Department of Agricultural and Resource Economics
University of California, Davis

Positive Mathematical Programming with Generalized Risk: A Revision

by

Quirino Paris

Working Paper No. 15-002

April 2015

Giannini Foundation of Agricultural Economics

Positive Mathematical Programming with Generalized Risk: A Revision

Quirino Paris

April 2015

Abstract

In 1956, Freund introduced the analysis of price risk in a mathematical programming framework. This paper generalizes the treatment of price risk preferences in a mathematical programming framework along the lines suggested by Meyer (1987) who demonstrated the equivalence of expected utility and a wide class of probability distributions that differ only by location and scale. This paper shows how to formulate a Positive Mathematical Programming (PMP) specification that allows the estimation of the risk preference parameters and calibrates the model to the base data within admissible small deviations. The PMP approach under generalized risk allows also the estimation of output supply elasticities and the response analysis of decoupled farm subsidies that, recently, has interested policy makers. The approach is applied to a sample of large farms. Not all farms produce all commodities.

Keywords: positive mathematical programming, generalized risk, output supply elasticities, policy analysis

JEL: C6

Quirino Paris is a professor in the Department of Agricultural and Resource Economics, University of California, Davis.

1. Introduction

The treatment of price risk in a mathematical programming framework has been confined either to an exponential utility function with constant absolute risk aversion or to a minimization of total absolute deviation (MOTAD) of income. The first approach, originally proposed by Freund (1956), appealed to the expected utility (EU) hypothesis and assumed that random prices were normally distributed. These assumptions lead to a mean-variance specification of expected net revenue defined as total expected revenue minus a risk premium. Such a premium corresponds to half the variance of revenue multiplied by a constant absolute risk aversion coefficient. This mathematical programming approach has serious limitations as only an unlikely entrepreneur may possess risk preferences that exhibit constant absolute risk aversion regardless of firm size, wealth endowment and risky market environment. The MOTAD approach was proposed by Hazell (1971) who justified its introduction with the difficult access to a quadratic programming computer software necessary to solve a mean-variance model. In contrast, according to Hazell (1971, p. 56), the MOTAD specification "has an important advantage over the mean-variance criterion in that it leads to a linear programming model in deriving the efficient mean-absolute deviation farm plans." The MOTAD model approximates a mean-standard deviation (MS) criterion but it says nothing about the economic agent's risk preferences with regard to either decreasing (constant, increasing) absolute or relative risk aversion.

The mean-standard deviation approach has a long history [Fisher (1906), Hicks (1933), Tintner (1941), Markowitz (1952), Tobin (1958)]. Meyer (1987) presented a remarkable reconciliation between the EU and the MS approaches that may be fruitfully applied in a positive mathematical programming (PMP) analysis of economic behavior under risk. The main objective of Meyer was to find consistency conditions between the EU and the MS approaches in such a way that an agent who ranks the available alternatives according to the value of some function defined over the first two moments of the random payoff would rank in the same way those alternatives by means of the expected value of some utility function defined over the same payoffs. It turns out that the location and scale condition is the crucial link to establish the consistency between the EU and the MS approaches. We reproduce here Meyer's argument (1987, p. 423):

"Assume a choice set in which all random variables Y_i (with finite means and variances) differ from one another only by location and scale parameters. Let X be the random variable obtained from one of the Y_i using the normalizing transformation $X = (Y_i - \mu_i)\sigma_i$ where μ_i and σ_i are the mean and standard deviation of Y_i. All Y_i, no matter which was selected to define X, are equal *in distribution* to $\mu_i + \sigma_i X$. Hence, the expected utility from Y_i for any agent with utility function $u(\ )$ can be written as

$$EU(Y_i) = \int_a^b u(\mu_i + \sigma_i x)dF(x) \equiv V(\mu_i, \sigma_i) \tag{1}$$

where a and b define the interval containing the support of the normalized random variable X."

"… under the location and scale condition, various popular and interesting hypotheses concerning absolute and relative risk-aversion measures in the EU setting can be translated into equivalent properties concerning $V(\mu_i, \sigma_i)$."

The structure of absolute risk is measured by the slope of the indifference curves in the (μ, σ) space that is represented as

(2) $$AR(\mu,\sigma) = \frac{-V_\sigma(\mu,\sigma)}{V_\mu(\mu,\sigma)}$$

where $V_\mu(\mu,\sigma)$ and $V_\sigma(\mu,\sigma)$ are first partial derivatives of the $V(\mu,\sigma)$ function. Some properties of this risk measure are:

1. Risk aversion is associated with $AR(\mu,\sigma) > 0$, risk neutrality with $AR(\mu,\sigma) = 0$ and risk propensity with $AR(\mu,\sigma) < 0$.

2. If $u(\mu + \sigma x)$ displays decreasing (constant, increasing) absolute risk aversion for all $\mu + \sigma x$, then $\frac{\partial AR(\mu,\sigma)}{\partial \mu} < (=,>)\ 0$ for all μ and $\sigma > 0$.

3. If $\mu + \sigma x$ displays increasing (constant, decreasing) relative risk aversion for all $\mu + \sigma x$, then $\frac{\partial AR(t\mu,t\sigma)}{\partial t} > (=,<)\ 0$ for $t > 0$.

Saha (1997) proposed a two-parameter MS utility function that conforms to Meyer's specification

(3) $$V(\mu,\sigma) = \mu^\theta - \sigma^\gamma$$

and assumed that $\theta > 0$. According to this MS utility function, the absolute risk measure (*AR*) is specified as

(4) $$AR(\mu,\sigma) = \frac{-V_\sigma(\mu,\sigma)}{V_\mu(\mu,\sigma)} = \frac{\gamma}{\theta}\mu^{(1-\theta)}\sigma^{(\gamma-1)}.$$

Hence, risk aversion, risk neutrality and risk propensity are associated with $\gamma > (=,<)\ 0$, respectively.

Decreasing, constant and increasing absolute risk aversion ($\gamma > 0$) is defined by

(5) $$\frac{\partial AR(\mu,\sigma)}{\partial \mu} = \frac{(1-\theta)\gamma}{\theta}\mu^{-\theta}\sigma^{(1-\gamma)} < (=,>)\ 0$$

and, therefore, by $\theta > 1$, $\theta = 1$, $\theta < 1$, respectively.

Decreasing, constant and increasing relative risk aversion is defined ($\gamma > 0$) by

(6) $$\frac{\partial AR(\mu,\sigma)}{\partial \mu}\Big|_{t=1} = (\gamma - \theta)AR < (=,>)\ 0$$

and, therefore, by $\theta > \gamma$, $\theta = \gamma$, $\theta < \gamma$, respectively.

