445316	0.433656	0.425235	0.418617	0.413153
1700	0.491722	0.467256	0.447969	0.436364	0.427983	0.421396	0.415958
1800	0.494014	0.469658	0.450457	0.438905	0.430562	0.424005	0.418592
1900	0.496172	0.47192	0.452801	0.441299	0.432991	0.426462	0.421071
2000	0.498211	0.474057	0.455016	0.44356	0.435285	0.428783	0.423414

Approximate Upper and Lower 95% Confidence Intervals For Geometric Brownian Motion

N/k	10	25	50	75	100	125	150
			Upper Confidence Limit				
100	0.661315	0.698327	0.727696	0.745445	0.7583	0.768423	0.776795
200	0.620849	0.655597	0.683168	0.699831	0.7119	0.721404	0.729263
300	0.598336	0.631823	0.658395	0.674454	0.686085	0.695244	0.702819
400	0.58286	0.615481	0.641365	0.657009	0.668338	0.677261	0.68464
500	0.571131	0.603096	0.62846	0.643788	0.65489	0.663633	0.670863
600	0.561724	0.593162	0.618108	0.633184	0.644103	0.652702	0.659813
700	0.553891	0.584891	0.609489	0.624355	0.635122	0.643601	0.650613
800	0.547194	0.57782	0.60212	0.616806	0.627443	0.635819	0.642747
900	0.541355	0.571653	0.595694	0.610224	0.620747	0.629034	0.635887
1000	0.536184	0.566193	0.590004	0.604395	0.614817	0.623025	0.629813
1100	0.531549	0.561298	0.584904	0.59917	0.609503	0.61764	0.624369
1200	0.527352	0.556867	0.580286	0.59444	0.604691	0.612763	0.619439
1300	0.523521	0.552821	0.57607	0.590121	0.600298	0.608312	0.614939
1400	0.519999	0.549102	0.572195	0.586151	0.596259	0.604219	0.610802
1500	0.516741	0.545661	0.56861	0.582478	0.592523	0.600433	0.606975
1600	0.513712	0.542463	0.565277	0.579064	0.58905	0.596914	0.603417
1700	0.510882	0.539475	0.562163	0.575875	0.585806	0.593626	0.600094
1800	0.508229	0.536674	0.559244	0.572884	0.582763	0.590543	0.596977
1900	0.505732	0.534037	0.556496	0.570069	0.5799	0.587642	0.594044
2000	0.503375	0.531547	0.553902	0.567412	0.577197	0.584903	0.591275
			Lower Confidence Limit				
100	0.338685	0.301673	0.272304	0.254555	0.2417	0.231577	0.223205
200	0.379151	0.344403	0.316832	0.300169	0.2881	0.278596	0.270737
300	0.401664	0.368177	0.341605	0.325546	0.313915	0.304756	0.297181
400	0.41714	0.384519	0.358635	0.342991	0.331662	0.322739	0.31536
500	0.428869	0.396904	0.37154	0.356212	0.34511	0.336367	0.329137
600	0.438276	0.406838	0.381892	0.366816	0.355897	0.347298	0.340187
700	0.446109	0.415109	0.390511	0.375645	0.364878	0.356399	0.349387
800	0.452806	0.42218	0.39788	0.383194	0.372557	0.364181	0.357253
900	0.458645	0.428347	0.404306	0.389776	0.379253	0.370966	0.364113
1000	0.463816	0.433807	0.409996	0.395605	0.385183	0.376975	0.370187
1100	0.468451	0.438702	0.415096	0.40083	0.390497	0.38236	0.375631
1200	0.472648	0.443133	0.419714	0.40556	0.395309	0.387237	0.380561
1300	0.476479	0.447179	0.42393	0.409879	0.399702	0.391688	0.385061

1400	0.480001	0.450898	0.427805	0.413849	0.403741	0.395781	0.389198
1500	0.483259	0.454339	0.43139	0.417522	0.407477	0.399567	0.393025
1600	0.486288	0.457537	0.434723	0.420936	0.41095	0.403086	0.396583
1700	0.489118	0.460525	0.437837	0.424125	0.414194	0.406374	0.399906
1800	0.491771	0.463326	0.440756	0.427116	0.417237	0.409457	0.403023
1900	0.494268	0.465963	0.443504	0.429931	0.4201	0.412358	0.405956
2000	0.496625	0.468453	0.446098	0.432588	0.422803	0.415097	0.408725

