Compensating Variation Without Apology?
Willingness-To-Pay and the Failure of Integrability

Richard E. Just and Daniel O. Gilligan
Agricultural and Resource Economics Department
University of Maryland
2200 Symons Hall
College Park, MD 20742

DRAFT
May 14, 1998

Abstract

Failure of integrability is shown to cause path-dependence of willingness-to-pay measures of welfare change. Using the linear expenditure system, effects of failure of integrability are negligible (substantial) for estimating income (price) elasticities. For single price changes, Hausman's approach to calculating willingness to pay from ordinary demands becomes subject to excessive errors of estimation. For multiple price changes, calculations of willingness to pay become path dependent. The empirical approach of Vartia to calculation of willingness to pay for multiple price changes thus involves an arbitrary choice of path. Furthermore, the Willig results justifying consumer surplus approximation fail.

It is well known that consumer surplus as a measure of consumer welfare suffers from path dependence.[1] Unless the restrictive assumption of constant marginal utility of income is met, the consumer surplus measure of welfare change associated with price changes for multiple goods or a price and income change will vary depending on the order of the price (and income) changes (i.e., on the path of integration). This ambiguity is avoided if welfare is measured by compensating variation (*CV*) or equivalent variation (*EV*). Both *CV* and *EV* are accepted as accurate willingness-to-pay measures of welfare change. However, an often overlooked fact demonstrated in this paper is that, depending on empirical implementation, all three of these measures are subject to a second form of path dependence—the failure of integrability. A system of demands is integrable when it is restricted so as to be consistent with a well-behaved utility function. That is, demands must be homogenous of degree zero in prices and income and must satisfy the budget constraint, Slutsky symmetry, and negative semi-definiteness of the Slutsky matrix. Hausman demonstrated for a single price change that exact measures of willingness-to-pay can be derived from the corresponding ordinary demand equation despite the failure of integrability. We consider further the lack of accuracy in estimation when a demand equation is not estmated as part of an integrable system. Moreover, we consider the application of such approaches to the measurement of willingness to pay for multiple prices change when demands do not meet the integrability restrictions.

A common practice in applied welfare analysis—particularly when time and funding are in short supply—is to borrow estimates of price and income elasticities from existing studies or to estimate a partial demand system to obtain approximate welfare results. Elasticity estimates so obtained typically fail to meet the requirements of integrability. This raises a number of practical concerns about the magnitude of the error introduced by estimating welfare changes without integrability as well as the extent of path dependence with multiple price changes. We demonstrate the effects of the failure of integrability on estimation of *CV* using the linear expenditure system as an example. The derivation of the analogous result for equivalent variation is trivially similar.

Willig developed a rule of thumb showing that the error in using consumer surplus as a measure of *CV* or *EV* is not large.[2] We also demonstrate that his results fail when integrability fails and, thus, his rule of thumb is not applicable to a major problem for which it was designed—the case where ad hoc demand specifications are used to estimate welfare change.

The results show, not surprisingly, that the variance of parameter estimators is larger when integrability is not imposed in estimation (the unconstrained or ad hoc case), and that the difference in variance between the unconstrained and constrained cases is increasing in income elasticity and number of demands in the system. While the benefits to estimating systems that satisfy integrability may be negligible for estimating income elasticities and own-price elasticities, they are considerable for estimating cross-price elasticities. In particular, errors in estimating price elasticities are shown to be an order of magnitude (or two) larger with unconstrained versus constrained estimation where the order is determined by the size of the demand system. While Hausman demonstrated that *CV* and *EV* can be measured accurately for a single price change using estimates of the corresponding ordinary demand, our results imply that, without system estmation of integrable demands, such willingness-to-pay measures are subject to greater errors of estimation for the single price change case which may or may not be considerable depending on the order of the system. More seriously, path dependence in the multiple price change case is shown to be subject to large empirical errors because of lack of precision in estimating cross-price elasticities.

Integrability

Integrability refers to the method by which underlying preferences can be recovered from a consistent system of ordinary demands. Let $q = (q_1,...,q_N)$ be a vector of consumer goods and $p = (p_1,..., p_N)$ the corresponding price vector. Total expenditure or income is denoted by m and $e(p,U)$ represents the expenditure function. The *i*th Hicksian and ordinary demand functions are $\bar{q}_i(p,U)$ and $q_i(p,m)$, respectively. In order for the system of (differentiable) ordinary demand equations to satisfy integrability, it must meet four conditions (Hurwicz and Uzawa):

(i) the budget constraint, $\sum_{i=1}^{N} p_i q_i(p,m) = m$;

(ii) homogeneity of degree zero in prices and income, $q_i(p,m) = q_i(\lambda\, p, \lambda\, m)$;

(iii) Slutsky symmetry, $\partial\bar{q}_i/\partial p_j = \partial\bar{q}_j/\partial p_i$, where $\partial\bar{q}_i/\partial p_j = \partial q_i/\partial p_j + q_j\, \partial q_i/\partial m$;

(iv) negative semi-definiteness of the Slutsky matrix, $\{\partial\bar{q}_i/\partial p_j\} \leq 0$, in matrix notation.