The risk analysis of Meyer (1987) admits all possible combinations of risk behavior. Saha (1997) listed these combinations for the MS utility function specified in relation (3). Table 1, for example, admits absolute risk aversion behavior that may be decreasing, when $\theta > 1$ and $\gamma > 0$, in association with either increasing relative risk aversion when $\gamma > \theta > 0$ or decreasing relative risk aversion when $\theta > \gamma > 0$. The meaning of decreasing absolute risk aversion relates to an economic agent who experiences a wealth increase and chooses to augment his investment – measured in absolute terms – in the risky asset. Decreasing relative risk aversion relates to an economic agent who experiences a wealth increase and chooses to increase the share of his investment in the risky asset. It is possible, therefore, for an economic agent to behave according to a decreasing absolute risk aversion framework and an increasing relative risk aversion scenario if the absolute amount of increase in the risky asset is not sufficient to increase also the share of that asset. In any given sample of economic agents' performances, therefore, the prevailing combination of risk preferences is an empirical question.

Table 1. Possible Combinations of Risk Behavior Under a MS Utility

		Relative Risk Aversion		
		Decreasing	Constant	Increasing
Absolute Risk Aversion	Decreasing	$\theta > 1$, $\theta > \gamma$	$\theta > 1$, $\theta = \gamma$	$\theta > 1$, $\theta < \gamma$
	Constant	$\theta = 1$, $\theta > \gamma$	$\theta = 1$, $\theta = \gamma$	$\theta = 1$, $\theta < \gamma$
	Increasing	$\theta < 1$, $\theta > \gamma$	$\theta < 1$, $\theta = \gamma$	$\theta < 1$, $\theta < \gamma$

The rest of the paper is organized as follows. Sections 2 and 3 discuss a novel PMP model that integrates a generalized risk analysis with an extension of calibration constraints involving observed prices of limiting inputs. This extension modifies the traditional PMP specification of calibration.n constraints involving observed levels of realized outputs. In particular, the extension avoids the user-determined perturbation parameters introduced by Howitt (1995a, 1995b) to guarantee that the dual variables of binding structural constraints will assume positive values. Section 4 defines and estimates a total cost function involving output quantities and limiting input prices. The derivatives of the cost function are used in calibrating models that are suitable for policy analysis. Section 5 discusses how to obtain endogenous (to a farm sample) output supply elasticities. Section 6 matches exogenous (to the farm sample) supply elasticities (available through econometric estimation, for example) with the endogenous supply elasticities. Section 7 defines two alternative calibrating equilibrium models that reproduce calibrating solutions that are identical to those ones obtained in section 3. Section 8 presents the empirical results of the more elaborate PMP and risky model applied to a sample of 14 farms when all farms produce all commodities. Section 9 deals with a more realistic sample of information where not all farms produce all commodities. Conclusions follow.

2. Generalized Risk in a PMP Framework

A Positive Mathematical Programming approach has been adopted frequently to analyze agricultural policy scenarios ever since Howitt proposed the methodology in 1995 (1995a, 1995b). Apparently, all the empirical applications of PMP that appeared in the literature to date dealt with economic scenarios in the absence of risk involving either prices or other parameters. In this section, we extend the PMP methodology to deal with generalized risk preferences and risky market output prices. Furthermore, we extend the PMP methodology to deal with calibration constraints involving observed prices of limiting inputs, say land. This extension modifies the traditional specification of calibration constraints and the notion of a calibrating solution, as explained further on.

Suppose N farmers produce J crops using I limiting inputs and a linear technology. Let us assume that, for each farmer, the $(J \times 1)$ vector of crops' market prices is a random variable $\tilde{\mathbf{p}}$ with mean $E(\tilde{\mathbf{p}})$ and variance-covariance matrix Σ_p. A $(J \times 1)$ vector $\mathbf{c}$ of accounting unit costs is also known. The $(I \times 1)$ vector $\mathbf{b}$ indicates farmer's availability of limiting resources. The matrix A of dimensions $(I \times J, I < J)$ specifies a linear technology. The $(J \times 1)$ vector $\mathbf{x}$ symbolizes the unknown output levels. Furthermore, farmer has knowledge of previously realized levels of outputs that are observed (by the econometrician) as $\mathbf{x}_{obs}$. Random wealth is defined by previously accumulated wealth, $\overline{w}$, augmented by the current random net revenue. Assuming a MS utility function under this scenario, mean wealth is defined as $\mu = [\overline{w} + (E(\underset{\cdot\cdot}{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]$ with standard deviation equal to $\sigma = (\mathbf{x}'\Sigma_p \mathbf{x})^{1/2}$.

Then, a primal PMP-MS model is specified as follows:

$$\text{(7)} \qquad \max_{\mathbf{x},\mathbf{h},\theta,\gamma} V(\mu,\sigma) = \mu^{\theta} - \sigma^{\gamma} = [\overline{w} + (E(\underset{\cdot\cdot}{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{\theta} - (\mathbf{x}'\Sigma_p \mathbf{x})^{\gamma/2}$$

subject to	$A\mathbf{x} \le \mathbf{b}$	dual variable $\mathbf{y}$
	$\mathbf{x} = \mathbf{x}_{obs} + \mathbf{h}$	dual variable $\boldsymbol{\lambda}$

where $\mathbf{h}$ is a vector of deviations from the realized and observed output levels. The first set of constraints forms the structural (technological) relations while the second set constitutes the calibration constraints. This specification of the calibration constraints differs from the traditional statement according to which ($\mathbf{x} \le \mathbf{x}_{obs} + \boldsymbol{\varepsilon}$) where $\boldsymbol{\varepsilon}$ is a user-determined vector of small positive numbers whose purpose is to allow the dual variables of binding structural constraints to take on positive values. In Howitt's words (1995a, p. 151): "The $\boldsymbol{\varepsilon}$ perturbation on the calibration constraints decouples the true resource constraints from the calibration constraints and ensures that the dual values on the allocable resources represent the marginal values of the resource constraints." This paper avoids the user-determined parameter $\boldsymbol{\varepsilon}$ of the traditional PMP methodology and allows the empirical data to reveal the components of the vector of deviations $\mathbf{h}$. Such deviations can take on either positive or negative values. To justify further the specification of the calibration constraints $\mathbf{x} = \mathbf{x}_{obs} + \mathbf{h}$, we note that the vector of realized output levels, $\mathbf{x}_{obs}$, has been "observed", that is measured, by persons other than the economic entrepreneur, say by an econometrician. It is likely, therefore, that the measured $\mathbf{x}_{obs}$ may either overstate or understate the true levels of realized outputs. The deviation vector $\mathbf{h}$ captures these likely measurement errors.

The dual constraints of problem (7) – derived by Lagrange method – turn out to be

(8) $$\gamma(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x} + A'\mathbf{y} + \boldsymbol{\lambda} \ge \theta[\overline{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(\theta-1)}[E(\tilde{\mathbf{p}}) - \mathbf{c}]\,.$$

The complexity of the estimation problem becomes clear by considering the nonlinearity of relation (8). Parameters θ and γ are unknown as are the output levels, $\mathbf{x}$, the deviations, $\mathbf{h}$, the dual variables, $\mathbf{y}$, and the Lagrange multipliers, $\boldsymbol{\lambda}$. Furthermore, it is often the case that also the market price of some input – say land – is known for the region surrounding the sample farms or even for a single farm. The PMP methodology, therefore, ought to use also this information, $\mathbf{y}_{obs}$, that will be treated in the form of the observed output levels as

(9) $$\mathbf{y} = \mathbf{y}_{obs} + \mathbf{u}$$

where $\mathbf{u}$ is an $(I \times 1)$ vector of deviations from the observed input prices.