Approximate Upper and Lower 99% Confidence Intervals For Geometric Brownian Motion

N/k	10	25	50	75	100	125	150
			Upper Confidence Limit				
100	0.710582	0.760508	0.800594	0.825015	0.842792	0.856845	0.8685
200	0.657271	0.703451	0.740529	0.763118	0.779562	0.79256	0.803341
300	0.627959	0.672081	0.707505	0.729086	0.744797	0.757215	0.767516
400	0.607959	0.650675	0.684971	0.705865	0.721075	0.733098	0.74307
500	0.592885	0.634542	0.667988	0.688364	0.703197	0.714922	0.724647
600	0.580847	0.621658	0.654424	0.674387	0.688918	0.700405	0.709933
700	0.570859	0.610969	0.643172	0.662791	0.677073	0.688362	0.697726
800	0.562347	0.601858	0.633581	0.652908	0.666976	0.678097	0.687321
900	0.554944	0.593935	0.62524	0.644312	0.658196	0.66917	0.678273
1000	0.548404	0.586935	0.617872	0.636719	0.650439	0.661285	0.67028
1100	0.542554	0.580675	0.611282	0.629928	0.643501	0.654231	0.66313
1200	0.537269	0.575018	0.605326	0.623791	0.637232	0.647857	0.65667
1300	0.532452	0.569863	0.599899	0.618198	0.631519	0.642049	0.650783
1400	0.528031	0.565131	0.594918	0.613065	0.626275	0.636718	0.645379
1500	0.523947	0.560761	0.590318	0.608324	0.621433	0.631794	0.640388
1600	0.520157	0.556704	0.586047	0.603923	0.616936	0.627223	0.635755
1700	0.516621	0.552919	0.582063	0.599818	0.612743	0.622959	0.631433
1800	0.513309	0.549375	0.578331	0.595973	0.608815	0.618966	0.627386
1900	0.510196	0.546043	0.574824	0.592358	0.605122	0.615212	0.623581
2000	0.50726	0.542901	0.571516	0.588949	0.60164	0.611672	0.619992
			Lower Confidence Limit				
100	0.289418	0.239492	0.199406	0.174985	0.157208	0.143155	0.1315
200	0.342729	0.296549	0.259471	0.236882	0.220438	0.20744	0.196659
300	0.372041	0.327919	0.292495	0.270914	0.255203	0.242785	0.232484
400	0.392041	0.349325	0.315029	0.294135	0.278925	0.266902	0.25693
500	0.407115	0.365458	0.332012	0.311636	0.296803	0.285078	0.275353
600	0.419153	0.378342	0.345576	0.325613	0.311082	0.299595	0.290067
700	0.429141	0.389031	0.356828	0.337209	0.322927	0.311638	0.302274
800	0.437653	0.398142	0.366419	0.347092	0.333024	0.321903	0.312679
900	0.445056	0.406065	0.37476	0.355688	0.341804	0.33083	0.321727
1000	0.451596	0.413065	0.382128	0.363281	0.349561	0.338715	0.32972
1100	0.457446	0.419325	0.388718	0.370072	0.356499	0.345769	0.33687
1200	0.462731	0.424982	0.394674	0.376209	0.362768	0.352143	0.34333
1300	0.467548	0.430137	0.400101	0.381802	0.368481	0.357951	0.349217

1400	0.471969	0.434869	0.405082	0.386935	0.373725	0.363282	0.354621
1500	0.476053	0.439239	0.409682	0.391676	0.378567	0.368206	0.359612
1600	0.479843	0.443296	0.413953	0.396077	0.383064	0.372777	0.364245
1700	0.483379	0.447081	0.417937	0.400182	0.387257	0.377041	0.368567
1800	0.486691	0.450625	0.421669	0.404027	0.391185	0.381034	0.372614
1900	0.489804	0.453957	0.425176	0.407642	0.394878	0.384788	0.376419
2000	0.49274	0.457099	0.428484	0.411051	0.39836	0.388328	0.380008