Any demand system satisfying these conditions is consistent with utility theory. Moreover, Shephard's lemma or Roy's identity provide expressions that, when integrated over prices, can be used to recover a concave, linearly homogenous expenditure function or indirect utility function, respectively, from which the direct utility function can be derived using duality theory.

To ensure that integrability conditions are satisfied, it is necessary to derive demand specifications from utility maximization and impose cross-equation parameter restrictions in estimation, or to impose all integrability conditions on arbitrary demand specifications during estimation. The former approach is used for the translog, generalized Leontief and other second-order flexible functional forms. The latter approach often leads to economically implausible restrictions on demand when ad hoc specifications are used. For example, LaFrance (1985, 1986) demonstrated that imposing the integrability conditions on an ad hoc system of linear or log-linear demands leads to extremely restrictive conditions on price and income elasticities (including zero income effects for all goods) that render these approaches useless for practical purposes.

The Linear Expenditure System

We use the Linear Expenditure System (LES), originally developed by Klein and Rubin, to demonstrate the implications of the failure of integrability for welfare measurement for several reasons. The LES is a system of linear demand equations of the form

$$q_i = \beta_i \frac{m}{p_i} + \sum_{j=1}^{N} \beta_{ij} \frac{p_j}{p_i}, \; i = 1,\ldots,N. \tag{1}$$

This system is linear in parameters and corresponds closely to the most common linear specification used for piecemeal or ad hoc estimation of individual demands. Furthermore, it is almost linear in variables, incorporating the minimum degree of nonlinearity in variables that allows imposition of homogeneity.[3] Because the unrestricted form of this system is similar to ad hoc demand specifications, examination of departures from integrability should provide useful insights for evaluating the implications of ad hoc practices. Nevertheless, the principles developed below are general and should be relevant to more general functional forms currently in use.

Stone derived restrictions on parameters of the demand specifications in (1) that are necessary to ensure integrability of the linear expenditure system. Imposing the budget constraint, homogeneity, and Slutsky symmetry leads to a system of linear expenditures given by

$$p_i q_i = \beta_i m + \alpha_i p_i - \beta_i \sum_{j=1}^{N} \alpha_j p_j, \qquad \sum_{i=1}^{N} \beta_i = 1, \quad i = 1,\ldots,N, \tag{2}$$

where the α_i are constants. While the general expenditure system implied by (1) requires the estimation of $N(N+1)$ parameters, the imposition of integrability in (2) results in $N(N-1)+1$ restrictions on the expenditure system, leaving only $2N-1$ parameters to be estimated. Imposition of the budget constraint requires that the sum of income coefficients across equations is one, $\sum_{i=1}^{N} \beta_i = 1$, and that the sum of price coefficients across equations is zero, $\alpha_i - \alpha_i \sum_{i=1}^{N} \beta_i = 0$. However, the latter N restrictions are redundant after imposition of the $N(N-1)$ restrictions needed for Slutsky symmetry. Symmetry requires that

$$\beta_{ij} = \beta_{kj} \frac{\beta_i}{\beta_k} = -\alpha_j \beta_i, \;\; \forall\, j \neq i, \;\; \text{and} \;\; \beta_{ii} = \beta_{ki} \frac{\beta_i - 1}{\beta_k} = \alpha_i (1 - \beta_i), \tag{3}$$

where k refers to an arbitrary equation in the system. Finally, negative semi-definiteness of the Slutsky matrix requires

$$m - \sum_{j=1}^{N} p_j \alpha_j > 0, \;\; \beta_i > 0, \; \forall i \tag{4}$$

(Deaton). Deaton and Muellbauer note that as a result of these restrictions, all goods in the linear expenditure system must be substitutes in consumption and all must be normal goods. These conditions are likely to be met for fairly aggregated commodity groupings.

Willingness-to-Pay Measures

Willingness-to-pay measures of welfare change can be derived from the restricted linear expenditure system in (2) by first recovering the corresponding utility function and indirect utility function. Geary demonstrated that the demands implicit in this expenditure system can be derived from a utility function of the form, $U(q) = \prod_{i=1}^{N} (q_i - \alpha_i)^{\beta_i}$. The corresponding indirect utility function is,

(6) $$V(p,m) = \left[1 - \sum_{j=1}^{N} \alpha_j \frac{p_j}{m}\right] \prod_{j=1}^{N} (m / p_j)^{\beta_j}, \qquad \sum_{j=1}^{N} \beta_{\mathrm{j}} = 1.$$

The indirect utility function provides a basis for defining the compensating variation, $V(p^1, m^1 - CV) = V(p^0, m^0)$, i.e., compensating variation, *CV*, is that amount of income that must be taken away from an individual following a price and income change to leave the individual's indirect utility unchanged. Using this definition, the measure of compensating variation for the restricted LES follows from (6),

(7) $$CV = m^1 - \sum_{j=1}^{N} \alpha_j p_j^1 - \left[m^0 - \sum_{j=1}^{N} \alpha_j p_j^0\right] \prod_{j=1}^{N} (p_j^1 / p_j^0)^{\beta_j}.$$

The expression for equivalent variation can be similarly derived as the solution to $V(p^1, m^1) = V(p^0, m^0 + EV)$.