Let W be a nonsingular diagonal matrix of dimensions $(J \times J)$ with positive diagonal terms equal to observed expected price $E(\tilde{p}_j) > 0$. And let V be a nonsingular diagonal matrix of dimensions $(I \times I)$ with positive diagonal terms $b_i / y_{obs,i} > 0$. The purpose of matrices W and V is twofold. First, to render homogeneous the units of measurement of all terms in the models defined below. Second, to weigh the deviations $\mathbf{h}$ and $\mathbf{u}$ according to the scale of the corresponding expected price and input size, respectively. Using a least-squares approach for the estimation of deviations $\mathbf{h}$ and $\mathbf{u}$, it turns out that, by the self-duality of least squares (LS), $\boldsymbol{\lambda} = W\mathbf{h}$ and $\boldsymbol{\psi} = V\mathbf{u}$, where $\boldsymbol{\psi}$ is the vector of Lagrange multipliers associated with constraints (9). To show this result, consider the following LS problem

(10) $$\min LS = \mathbf{h}'W\mathbf{h}/2 + \mathbf{u}'V\mathbf{u}/2$$

(11) subject to $$\mathbf{x} = \mathbf{x}_{obs} + \mathbf{h} \qquad \text{dual variable } \boldsymbol{\lambda}$$

$$\mathbf{y} = \mathbf{y}_{obs} + \mathbf{u} \qquad \text{dual variable } \boldsymbol{\psi}.$$

The corresponding Lagrange function and first order necessary conditions with respect to $\mathbf{h}$ and $\mathbf{u}$ are

(12) $$L = \mathbf{h}'W\mathbf{h}/2 + \mathbf{u}'V\mathbf{u}/2 + \boldsymbol{\lambda}'(\mathbf{x} - \mathbf{x}_{obs} - \mathbf{h}) + \boldsymbol{\psi}'(\mathbf{y} - \mathbf{y}_{obs} - \mathbf{u})$$

(13) $$\frac{\partial L}{\partial \mathbf{h}} = W\mathbf{h} - \boldsymbol{\lambda} = \mathbf{0}$$

(14) $$\frac{\partial L}{\partial \mathbf{u}} = V\mathbf{u} - \boldsymbol{\psi} = \mathbf{0}$$

with the result that $\boldsymbol{\lambda} = W\mathbf{h}$ and $\boldsymbol{\psi} = V\mathbf{u}$ as claimed above.

A crucial issue concerns parameters θ and γ. On the one hand, we assume that an economic entrepreneur wishes to maximize her utility of wealth while minimizing the disutility of its risk. On the other hand, it is a fact that high levels of current income (a component of wealth) are associated with high risk of losses. Another fact is that this entrepreneur has already made her choice of a production plan, $\mathbf{x}_{obs}$, in the face of output price risk. It is also likely that she does not know (or that she is not even aware of) parameters θ and γ. The challenge, therefore, is to infer – from her decisions – the values of parameters θ and γ that could explain the behavior of this entrepreneur in a rational fashion.

We assume that this entrepreneur is risk averse, implying that $\theta > 0$ and $\gamma > 0$. Furthermore, for any given level of expected wealth, a high level of utility will be achieved with the highest admissible level of parameter θ, where admissibility depends on the technology, the limiting input constraints, the observed production plan and the observed input prices. An alternative viewpoint, one that mimics the relationship between high levels of random wealth and high levels of its standard deviation, would postulate that high levels of utility (of wealth) are associated with high levels of its risk disutility. Therefore, for any given level of the standard deviation of wealth, the parameter γ should acquire the highest admissible value, given the technology, the observed production plan and input prices.

3. Phase I PMP Model

For estimation purposes, therefore, the squares of parameters θ and γ will be maximized together with the minimization of deviations $\mathbf{h}$ and $\mathbf{u}$ in a least-squares objective function subject to relevant primal and dual constraints and their associated complementary slackness conditions. This task leads to the following phase I model

(15) $$\min LS = \mathbf{h}'W\mathbf{h}/2 + \mathbf{u}'V\mathbf{u}/2 - \theta^2 - \gamma^2$$

subject to

(16) $$A\mathbf{x} \leq \mathbf{b} + V\mathbf{u}$$

(17) $$\theta[\overline{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(\theta-1)}[E(\tilde{\mathbf{p}}) - \mathbf{c}] \leq A'\mathbf{y} + W\mathbf{h} + \gamma(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x}$$

(18) $$\mathbf{x} = \mathbf{x}_{obs} + \mathbf{h}$$

(19) $$\mathbf{y} = \mathbf{y}_{obs} + \mathbf{u}$$

(20) $$\mathbf{y}'(\mathbf{b} + V\mathbf{u} - A\mathbf{x}) = 0$$

(21) $$\mathbf{x}'\{A'\mathbf{y} + W\mathbf{h} + \gamma(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x} - \theta[\overline{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(\theta-1)}[E(\tilde{\mathbf{p}}) - \mathbf{c}]\} = 0$$

with $\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \theta \geq 0, \gamma \geq 0$, $\mathbf{h}$ and $\mathbf{u}$ free.

With the specification of the calibration constraints as in relations (18) and (19), the notion of a PMP calibrating solution differs from the traditional concept according to which the optimal calibrating solution is equal to the observed output levels, that is, $\mathbf{x}^* \cong \mathbf{x}_{obs}$, as the perturbation vector $\boldsymbol{\varepsilon}$ contains very small (user determined) positive numbers. Critics of PMP have judged this solution as being tautological. With the methodology proposed in this paper, a calibrating solution $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ will not, in general, be exactly equal to the corresponding vectors of the observed production plan and input prices $(\mathbf{x}_{obs}, \mathbf{y}_{obs})$. The objective of model (15)-(21), therefore, is to minimize the deviations $\mathbf{h}$ and $\mathbf{u}$ in the amount allowed by the technological and risky environments facing farmers. Hence, the specification (15)-(21) takes on the features of an econometric estimation problem.

Constraints (16) represent the structural (technological) relations of input demand being less-than-or-equal to the effective input supply. Constraints (17) represent the dual relations with marginal utility of the production plan being less-than-or-equal to its marginal cost. Here marginal cost has two parts: the marginal cost due to limiting and variable inputs, $A'\mathbf{y}+W\mathbf{h}$, and the marginal cost of output price risk, $\gamma(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x}$. Constraints (18) and (19) are the calibration relations. Constraints (20) and (21) are complementary slackness conditions. Because constraints (16)-(21) represent primal and dual relations and their complementary slackness conditions, any feasible solution of relations (16)-(21) constitutes an admissible economic equilibrium that is consistent with the behavior of decision making under price risk. The complexity of the model constraints may admit local optima. The GAMS software used in the empirical analysis includes the solver BARON (Branch And Reduce Optimization Navigator) for the global solution of nonlinear problems. The user manual states (2015): "… BARON implements deterministic and global optimization algorithms of the branch-and-bound type that are guaranteed to provide global optima under fairly general assumptions. These assumptions include the existence of finite lower and upper bounds on nonlinear expressions to be solved." Hence, using the Baron solver, it is possible to find equilibrium solutions that are close to the global optimum.