End Notes

[1] The difference can be attributed to the scaling attributes of the power law. For example, define $VAR=\sigma^2 k^{2H}$, as the power law for a fractional Brownian motion with a time step, k. Suppose that σ=.30 and k=5 days. When H=.3, VAR =.2358. What time step is required to double the variance to .472? The solution is 15.87, more than three times the number of days to achieve a variance of .236. Likewise, if H=.7, VAR = .857. To double variance to 1.713 under this power law, requires a time step of only 8.2 days, considerably less than twice the original number of days. However, when H=.5, VAR =.45. Doubling variance to .90 requires 10 days, exactly double that of the original 5 days. Since variance increases in direct proportion to k, that is $r_1=r_2=r_3$, it is self-similar for H=.5, but because the relationship is derived from a power law, it is also self-affine. In contrast, since $r_1 \neq r_2 \neq r_3$ for $H \neq .5$ the series is self-affine, but not self-similar.

[2] The variance ratio approach is similar to other approaches to measuring linear or non-linear dynamics in a time series. For example a common approach to measuring time irregularity is through the spectral density function which in its generic form is given by $G(k) = g(k)\alpha k^{-d}$ (see for example Higuchi 1988.1990). In Higuchi, g(k) is measured by the mean of the observed differences in prices over a time step k ($k = \Delta t$), α is an arbitrary constant, and d is the characteristic exponent. It is the characteristic exponent that measures the irregularity of a time series. Higuchi (1988,1990) has shown that in this form, d is a fairly reasonable and stable measure of the fractal dimension of a time series. As a fractal dimension, d=1.5 lies midway between a one-dimensional line and a 2-dimensional plain in Euclidian space, and at this point the underlying process is a Brownian motion. The properties of the spectral density when d=1.5 are consistent with the properties of the Hurst rescaled range for H=.5, and the fractional Brownian motion at H=.5. Furthermore, G(k) is self-affine over the domain of k, but the degree of self-affinity, or self-similarity, will depend on the nature of the underlying time series. The general approach to determining the fractal dimension, d, is to map ln[G(k)] against ln[k] and assign to d the value of this slope.

[3] I have not found previous research that supported Monte Carlo estimates of the asymptotic standard deviations of H and β_1. However, in Fama and French (1988) a similar approach is used to estimate the standard errors of first order autocorrelation coefficients. Qualitatively they are able to support the conjecture that stock price movements have stationary and random components. However, when their specific tests were assessed using standard errors from Monte Carlo simulations they found that the null hypothesis (of non-stationarity in prices) was difficult to reject. In fact they speculate that the large standard errors in a pure random walk may make such hypotheses altogether untestable (Fama and French, 1988, page 257). A wide acceptance region for unit roots in time series data has also been discussed by Kwiatkowski et al (1992) and critical values for fractional cointegration generated from Monte Carlo methods are described in Sephton (2002). Panas (2001) uses a bootstrapping method to estimate standard errors for stocks traded on the Athens Stock Exchange and is able to reject the null for 11 of 13 stocks. In an application to self-similar properties in ethernet traffic, Leland, Taqqu, Willinger and Wilson (1994) apply numerical techniques to obtain confidence intervals. However, in their model the confidence intervals were constructed around the estimate of H, whereas in the current study the confidence intervals were constructed around a fixed point of H=.5. In some disciplines of the social sciences and humanities a surrogate approach has been used. The surrogate approach

repeatedly randomizes observations from a particular sample to remove all correlations. The H values are then calculated for each surrogate, and their sample standard deviations used in the statistical tests. (see West and Griffin (1998) and West, Hamilton and West (1999); See Rangarajan and Ding (2000) for a critique)

[4] The R-S estimates in Table 3 are different than those presented in column 1. Qualitatively, corn, fluid milk, pork bellies, soybeans, sugar and Winnipeg oats display persistent tendencies with H > .5. However only Winnipeg oats (.566) and perhaps fluid milk (.540) are sufficiently higher than .5 to warrant concern. The remaining 11 commodities have R-S estimated H ≤ .5. Winnipeg wheat (.499) and rapeseed (.494) are virtually identical to .5 and would thus be characterized as having a pure random walk. The remaining futures prices again display mean-reverting tendencies. Qualitatively the main conclusion is that the Mandelbrot-Hurst approach provides results that are not inconsistent with the variance ratio approach.