Path Dependence of Compensating Variation

In order to demonstrate the path dependence of *CV* for the LES when integrability fails, we derive the expression for *CV* for the unrestricted LES in (1) using Hause's method. Shephard's Lemma relates the expenditure function to the *i*th ordinary demand equation in (1),

(8) $$\frac{\partial e_i(p, U_i)}{\partial p_i} = q_i[p, e_i(p, U_i)] = \frac{\beta_i}{p_i} e_i(p, U_i) + \sum_{j=1}^{N} \beta_{ij} \frac{p_j}{p_i}.$$

Note that a subscript is attached to the expediture function denoting the demand from which it is derived because non-integrable ordinary demands will yield path dependent differential equations describing the expenditure function. Solving the differential equation in (8) for $e_i(p, U_i)$ yields an expression for how the unrestricted LES expenditure function depends on the *i*th price,

(9) $$e_i(p, U_i) = U_i p_i^{\beta_i} + \frac{1}{\beta_i}\left[\frac{\beta_{ii} p_i}{1 - \beta_i} - \sum_{j=1}^{N} \beta_{ij} p_j\right].$$

Hausman refers to expenditure functions derived in this way as "quasi-expenditure functions," since they do not satisfy the integrability conditions. If (9) were derived from an integrable demand system, the CV of a single price change, say p_i from p_i^0 to p_i^1, would be $CV = m - e(p_1^0,..., p_i^1,..., p_N^0, U_0)$. However, when demands are not integrable, the N expenditure functions derived from N ordinary demands according to (9) are mutually incompatible, which is the source of path dependence of willingness-to-pay measures when integrability fails.

To illuminate the path dependence problem, consider two alternative rectangular paths for a two-price-change case. Let Path L_1 refer to a change in price p_1 followed by a change in price p_2. Path L_2 refers to the price changes taken in opposite order. With Marshallian consumer surplus, path dependence is shown simply by demonstrating a difference in consumer surplus change calculated along the two paths (where all other prices are dropped for simplicity of notation),

$$Path\ L_1: \quad \Delta CS_1 = -\int_{p_1^0}^{p_1^1} q_1(p_1, p_2^0, m)\, dp_1 - \int_{p_2^0}^{p_2^1} q_2(p_1^1, p_2, m)\, dp_2$$

and

$$Path\ L_2: \quad \Delta CS_2 = -\int_{p_2^0}^{p_2^1} q_2(p_1^0, p_2, m)\, dp_2 - \int_{p_1^0}^{p_1^1} q_1(p_1, p_2^1, m)\, dp_1 ,$$

when $\partial q_i / \partial p_j \neq \partial q_j / \partial p_i$. Willingness-to-pay measures are represented similarly by the change in Hicksian consumer surplus. Neither CV nor EV are path dependent in theory because

$$Path\ L_1: \quad \Delta H_1 = -\int_{p_1^0}^{p_1^1} \bar{q}_1(p_1, p_2^0, U)\, dp_1 - \int_{p_2^0}^{p_2^1} \bar{q}_2(p_1^1, p_2, U)\, dp_2$$

and

$$Path\ L_2: \quad \Delta H_2 = -\int_{p_2^0}^{p_2^1} \bar{q}_2(p_1^0, p_2, U)\, dp_2 - \int_{p_1^0}^{p_1^1} \bar{q}_1(p_1, p_2^1, U)\, dp_1$$

are identical given Slutsky symmetry, $\partial \bar{q}_i / \partial p_j = \partial \bar{q}_j / \partial p_i$. In practice, however, when individual compensated demands are inferred from corresponding ordinary demands that do not satisfy integrability, Slutsky symmetry will typically fail.

Suppose one employs independent estimates of ordinary demands, $q_i(p,m)$, i=1,…,N, and computes the Hicksian surplus change by inferring the associated compensated demands. In principle, this is done by using the Slutsky equation or, equivalently, by solving the differential equation $\partial e / \partial p_i = q_i[p, e(p,U)]$ for $e(p,U)$, then finding $\bar{q}_i = \partial e / \partial p_i$ and imposing the boundary condition implicit in $\bar{q}_i(p,U) = q_i(p,m)$.