4. Phase II PMP Model

Phase II of the PMP methodology deals with the estimation of a cost function that embodies all the technological and behavioral information revealed in phase I. Typically, a marginal cost function expresses a portion of the dual constraints in a phase I PMP model. In the absence of risk, PMP marginal cost is defined as $A'\mathbf{y}+W\mathbf{h}+\mathbf{c}$, where $A'\mathbf{y}$ stands for the marginal cost due to limiting inputs and $W\mathbf{h}+\mathbf{c}$ for the effective marginal cost due to variable outputs. In the risky price case, marginal cost is given by the right-hand-side of relation (17) where all the elements are measured in utility units. We desire to obtain a dollar expression of marginal cost, as in the familiar relation $MC \geq E(\tilde{\mathbf{p}})$. To achieve this result, the elements of relation (17) will be divided by the term $\theta[\overline{w}+(E(\tilde{\mathbf{p}})-\mathbf{c})'\mathbf{x}]^{(\theta-1)}$ to write

(22) $$MC \geq E(\tilde{\mathbf{p}})$$

$$\mathbf{c}+\frac{1}{\theta}[\overline{w}+(E(\tilde{\mathbf{p}})-\mathbf{c})'\mathbf{x}]^{(1-\theta)}[A'\mathbf{y}+W\mathbf{h}]+\frac{\gamma}{\theta}[\overline{w}+(E(\tilde{\mathbf{p}})-\mathbf{c})'\mathbf{x}]^{(1-\theta)}(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x} \geq E(\tilde{\mathbf{p}})$$

In relation (22), all the terms are measured in dollars. The marginal cost due to limiting and variable inputs is given by $\left\{\mathbf{c}+\frac{1}{\theta}[\overline{w}+(E(\tilde{\mathbf{p}})-\mathbf{c})'\mathbf{x}]^{(1-\theta)}[A'\mathbf{y}+W\mathbf{h}]\right\}$. The marginal cost due to risky output prices is given by. $\left\{\frac{\gamma}{\theta}[\overline{w}+(E(\tilde{\mathbf{p}})-\mathbf{c})'\mathbf{x}]^{(1-\theta)}(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x}\right\}$.

The cost function selected to synthesize the technological and behavioral relations of phase I is expressed as a modified Leontief cost function such as

(23) $$C(\mathbf{x},\mathbf{y}) = (\mathbf{f}'\mathbf{x})(\mathbf{g}'\mathbf{y})+(\mathbf{g}'\mathbf{y})(\mathbf{x}'Q\mathbf{x})/2+(\mathbf{f}'\mathbf{x})[(\mathbf{y}^{1/2})'G\mathbf{y}^{1/2}].$$

A cost function is linear homogeneous and concave in input prices, $\mathbf{y}$. Therefore, matrix G is negative semidefinite. Furthermore, a cost function is increasing in output levels. Thus, matrix Q is positive semidefinite. Parameters $\mathbf{f}$ and $\mathbf{g}$ give flexibility to the cost function.

The marginal cost function associated with cost function (23) is given by

(24) $$MC_{\mathbf{x}} = \frac{\partial C}{\partial \mathbf{x}} = \mathbf{f}(\mathbf{g}'\mathbf{y})+(\mathbf{g}'\mathbf{y})Q\mathbf{x}+\mathbf{f}[(\mathbf{y}^{1/2})'G\mathbf{y}^{1/2}]$$

The derivative of the cost function with respect to input prices corresponds to Shephard lemma that produces the demand function for inputs:

(25) $$\frac{\partial C}{\partial \mathbf{y}} = (\mathbf{f}'\mathbf{x})\mathbf{g} + \mathbf{g}(\mathbf{x}'Q\mathbf{x})/2 + (\mathbf{f}'\mathbf{x})[\Delta(\mathbf{y}^{-1/2})'G\mathbf{y}^{1/2}] = A\mathbf{x}$$

where $\Delta(\mathbf{y}^{-1/2})$ represents a diagonal matrix with elements $y_i^{-1/2}$ on the main diagonal.

With knowledge of the solution components resulting from the phase I model (15)-(21), $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{h}},\hat{\mathbf{u}},\hat{\theta},\hat{\gamma}$, a phase II model's goal is to estimate the parameters of the cost function, $\mathbf{f},\mathbf{g},Q,G$. This task is accomplished by means of the following specification

(26) $$\min Aux = \mathbf{d}'\mathbf{d}/2 + \mathbf{r}'\mathbf{r}/2$$

subject to

(27) $$\mathbf{f}(\mathbf{g}'\hat{\mathbf{y}}) + (\mathbf{g}'\hat{\mathbf{y}})Q\hat{\mathbf{x}} + \mathbf{f}[(\hat{\mathbf{y}}^{1/2})'G\hat{\mathbf{y}}^{1/2}] =$$

$$\mathbf{c} + \frac{1}{\hat{\theta}}[\overline{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\hat{\mathbf{x}}]^{(1-\hat{\theta})}[A'\hat{\mathbf{y}} + W\hat{\mathbf{h}}] + \frac{\hat{\gamma}}{\hat{\theta}}[\overline{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\hat{\mathbf{x}}]^{(1-\hat{\theta})}(\hat{\mathbf{x}}'\Sigma_p\hat{\mathbf{x}})^{(\hat{\gamma}/2-1)}\Sigma_p\hat{\mathbf{x}} + \mathbf{d} \geq E(\tilde{\mathbf{p}})$$

(28) $$(\mathbf{f}'\hat{\mathbf{x}})\mathbf{g} + \mathbf{g}(\hat{\mathbf{x}}'Q\hat{\mathbf{x}})/2 + (\mathbf{f}'\hat{\mathbf{x}})[\Delta(\hat{\mathbf{y}}^{-1/2})'G\hat{\mathbf{y}}^{1/2}] = A\hat{\mathbf{x}} + \mathbf{r}$$

(29) $$Q = LDL'$$

(30) $$QQ^{-1} = I$$

with $\mathbf{f}'\hat{\mathbf{x}} > 0, \mathbf{g}'\hat{\mathbf{y}} > 0, D > 0$, $\mathbf{f}$ and $\mathbf{g}$ free, $\mathbf{d} \geq \mathbf{0}, \mathbf{r} \geq \mathbf{0}$. The nonnegative vector variables $\mathbf{d} \geq \mathbf{0}, \mathbf{r} \geq \mathbf{0}$ perform the role of pseudo slack variables necessary to provide the GAMS solver with an objective function to optimize. The optimal value of $\mathbf{d} \geq \mathbf{0}, \mathbf{r} \geq \mathbf{0}$ is identically equal to zero for all the pseudo slack variables.

Relation (27) represents $MC \geq E(\tilde{\mathbf{p}})$. Relation (28) is Shephard lemma. Relation (29) is the Cholesky factorization of the Q matrix with D as a diagonal matrix with positive elements on the main diagonal and L is a unit lower triangular matrix. Relation

(30) defines the inverse of the Q matrix. This constraint assumes relevancy for computing the supply elasticities of the various outputs. Relations $\mathbf{f}'\hat{\mathbf{x}} > 0$ and $\mathbf{g}'\hat{\mathbf{y}} > 0$ guarantee that the cost function is increasing in output and decreasing in input prices. Any feasible solution of model (27)-(30) is an admissible cost function for representing the economic agent's decisions under price risk.

5. PMP with Generalized Risk and Output-Supply Elasticities

It may be of interest to estimate price supply elasticities for the various commodity outputs involved in a PMP-MS approach. The supply function for outputs is derivable from relation (24) by equating it to the expected market output prices, $E(\tilde{\mathbf{p}})$, and inverting the marginal cost function:

$$\mathbf{x} = -Q^{-1}\mathbf{f} - Q^{-1}\mathbf{f}[(\mathbf{y}^{1/2})G\mathbf{y}^{1/2}]/(\mathbf{g}'\mathbf{y}) + [1/(\mathbf{g}'\mathbf{y})]Q^{-1}E(\tilde{\mathbf{p}}) \tag{31}$$

that leads to the supply elasticity matrix

$$\Xi = \Delta[E(\tilde{\mathbf{p}})]\frac{\partial \mathbf{x}}{\partial E(\tilde{\mathbf{p}})}\Delta[(\mathbf{x}^{-1})] = \Delta[E(\tilde{\mathbf{p}})]Q^{-1}\Delta[(\mathbf{x}^{-1})]/(\mathbf{g}'\mathbf{y}) \tag{32}$$

where matrices $\Delta[E(\tilde{\mathbf{p}})]$ and $\Delta[\mathbf{x}^{-1}]$ are diagonal with elements $E(\tilde{p}_j)$ and x^{-1} on the main diagonals, respectively. Relation (32) includes all the own- and cross-price elasticities for all the output commodities admitted in the model.

6. Endogenous and Disaggregated Output-Supply Elasticities

PMP has been applied frequently to analyze farmers' behavior to changes in agricultural policies. A typical empirical setting is to map out several areas in a region (or state) and to assemble a representative farm for each area (or to treat each area as a large farm). When supply elasticities are exogenously available (say the own-price elasticities of crops) at the regional (or state) level (via econometric estimation or other means), a connection of all area models can be specified by establishing a weighted sum of all the areas endogenous own-price elasticities and the given regional elasticities. The weights are the share of each area's expected revenue over the total expected revenue of the region.

Let us suppose that exogenous own-price elasticities of supply are available at the regional level for all the J crops, say $\bar{\eta}_j, j = 1,...,J$. Then, the relation among these exogenous own-price elasticities and the corresponding areas' endogenous elasticities can be established as a weighted sum such as

$$\bar{\eta}_j = \sum_{n=1}^{N} w_{nj}\eta_{nj} \tag{33}$$

where the weights are the areas' expected revenue shares in the region (state)

$$w_{nj} = \frac{E(\tilde{p}_{nj})x_{nj}}{\sum_{t=1}^{N} E(\tilde{p}_{tj})x_{tj}} \tag{34}$$

$$\eta_{nj} = E(\tilde{p}_{nj})Q_n^{jj}x_{nj}^{-1} / (\mathbf{g}_n'\mathbf{y}_n) \tag{35}$$

where Q_n^{jj} is the *j*th element on the main diagonal in the inverse of the Q_n matrix.

The phase II model that executes the estimation of the cost function parameters and the disaggregated (endogenous) output supply elasticities for a region (state) that is divided into *N* areas takes on the following specification:

$$\min Aux = \sum_{n=1}^{N} \mathbf{d}_n'\mathbf{d}_n / 2 + \sum_{n=1}^{N} \mathbf{r}_n'\mathbf{r}_n / 2 \tag{36}$$

subject to

$$\mathbf{f}_n(\mathbf{g}_n'\hat{\mathbf{y}}_n) + (\mathbf{g}_n'\hat{\mathbf{y}}_n)Q_n\hat{\mathbf{x}}_n + \mathbf{f}_n[(\hat{\mathbf{y}}_n{}^{1/2})'G_n\hat{\mathbf{y}}_n{}^{1/2}] = \tag{37}$$

$$\mathbf{c}_n + \frac{1}{\hat{\theta}_n}[\overline{w}_n + (E(\tilde{\mathbf{p}}_n) - \mathbf{c}_n)'\hat{\mathbf{x}}_n]^{(1-\hat{\theta}_n)}[A_n'\hat{\mathbf{y}}_n + W_n\hat{\mathbf{h}}_n]$$

$$+\frac{\hat{\gamma}_n}{\hat{\theta}_n}[\overline{w}_n + (E(\tilde{\mathbf{p}}_n) - \mathbf{c}_n)'\hat{\mathbf{x}}_n]^{(1-\hat{\theta}_n)}(\hat{\mathbf{x}}_n'\Sigma_p\hat{\mathbf{x}}_n)^{(\hat{\gamma}_n/2-1)}\Sigma_p\hat{\mathbf{x}}_n + \mathbf{d}_n \geq E(\tilde{\mathbf{p}}_n)$$

(38) $$(\mathbf{f}_n'\hat{\mathbf{x}}_n)\mathbf{g}_n + \mathbf{g}_n(\hat{\mathbf{x}}_n'Q_n\hat{\mathbf{x}}_n)/2 + (\mathbf{f}_n'\hat{\mathbf{x}}_n)[\Delta(\hat{\mathbf{y}}_n^{-1/2})'G_n\hat{\mathbf{y}}_n] = A_n\hat{\mathbf{x}}_n + \mathbf{r}_n$$

(39) $$Q_n = L_nD_nL_n'$$

(40) $$Q_nQ_n^{-1} = I$$

(41) $$\Xi_n = \Delta[E(\tilde{\mathbf{p}}_n)]Q_n^{-1}\Delta[(\mathbf{x}_n^{-1})]/(\mathbf{g}_n'\mathbf{y}_n)$$ endogenous own- and cross-price elasticities

(43) $$w_{nj} = \frac{E(\tilde{p}_{nj})\hat{x}_{nj}}{\sum_{t=1}^{N}E(\tilde{p}_{tj})\hat{x}_{tj}}$$ expected revenue weights

(43) $$\eta_{nj} = E(\tilde{p}_{nj})Q_n^{jj}\hat{x}_{nj}^{-1}/(\mathbf{g}_n'\hat{\mathbf{y}}_n)$$ own-price elasticities

(44) $$\overline{\eta}_j = \sum_{n=1}^{N} w_{nj}\eta_{nj}$$ disaggregation of exogenous elasticities

with $D_n > 0, \mathbf{g}_n$ and $\mathbf{f}_n$ free and $\mathbf{f}_n'\hat{\mathbf{x}}_n > 0$, $\mathbf{g}_n'\hat{\mathbf{y}}_n > 0$, $\mathbf{d}_n \geq \mathbf{0}, \mathbf{r}_n \geq \mathbf{0}$.