Consider the two-price-change case where estimates of ordinary demands follow the unrestricted LES in (1). Solving the differential equation in (8) obtains an associated expenditure function as in (9). The associated Hicksian demands are

(10) $$\bar{q}_i = \frac{\partial e_i(p,U_i)}{\partial p_i} = U_i \beta_i p_i^{\beta_i - 1} + \frac{\beta_{ii}}{1-\beta_i}, \quad i=1,2.$$

If p_1 changes first (as in Path L_1), the boundary condition for the initial segment of the path is $\bar{q}_1(p_1^0, p_2^0, U_1^0) = q_1(p_1^0, p_2^0, m_0)$, which implies

(11) $$U_1^0 = (p_1^0)^{-\beta_2}\left[m - \frac{\beta_{11} p_1^0}{1-\beta_1} + \frac{1}{\beta_1}\sum_{j\neq 1}\beta_{1j} p_j^0\right].$$

Then the CV of the change in p_1 for the unrestricted LES is obtained by substituting (11) into (9) and evaluating at the appropriate price levels,

(12) $$\begin{aligned} CV_{01} &= -\int_{p_1^0}^{p_1^1} \bar{q}_1(p_1, p_2^0, U_1^0)\, dp_1 = e_1(p_1^0, p_2^0, U_1^0) - e_1(p_1^1, p_2^0, U_1^0) \\ &= m - \frac{\beta_{11} p_1^1}{1-\beta_1} + \frac{1}{\beta_1}\sum_{j\neq 1}\beta_{1j} p_j^0 - \left[m - \frac{\beta_{11} p_1^0}{1-\beta_1} + \frac{1}{\beta_1}\sum_{j\neq 1}\beta_{1j} p_j^0\right] R_1^{\beta_1}, \end{aligned}$$

where $R_i = p_i^1 / p_i^0$ for simplicity of notation. The latter equality follows because, by the budget constraint, $m = e_1(p_1^0, p_2^0, U_1^0)$.

The boundary condition for computing CV along the second segment of the path (in which p_2 changes) requires that utility be held at the initial level (prior to the change in p_1) by deducting from income the compensating variation CV_{01} of the first price change, i.e., the appropriate boundary condition for the second segment of the path is $\bar{q}_1(p_1^1, p_2^0, U_1^0) = q_1(p_1^1, p_2^0, m_0 - CV_{01})$. Solving this condition,

(13) $$U_2^0 = (p_2^0)^{-\beta_2}\left[m - CV_{01} - \frac{\beta_{22} p_2^0}{1-\beta_2} + \frac{1}{\beta_2}\sum_{j\neq 2}\beta_{2j} p_j^1\right],$$

where for simplicity of notation $p_j^1 = p_j^0,\ j = 3,\ldots,N$. Thus, the CV of the second segment of Path L_1 is

(14) $$\begin{aligned} CV_{12} &= -\int_{p_2^0}^{p_2^1} \bar{q}_2(p_1^1, p_2, U_2^0)\, dp_2 = e_2(p_1^1, p_2^0, U_2^0) - e_2(p_1^1, p_2^1, U_2^0) \\ &= m - CV_{01} - \frac{\beta_{22} p_2^1}{1-\beta_2} + \frac{1}{\beta_2}\sum_{j\neq 2}\beta_{2j} p_j^1 - \left[m - CV_{01} - \frac{\beta_{22} p_2^0}{1-\beta_2} + \frac{1}{\beta_2}\sum_{j\neq 2}\beta_{2j} p_j^1\right] R_2^{\beta_2}. \end{aligned}$$

Finally, the CV of the two price changes along Path L_1 is the sum of the compensating variations of the individual price changes,

(15) $$\begin{aligned} CV_{012} = CV_{01} + CV_{12} &= m(1 - R_1^{\beta_1} R_2^{\beta_2}) - \frac{\beta_{11} p_1^1}{1-\beta_1}(1 - R_1^{\beta_1 - 1}) R_2^{\beta_2} - \frac{\beta_{22} p_2^1}{1-\beta_2}(1 - R_2^{\beta_2 - 1}) \\ &\quad + \frac{1}{\beta_1}\sum_{j\neq 1}\beta_{1j} p_j^0 (1 - R_1^{\beta_1}) R_2^{\beta_2} + \frac{1}{\beta_2}\sum_{j\neq 2}\beta_{2j} p_j^0 (1 - R_2^{\beta_2}). \end{aligned}$$

The CV associated with the same price changes taken in opposite order (along Path L_2) yields an expression, CV_{021}, that is the same as (15) except all 1 and 2 subscripts are interchanged.