7. Calibrating Equilibrium Models

With the parameter estimates of the cost function derived from either phase II model (26)-(30) or model (36)-(44), $\hat{\mathbf{f}}_n, \hat{\mathbf{g}}_n, \hat{Q}_n, \hat{G}_n$, it is possible to set up a calibrating equilibrium model to be used for policy analysis. Such a model takes on the following economic equilibrium specification

(45) $$\min CSC = \mathbf{y}'\mathbf{z}_p + \mathbf{x}'\mathbf{z}_d = 0$$

subject to

(46) $$(\hat{\mathbf{f}}'\mathbf{x})\hat{\mathbf{g}} + \hat{\mathbf{g}}(\mathbf{x}'\hat{Q}\mathbf{x})/2 + (\hat{\mathbf{f}}'\mathbf{x})[\Delta(\mathbf{y}^{-1/2})'\hat{G}\mathbf{y}^{1/2}] + \mathbf{z}_p = \mathbf{b} + V\hat{\mathbf{u}}$$

(47) $$\hat{\mathbf{f}}(\hat{\mathbf{g}}'\mathbf{y}) + (\hat{\mathbf{g}}'\mathbf{y})\hat{Q}\mathbf{x} + \hat{\mathbf{f}}[(\mathbf{y}^{1/2})'\hat{G}\mathbf{y}^{1/2}] = E(\tilde{\mathbf{p}}) + \hat{\mathbf{z}}_d$$

with $\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{z}_p \geq \mathbf{0}, \mathbf{z}_d \geq \mathbf{0}$. The objective function represents the complementary slackness conditions (*CSC)* of constraints (46) and (47) with an optimal value of zero. The variables $\mathbf{z}_p$ and $\mathbf{z}_d$ are surplus variables of the primal and the dual constraints, respectively. The solution of model (45)-(47) calibrates precisely the solution obtained from the phase I model (15)-(21), that is, $\hat{\mathbf{x}}_{LS} = \hat{\mathbf{x}}_{CSC}$ and $\hat{\mathbf{y}}_{LS} = \hat{\mathbf{y}}_{CSC}$. Note that the matrix of fixed technical coefficients A does not appear in either constraint (46) or (47). The calibrating model, then, can be used to trace the production and revenue response to changes in the expected output prices, subsidies and the supply of limiting inputs in a more flexible technical framework.

An alternative calibrating equilibrium model is suitable for dealing with a crucial aspect of a risky policy scenario. Wealth is the anchoring measure of risk preferences of an economic agent. As illustrated above, wealth is composed of accumulated income (or exogenous income) and net revenue derived from the current production cycle as in $[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]$ where $\bar{w}$ measures the amount of exogenous income. Agricultural policies in many countries deal with subsidies to farmers for cultivating crops. These subsidies may or may not be coupled to the level of crop production. Subsidies that are decoupled from the crop production decisions of farmers constitute exogenous income and end up in the $\bar{w}$ term of wealth that becomes an important target of policy makers. The $\bar{w}$ term, then, must appear in the calibrating model to allow the representation of decoupled subsidies as in the following specification

(48) $$\min CSC = \mathbf{y}'\mathbf{z}_p + \mathbf{x}'\mathbf{z}_d = 0$$

subject to

(49) $$(\hat{\mathbf{f}}'\mathbf{x})\hat{\mathbf{g}} + \hat{g}(\mathbf{x}'\hat{Q}\mathbf{x})/2 + (\hat{\mathbf{f}}'\mathbf{x})[\Delta(\mathbf{y}^{-1/2})\hat{G}\mathbf{y}^{1/2}] + \mathbf{z}_p = \mathbf{b} + V\hat{\mathbf{u}}$$

(50)

$$\mathbf{c} + \frac{1}{\hat{\theta}}[\overline{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(1-\hat{\theta})}[A'\mathbf{y} + W\hat{\mathbf{h}}] + \frac{\hat{\gamma}}{\hat{\theta}}[\overline{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(1-\hat{\theta})}(\mathbf{x}'\Sigma_p\mathbf{x})^{(\hat{\gamma}/2-1)}\Sigma_p\mathbf{x} = E(\tilde{\mathbf{p}}) + \mathbf{z}_d$$

with $\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{z}_p \geq \mathbf{0}, \mathbf{z}_d \geq \mathbf{0}$. Also the solution of model (48)-(50) calibrates precisely the solution obtained from the phase I model (15)-(21), that is, $\hat{\mathbf{x}}_{LS} = \hat{\mathbf{x}}_{CSC}$ and $\hat{\mathbf{y}}_{LS} = \hat{\mathbf{y}}_{CSC}$.

8. Empirical Implementation of PMP-MS with Supply Elasticities

The PMP-MS approach described in previous sections was applied to a sample of $N = 14$ representative farms. There are four crops: sugar beet, soft wheat, corn and barley. There is only one limiting input: land. In this sample, all farms produce all crops. A more realistic sample where some farms produce only some crops will be presented in section 9. Phase I model (36)-(44) was initially solved using the Conopt3 solver of GAMS. Then, an extensive analysis of a global optimum was performed using the BARON solver and randomly selected (by BARON) initial points. The BARON solver consumed hours of cpu time but, in the end, it found a best solution that is identical to the solution found by the Conopt3 solver.

Table 2 shows the variance-covariance matrix of the market output prices.

Table 2. Variance-Covariance Matrix of Market Output Prices

	Sugar Beet	Soft Wheat	Corn	Barley
Sugar Beet	0.0024719	-0.0164391	-0.0117184	-0.0121996
Soft Wheat	-0.0164391	0.2386034	0.1821288	0.2049011
Corn	-0.0117184	0.1821288	0.1530464	0.1610119
Barley	-0.0121996	0.2049011	0.1610119	0.1830829

Tables 3 and 4 present the observed output levels and input prices ($\mathbf{x}_{obs}, \mathbf{y}_{obs}$). They also exhibit the percent deviation of the solution $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ of model (15)-(21) from the corresponding targets.

Table 3. Observed Output Levels, $\mathbf{x}_{obs}$, and Percent Deviation (dev) of the LS Calibrated Solution, $\hat{\mathbf{x}}$

	Sugar Beet	Soft Wheat	Corn	Barley	Sugar Beet	Soft Wheat	Corn	Barley
Farm	$\mathbf{x}_{obs}$	$\mathbf{x}_{obs}$	$\mathbf{x}_{obs}$	$\mathbf{x}_{obs}$	% dev	% dev	% dev	% dev
1	1133.4240	305.4032	341.3693	18.2398	0.0300	-0.0289	0.0224	-0.1037
2	3103.7830	861.7445	478.4465	59.8025	0.0113	-0.0029	-0.0056	0.0159
3	1547.9780	450.7937	881.9748	7.6887	0.0242	-0.0075	-0.0049	0.5114
4	3488.3540	821.3934	1493.332	51.1247	0.0106	-0.0077	0.0004	0.0909
5	959.1102	468.2848	478.9261	28.2406	0.0284	-0.0275	0.0117	0.0880
6	942.2039	801.1288	1283.591	152.581	0.0349	-0.0078	0.0011	0.0168
7	1600.7310	695.8293	899.4739	66.9718	0.0251	-0.0052	0.0023	-0.0077
8	3507.5490	1212.8550	1237.584	98.0497	0.0100	0.0015	0.0021	-0.0611
9	1050.5370	332.3773	498.0150	63.6696	0.0386	0.0025	-0.0014	-0.0519
10	3473.6780	952.5199	774.7402	84.0070	0.0114	-0.0059	0.0018	0.0256
11	1245.7220	765.1689	501.9673	59.5366	0.0305	-0.0075	0.0052	0.0143
12	3276.1450	1100.1680	742.9419	177.974	0.0081	-0.0083	0.0026	0.0204
13	877.0970	380.9171	564.6091	76.2122	0.0319	-0.0325	0.0092	0.0340
14	1430.9460	768.6901	1309.392	67.7906	0.0229	-0.0084	0.0012	0.0363

Table 4. Deviations of $\hat{\mathbf{y}}$ from $\mathbf{y}_{obs}$: vector $\hat{\mathbf{u}}$

	Absolute Deviation	Observed Land Prices	Percent Deviation
Farm	$\hat{\mathbf{u}}$	$\mathbf{y}_{obs}$	%
1	-0.0002373	4.42	-0.0054
2	-0.0000025	4.38	-0.0001
3	0.0001239	6.98	0.0018
4	0.0000524	5.73	0.0009
5	-0.0001557	4.40	-0.0035
6	0.0000049	1.86	0.0003
7	0.0000682	3.65	0.0019
8	0.0000039	3.36	0.0001
9	0.0000708	2.75	0.0026
10	0.0000440	4.28	0.0010
11	0.0000277	3.28	0.0008
12	-0.0000253	1.93	-0.0013
13	-0.0000716	2.32	-0.0031
14	0.0000062	4.03	0.0002

All the deviations $\hat{\mathbf{h}}$ in Table 3 are below 1 percent. Hence, the calibrating solution $\hat{\mathbf{x}}$ is satisfactorily close to the observed output levels $\mathbf{x}_{obs}$. Similarly, the deviations $\hat{\mathbf{u}}$ in Table 4 are well below 1 percent.

Table 5 presents the estimates of the parameters θ and γ of the MS utility function.

Table 5. Estimates of θ and γ

Farm	Parameter θ	Parameter γ
1	1.1843	1.4338
2	1.1474	1.3804
3	1.1916	1.3817
4	1.1844	1.4010
5	1.1502	1.3463
6	1.1455	1.3546
7	1.1608	1.3834
8	1.1773	1.4012
9	1.1710	1.4369
10	1.1627	1.3869
11	1.1702	1.3854
12	1.1344	1.3393
13	1.1457	1.3503
14	1.1504	1.3572

The sample is composed of relatively homogeneous farms. Hence, the limited range of variations of the MS utility parameters is not a surprise. All farmers exhibit decreasing absolute risk aversion, $\theta > 1$, and increasing relative risk aversion, $\theta < \gamma$. This combination of risk behavior is admissible by the MS utility.

The estimated parameters of the cost function are reported in Tables 6 and 7. For reasons of space, only three Q matrices are reported.

Table 6. Intercepts $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$ and $\hat{G}$ Matrix of the Marginal Cost and Input Demand Functions

Farm	$\hat{\mathbf{f}}$							
	Sugar Beet	Soft Wheat	Corn	Barley	$\hat{\mathbf{g}}$	$\hat{G}$	$\hat{\mathbf{f}}'\hat{\mathbf{x}}$	$\hat{\mathbf{g}}'\hat{\mathbf{y}}$
1	0.0364	0.0124	-0.0436	-0.2548	0.00235	-6.9714	25.5186	0.01038
2	0.0422	0.0148	-0.0824	-0.2223	0.00151	-5.4745	91.1059	0.00663
3	0.0277	0.0411	-0.0614	-0.0195	0.00140	-6.7432	7.0695	0.00977
4	0.0175	0.0164	-0.0262	-0.0286	0.00084	-11.2156	33.9950	0.00483
5	0.0322	0.0008	-0.0302	-0.0707	0.00693	-9.5411	14.8326	0.03049
6	0.0135	0.1666	0.2589	-0.3812	0.01469	-5.1733	420.3299	0.02733
7	0.0378	0.0075	0.0559	0.1214	0.00427	-4.9656	124.1857	0.01558
8	0.0195	0.2246	-0.1346	0.1202	0.00173	-6.8739	185.9156	0.00580
9	1.3944	0.1488	0.3717	-0.0350	0.00538	-0.3180	1697.7630	0.01480
10	0.0323	-0.0084	-0.0321	-0.0535	0.00151	-6.8887	74.8985	0.00648
11	0.0883	0.0001	-0.0706	-0.1876	0.00726	-8.1445	63.5781	0.02383
12	0.0011	0.0519	-0.1482	1.2355	0.00049	-14.0441	170.5294	0.00095
13	0.0624	-0.1049	0.1739	-0.1844	0.01177	-6.3013	98.8884	0.02730
14	0.0364	0.0124	-0.0436	-0.2548	0.00382	-7.4292	85.1550	0.01540

Table 7. Matrices $\hat{Q}$ and $\hat{D}$ for Three Farms

	Matrix $\hat{Q}$				Matrix $\hat{D}$			
Farm 1	Sugar Beet	Soft Wheat	Corn	Barley	Sugar Beet	Soft Wheat	Corn	Barley
S. Beet	0.16237	0.03079	-0.17470	0.69499	0.16237			
S.Wheat	0.03079	1.63198	-0.62077	-4.30646		1.62615		
Corn	-0.17470	-0.62077	1.45275	-2.15787			1.05243	
Barley	0.69499	-4.30646	-2.15787	38.44066				14.72116
Farm 2								
S. Beet	0.16890	-0.29596	-0.09062	-0.01211	0.16890			
S.Wheat	-0.29596	1.70334	-0.24643	-0.16048		1.18476		
Corn	-0.09062	-0.24643	1.57655	-3.18253			1.38933	
Barley	-0.01211	-0.16048	-3.18253	20.80496				13.16816
Farm 3								
S. Beet	1.94306	0.73345	-3.59200	0.12596	1.94306			
S.Wheat	0.73345	1.77257	-1.70656	-0.55029		1.49572		
Corn	-3.59200	-1.70656	7.06136	0.10905			0.33884	
Barley	0.12596	-0.55029	0.10905	8.76959				8.40237

All 14 farms achieved a nonsingular $\hat{Q}$ matrix. This feature is instrumental in defining the matrix of endogenous supply elasticities. Table 8 presents the endogenous own- and cross-price supply elasticities for three farms.