Because (15) is not symmetric with respect to prices, the two measures of compensating variation are clearly not equivalent. Thus, the compensating variation is path dependent. However, substituting (3) into (15) yields

$$(16)\qquad CV_{012} = CV_{01} + CV_{12} = \left[m - \sum_{j>2} \alpha_j p_j^0 \right] (1 - R_1^{\beta_1} R_2^{\beta_2}) - \sum_{j=1}^{2} \alpha_j p_j^1 + . \sum_{j=1}^{2} \alpha_j p_j^0 R_1^{\beta_1} R_2^{\beta_2}$$

which is both symmetric with respect to subscripts 1 and 2 and consistent with (7). In other words, path independence of compensating variation holds if and only if the ordinary demands, from which compensated demands are derived, are integrable. Of course, similar results can be demonstrated regarding path dependence of equivalent variation.

These results have serious implications for any attempt to derive measures of exact welfare change from an ad hoc demand or system of non-integrable demand equations. This demonstration also underscores the restrictiveness of Hausman's result. Hausman argues that deriving welfare measures from observed market demands is better than starting from a specification for the expenditure or indirect utility function because of the econometric benefits of finding the best functional form that fits the individual demand equation. However, this claim is only applicable to the single price change case and, as shown below, ignores the accuracy in estimation of an individual demand that is possible through system estimation.

When more than one price changes, parameter estimates from multiple demand equations are required. Path independence of willingness to pay measures is attained only by deriving a demand system from a common expenditure or indirect utility function and imposing implied cross-equation parameter restrictions in estimation, or by imposing all integrability restrictions on an arbitary specification of the demand system in estimation. With these considerations, the results of Hausman and Vartia are of questionable usefulness for measuring willingness to pay for multiple price changes. In particular, neither repetitive application of Hausman's approach nor application of Vartia's approach accounts for path dependency. As Vartia acknowledges, his derivation assumes integrability. LaFrance's results show that imposing integrability on arbitrary specifications of demands can result in implausible restrictions. Thus, the only practical way to impose integrability and thus attain path dependence with minimal flexibility is to begin with an expenditure or indirect utility function. But in this case, willingness to pay can be determined directly from the expenditure or indirect utility function upon estimation of the demand parameters. Thus, the Hausman and Vartia methods are not needed.

Failure of Willig Approximations

Another important implication of the failure of integrability is that Willig bounds on errors in consumer surplus as an approximation of compensating and equivalent variation are also not valid. Willig does not mention the dependence of his results on integrability, but this result is easily demonstrated using the LES as an example.

Because the change in consumer surplus is path dependent even when integrability holds due to nonconstant marginal utility of income, any expression for consumer surplus change depends on the path of integration. Consider a path in which income is first changed and then price changes are imposed sequentially from initial to final values. Let $\hat{p}^j \equiv (p_1^1,\ldots,p_j^1,p_{j+1}^0,\ldots,p_N^0)$ and $\breve{p}^j(p_j) \equiv (p_1^1,\ldots,p_{j-1}^1,p_j,p_{j+1}^0,\ldots,p_N^0)$ where $\hat{p}^j$ is a vector representing the *j*th point along a path of integration and $\breve{p}^j$ is a function of p_j along the *j*th segment of the path. A general expression for the change in consumer surplus associated with an income change and price changes from $\hat{p}^0$ to $\hat{p}^N$ is

$$\Delta CS = \int_{m^0}^{m^1} dm - \sum_{i=1}^{N} \int_{\hat{p}^{i-1}}^{\hat{p}^i} q_i(\breve{p}^i, m^1)\, dp_i \,.$$

The corresponding expression for the change in consumer surplus for the restricted (integrable) LES in (2) is

(17) $$\Delta CS = -\sum_{i=1}^{N} \left\{ \ln(R_i)\beta_i \left[m - \sum_{j<i} \alpha_j p_j^1 - \sum_{j>i} \alpha_j p_j^0 \right] + \alpha_i (1-\beta_i)(p_i^1 - p_i^0) \right\}$$

assuming no change in income. Similarly, the change in consumer surplus for the unrestricted LES in (1) is given by

(18) $$\Delta CS = -\sum_{i=1}^{N} \left\{ \ln(R_i) \left[\beta_i m + \sum_{j<i} \beta_{ij} p_j^1 + \sum_{j>i} \beta_{ij} p_j^0 \right] + \beta_{ii}(p_i^1 - p_i^0) \right\}.$$

Willig's result shows that the error in approximating willingness-to-pay welfare measures by the change in consumer surplus is bounded by functions of the change in consumer surplus and income elasticity of demand. However, the path dependence associated with nonconstant marginal utility of income for the change in consumer surplus present in expression (17) and is compounded in (18) by the failure of integrability. This second source of path dependence leads to failure of the Willig bounds.

When integrability fails, Willig's results are invalid for two reasons. First, failure to restrict the demand system to satisfy integrability introduces errors in consumer surplus as evidenced by the difference between (17) and (18). This error enters nonlinearly in the derivation of Willig's error bounds. Second, failure of integrability leads to errors in the measurement of income elasticities. In the LES, the income elasticity of demand for the *i*th good is $\eta_i = \beta_i / s_i$ where s_i is the *i*th budget share. Because, as demonstrated below, errors in unrestricted estimation of the β_i are likely to be comparable to restricted (integrable) estimation of demands, this second source of error may not be serious.

The magnitude of the error in the Willig results will depend on how seriously unrestricted demand estimates (that do not impose integrability) depart from restricted demand estimates (that impose integrability). In the following section, we take a step toward quantifying this comparison.

Assessment of Econometric Error from Failure to Impose Integrability

The LES can be used to demonstrate the effects of errors in parameter estimation when integrability is not imposed. That is, if demands as in (1) are estimated when demands as in (2) apply in reality, then the parameter estimators will satisfy integrability in expectations. Note that estimation of (1) corresponds to independent estimation of individual demand equations as might be the case when the results of independent studies are combined and used to assess welfare effects.

Assume data are available for T periods on N commodity demands. In the LES, a typical and convenient addition of disturbances for econometric purposes follows the form,

$$p_{it}q_{it} = \beta_i m_t + \sum_{j=1}^{N} \beta_{ij} p_{jt} + \varepsilon_{it}\ , \qquad i = 1,\dots,N, \qquad E(\varepsilon_{it}) = 0, \quad E(\varepsilon_{it}\varepsilon_{jt}) = \sigma_{ij}\,. \tag{19}$$

Let

$$Y_i = \begin{bmatrix} p_{i1}q_{i1} \\ \vdots \\ p_{iT}q_{iT} \end{bmatrix}, \quad \mathrm{X} = \begin{bmatrix} m_1 & p_{11} & \cdots & p_{N1} \\ \vdots & \vdots & & \vdots \\ m_T & p_{1T} & \cdots & p_{NT} \end{bmatrix}, \quad B_i = \begin{bmatrix} \beta_i \\ \beta_{i1} \\ \vdots \\ \beta_{iN} \end{bmatrix}, \quad \text{and} \quad \varepsilon_i = \begin{bmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{bmatrix}.$$

Then in matrix notation, (19) becomes

$$Y_i = XB_i + \varepsilon_i\,, \qquad i = 1,\dots,N\,. \tag{20}$$

Unrestricted estimation of this model by ordinary least squares yields parameter estimates $\hat{B}_i = (X'X)^{-1}X'Y_i$, $i = 1,\dots,N$, with $Cov(\hat{B}_i) = \sigma_{ii}(X'X)^{-1}$. If the true demands are integrable, they take the form

$$p_i q_i = \beta_i m + \alpha_i p_i - \beta_i \sum_{j=1}^{N} \alpha_j p_j + \varepsilon_{it}\,, \qquad \sum_{i=1}^{N} \beta_i = 1, \quad i = 1,\dots,N\,. \tag{21}$$

Then with regression following (19), $\hat{\beta}_{ij}$ estimates $-\beta_j\alpha_j$ and $\hat{\beta}_{jj}$ estimates $(1-\beta_j)\alpha_j$.

While estimation of the unrestricted demand system in (19) results in only one estimate of each β_i, because each β_i appears in only one equation, there are potentially N different estimators of each α_j, one implied by each equation in (19). If the primary source of errors in estimation of (19) is errors in estimation of the β_i, then little may be gained by integrable system estimation as in (21). However, if the primary source of errors from estimation of (19) is conflicting implications for the α_j, then imposing integrability as in (21) has substantial econometric benefits. Said another way, if estimates of $\hat{\beta}_{ij}$ from equation (19) lead to widely varying implied estimates of α_j, then the path dependence of compensating and equivalent variation calculations is likely to be large.

In order to focus attention on the more interesting case in which errors in estimation of α_j are relatively important compared to errors in estimation of β_i, assume for the moment that the β_i, $i = 1,\ldots,N$, are known. Estimation of (19) then yields N different estimators of α_j given by $\hat{\beta}_{ij} \big/ \tilde{\beta}_i$, $i = 1,\ldots,N$ where for convenience $\tilde{\beta}_i = -\beta_i$ for $i \neq j$ and $\tilde{\beta}_j = 1 - \beta_j$ otherwise. Each estimator of α_j has variance

$$V(\hat{\beta}_{ij} \big/ \tilde{\beta}_i) = \sigma_{ii} w_j \big/ \tilde{\beta}_i^2 , \tag{22}$$

where w_j is the (j+1)th diagonal element of $(X'X)^{-1}$. An efficient estimator of α_j can be found by using $Cov(\hat{B}_{\bullet i}, \hat{B}_{\bullet j}) = \sigma_{ij}(X'X)^{-1}$ which implies

$$Cov\left(\hat{\beta}_{\bullet j}\right) \equiv Cov \begin{bmatrix} \hat{\beta}_{1j} \\ \vdots \\ \hat{\beta}_{Nj} \end{bmatrix} = w_j \Sigma$$

where Σ is the covariance matrix, $\Sigma = \{\sigma_{ij}\}$.

If the true demand system satisfies integrability as in (21) with $\sum_{i=1}^{N} \beta_i = 1$, then $E(Z\hat{\beta}_{\bullet j}) = e\alpha_j$ where $e' = (1,\ldots,1)$ and Z is a diagonal matrix with $Z^{-1} = diag\{\tilde{\beta}_1, \ldots, \tilde{\beta}_N\}$. A suitable regression model for estimating α_j is $Z\hat{\beta}_{\bullet j} = e\alpha_j + \delta_j$, $E(\delta_j) = 0$, $Cov(\delta_j) = w_j Z\Sigma Z$, for which an efficient estimator is the Aitken estimator,

$$\hat{\alpha}_j = [e'(Z\Sigma Z)^{-1} e]^{-1} e'(Z\Sigma Z)^{-1} Z\hat{\beta}_{\bullet j} = \left[\sum_{i=1}^{N} \sum_{k=1}^{N} \tilde{\beta}_i \tilde{\beta}_k \sigma^{ik} \right]^{-1} \sum_{i=1}^{N} \sum_{k=1}^{N} \tilde{\beta}_i \sigma^{ik} \hat{\beta}_{kj} , \tag{23}$$

where $\Sigma^{-1} = \{\sigma^{ij}\}$. The corresponding variance is

$$V(\hat{\alpha}_j) = w_j \left[\sum_{i=1}^{N} \sum_{k=1}^{N} \tilde{\beta}_i \tilde{\beta}_k \sigma^{ik} \right]^{-1} \sum_{i=1}^{N} \sum_{k=1}^{N} \tilde{\beta}_i \sigma^{ik} \hat{\beta}_{kj} . \tag{24}$$

The variance of the efficient estimator in (24) can be compared to the variance of independent estimators in (22) to determine the magnitude of the error in estimation of the α_j when integrability is not imposed. To facilitate simple comparison, suppose $\sigma_{ij} = 0 \ \forall \ i \neq j$ so $\sigma^{ij} = 0$ and $\sigma^{ii} = 1/\sigma_{ii}$. Suppose also that the standard error of expenditure equation disturbances is proportional to income coefficients, i.e., $\sigma_{ii} = k\beta_i^2$ for some k. This assumption is consistent with the plausible case where larger expenditure errors are made on those commodities for which expenditures are larger. The variances of α_j estimators from (22) are thus

$$V(\hat{\beta}_{ij} / \tilde{\beta}_i) = \frac{\sigma_{ii} w_j}{\beta_i^2} = kw_j , \quad \text{for } i \neq j, \tag{25}$$

$$(26) \qquad V(\hat{\beta}_{ii}/\tilde{\beta}_i) = \frac{\sigma_{ii} w_i}{(1-\beta_i)^2} = kw_i \left[\frac{\beta_i}{1-\beta_i}\right]^2, \text{ for } i=j$$

whereas the variance of the efficient estimator is

$$(27) \qquad V(\hat{\alpha}_j) = w_j \left[\sum_{i=1}^{N} \tilde{\beta}_i^2 \frac{1}{\sigma_{ii}}\right]^{-1} = kw_j \left\{N - 1 + \left[\frac{1-\beta_j}{\beta_j}\right]^2\right\}^{-1}.$$

Comparing (26) and (27) reveals that unconstrained estimation causes an *N*-fold increase in the variance of the own-price elasticities if $\beta_i = 1/2$. To assist in interpreting results, note that

$$\beta_i = \frac{\partial p_i q_i}{\partial m} = p_i \frac{\partial q_i}{\partial m} = \frac{p_i q_i}{m} \eta_i = s_i \eta_i$$

where η_i is the income elasticity and s_i is the *i*th budget share. Generally, one can show that (27) is less than (26) for $0 < \beta_i < 1$ and that the difference is increasing in β_i and *N*. If *N* = 20 and $\beta_i = 1/20$, on the other hand, the ratio of (26) to (27) is only 20/19. Thus, if income elasticities tend to cluster around 1, accuracy is improved only about 5 percent with constrained estimation when budget shares are about 5 percent. As a general rule of thumb, if $\beta_i = 1/N$, then the ratio of (26) to (27) is $N/(N-1)$, or more generally, if $\beta_i = \zeta/N$, the ratio is $1 + (N-1)\zeta^2/(N-\zeta)^2$. Overall, comparison of (26) and (27) reveals that the variance of errors in estimates of own price coefficients is somewhat lower when integrability is imposed as in (27) but the difference is not great for moderate-sized systems of demands.

Turning to cross-price coefficients, however, the relevant comparison is between (25) and (27). If $\beta_i = 1/2$, then (25) represents an *N*-fold increase in variance. In the more plausible case where on average $\beta_i = 1/N$, (25) represents *N*(*N* – 1)-fold increase in variance over the efficient system estimator. More generally, if $\beta_i = \zeta/N$, then ratio of (25) to (27) is $N - 1 + (N-\zeta)^2/\zeta^2$. Clearly the errors for estimating cross-price coefficients tend to be much larger when the rest of the demand system and associated integrability coefficients are not considersd. These results show for moderate-sized systems the variances of cross-price coefficients differ by about a second-order of magnitude in the number of demands in the system.[4]

Conclusions

This paper has investigated the problem of path dependence that occurs in willingness-to-pay measures of welfare change when integrability does not hold in the system of demands from which it is derived. For measures of the welfare benefits associated with changes in more than one price, the failure of integrability introduces error in willingness-to-pay measures. Furthermore, Willig's error bounds on consumer surplus as an approximation of willingness to pay also fail with failure of integrability. These problems are serious for practical policy evaluation studies that rely on piecemeal estimates of demands or elasticities from various other studies.

For problems that involve welfare measurement for multiple price changes, the results of this paper suggest a considerable benefit to estimation of integrable systems of demands, which for practical purposes and minimal flexibility implies estimation of a demand system derived from an explicit specification of the expenditure or indirect utility function. Without imposing the associated integrability, estimates of individual demands may be unbiased, e.g., if the functional specification for the individual demand is appropriate. However, the variances of errors in estimation of key price elasticities are likely to be larger by an order of magnitude if demands are not estimated as a system with all necessary cross-equation parameter restrictions imposed. Furthermore, the researcher will be in a position to influence the magnitude of estimated compensating or equivalent variation by the choice of order in which price changes are considered.

Footnotes

---

[1] For the purposes of this paper, consumer surplus is defined as the area under the ordinary demand curve and above price following Just, Hueth, and Schmitz. Compensating and equivalent variation then follow the parallel Hicksian surplus defined as the area behind Hicksian demand curves and above price with utility held constant at the initial or subsequent levels, respectively.

[2] Willig demonstrated his rule of thumb for the case of a single price change, but mentions similar results for multiple price changes.

[3] Because this system incorporates minimal nonlinearity in variables to satisfy homogeneity, it is far less restrictive than the linear or log-linear demands analyzed by LaFrance (1985, 1986).

[4] Similar conclusions apply to estimation of both own- and cross-price elasticities because they are proportional to the LES price coefficients. In the unrestricted LES in equation (19), own-price elasticities are given by $\varepsilon_{ii} = \beta_{ii}/q_i - 1$, while cross-price elasticities take the form $\varepsilon_{ij} = (p_j / p_i)(\beta_{ij} / q_i)$. Since $\hat{\beta}_{ij}$ estimates $-\beta_i\alpha_j$ and $\hat{\beta}_{ii}$ estimates $(1-\beta_i)\alpha_j$, the implied estimates of own- and cross-price elasticity consistent with the restricted LES are $\varepsilon_{ii} = (1-\beta_i)\alpha_i / q_i - 1$ and $\varepsilon_{ij} = -(p_j / p_i)(\beta_i\alpha_j / q_i)$, respectively. Thus, the error in estimation of own- and cross-price elasticities is proportional to the error in estimation of α_j discussed above.

References

Deaton, Angus. *Models and Projections of Demand in Post-War Britain*. New York: John Wiley & Sons, 1975.

Deaton, Angus, and John Muellbauer. *Economics and Consumer Behavior*. New York: Cambridge University Press, 1980.

Geary, R. C. "A Note on 'A Constant-Utility Index of the Cost of Living'." *Review of Economic Studies* 18, no. 1 (1949-50): 65-66.

Hause, John C. "The Theory of Welfare Cost Measurement." *Journal of Political Economy* 83, no. 6 (December 1975): 1145-82.

Hausman, Jerry A. "Exact Consumer's Surplus and Deadweight Loss`." *American Economic Review* 71, no. 4 (September 1981): 662-76.

Just, Richard E., Darrell L. Hueth, and Andrew Schmitz. *The Economics of Public Policy Evaluation*. forthcoming.

Klein, L. R., and H. Rubin. "A Constant-Utility Index of the Cost of Living." *Review of Economic Studies* 15 (1948-49): 84-87.

LaFrance, Jeffrey T. "Linear Demand Fucntions in Theory and Practice." *Journal of Economic Theory* 37, no. 1 (October 1985): 147-66.

———. "The Structure of Constant Elasticity Demand Models." *American Journal of Agricultural Economics* 68, no. 3 (August 1986): 543-52.

Stone, Richard. "Linear Expenditure Systems and Demand Analysis: An Application to the Pattern of British Demand." *Economic Journal* 64, no. 255 (September 1954): 511-27.

Vartia, Yrjo O. "Efficient Methods of Measuring Welfare Change and Compensated Income in Terms of Ordinary Demand Functions," *Econometrica* 51 (1983):79-98.

Willig, Robert D. "Consumer's Surplus Without Apology." *American Economic Review* 66, no. 4 (September 1976): 589-97.