Table 8. Endogenous Own- and Cross-Supply Elasticities for Three Farms

Farm 1	Sugar Beet	Soft Wheat	Corn	Barley
S. Beet	0.25	-0.03	0.07	-0.27
S. Wheat	-0.05	1.15	0.65	2.89
Corn	0.10	0.70	0.88	2.12
Barley	-0.02	0.16	0.11	0.72
Farm 2				
S. Beet	0.24	0.17	0.20	0.28
S. Wheat	0.23	0.43	0.32	0.45
Corn	0.16	0.18	0.87	1.09
Barley	0.03	0.03	0.13	0.44
Farm 3				
S. Beet	0.25	0.08	0.23	-0.76
S. Wheat	0.14	0.42	0.17	0.91
Corn	0.66	0.29	0.66	-1.77
Barley	-0.02	0.01	-0.02	3.15

We stipulated that regional, exogenous own-price supply elasticities were available in the magnitude of 0.5 for sugar beet, 0.4 for soft wheat, 0.6 for corn and 0.3 for barley. The endogenous own-price elasticities of all farms were aggregated to be consistent with the regional exogenous elasticities according to relation (44). Table 9 presents the farms' own-price supply elasticities and the revenue weights used in the aggregation relation.

Table 9. Disaggregation/Aggregation of the Regional, Exogenous Supply Elasticities.

	Exogenous Own-Supply Elasticities					Revenue Weights			
Farms	Sugar Beet:0.5	Soft Wheat:0.4	Corn: 0.6	Barley: 0.3		Sugar Beet	Soft Wheat	Corn	Barley
1	0.25	1.15	0.88	0.72		0.0406	0.0291	0.0295	0.0165
2	0.24	0.43	0.87	0.44		0.1334	0.0937	0.0489	0.0633
3	0.25	0.42	0.66	3.15		0.0527	0.0446	0.0698	0.0082
4	0.15	0.41	0.64	1.14		0.0999	0.0893	0.1383	0.0548
5	0.13	0.37	0.40	0.77		0.0327	0.0413	0.0386	0.0262
6	0.09	0.24	0.30	0.12		0.0372	0.0828	0.1151	0.1595
7	0.14	0.43	0.34	0.34		0.0502	0.0689	0.0769	0.0604
8	0.14	0.27	0.49	0.28		0.1288	0.1294	0.1022	0.0910
9	0.13	0.23	0.39	0.15		0.0377	0.0336	0.0426	0.0565
10	0.17	0.45	0.52	0.34		0.1026	0.0930	0.0649	0.0828
11	0.10	0.38	0.48	0.37		0.0424	0.0737	0.0417	0.0538
12	2.33	0.57	2.08	0.06		0.1554	0.1079	0.0685	0.1862
13	0.10	0.35	0.30	0.13		0.0299	0.0335	0.0455	0.0692
14	0.10	0.28	0.38	0.22		0.0564	0.0795	0.1174	0.0716

9. When not All Farms Produce All Crops

Empirical reality compels a further consideration of the above methodology in order to deal with farm samples where not all farms produce all commodities. It turns out that very little must be changed for obtaining a calibrating solution in the presence of missing commodities, their prices and the corresponding technical coefficients. Using the GAMS software, it is sufficient to condition the various constraints of phase I, phase II and phase III models by the nonzero observations of the output levels.

To exemplify, suppose that the farm sample displays the following Table 10 of observed crop levels.

Table 10. Observed Output Levels, $\bar{\mathbf{x}}$, with non produced commodities

	Sugar Beet	Soft Wheat	Corn	Barley
Farm	$\bar{\mathbf{x}}$	$\bar{\mathbf{x}}$	$\bar{\mathbf{x}}$	$\bar{\mathbf{x}}$
1	1133.4240	0	341.3693	18.2398
2	3103.7830	861.7445	0	59.8025
3	0	450.7937	881.9748	0
4	3488.3540	821.3934	1493.332	51.1247
5	959.1102	468.2848	0	28.2406
6	942.2039	801.1288	1283.591	152.581
7	1600.7310	0	899.4739	66.9718
8	0	1212.8550	1237.584	98.0497
9	1050.5370	332.3773	0	63.6696
10	3473.6780	952.5199	774.7402	0
11	0	765.1689	501.9673	59.5366
12	3276.1450	1100.1680	0	177.974
13	877.0970	380.9171	564.6091	76.2122
14	1430.9460	0	1309.392	0

Other missing information deals with prices and unit accounting costs associated with the zero-levels of crops. Furthermore, the technical coefficients of farms not producing the observed crops also equal to zero. Hence, we can state that, for $t = 1,...,T$, the number of farms, and $j = 1,...,J$, the number of crops, if $\bar{x}_{tj} = 0$, also $p_{tj} = 0$, $c_{tj} = 0$ and $A_{tij} = 0$.

Furthermore, suppose that only one input, land, is involved in this farm sample. Then, the land price is observed for all farms. The procedure to deal with this type of sample data consists in conditioning the relevant constraints on the positive values of the output levels. In GAMS, this procedure requires a conditional statement using the $ sign option.

The BARON solver of the GAMS software found a best solution that is identical to the solution found by the Conopt3 solver in less than 15 minutes of computing time. Tables 11 and 12 present the primal and dual solutions and the percent deviations from the target levels. Except for one cell, all deviations are below the one percent.

Table 11. Estimated LS Solution, $\hat{\mathbf{x}}$, and Percent Deviation (dev) from the Observed Levels, $\mathbf{x}_{obs}$ with Zero Levels for Some Crops and Some Farms

	Sugar Beet	Soft Wheat	Corn	Barley	Sugar Beet	Soft Wheat	Corn	Barley
Farm	$\hat{\mathbf{x}}$	$\hat{\mathbf{x}}$	$\hat{\mathbf{x}}$	$\hat{\mathbf{x}}$	% dev	% dev	% dev	% dev
1	1134.621	0	341.562	17.927	0.1056	0	0.0565	-1.7147
2	3104.862	861.636	0	59.784	0.0348	-0.0126	0	-0.0310
3	0	451.144	881.524	0	0	0.0777	-0.0511	0
4	3489.369	821.156	1493.423	51.243	0.0291	-0.0289	0.0061	0.2323
5	960.002	467.880	0	28.452	0.0929	-0.0864	0	0.7473
6	943.163	800.932	1283.650	152.650	0.1018	-0.0246	0.0046	0.0451
7	1601.983	0	899.502	66.857	0.0782	0	0.0031	-0.1710
8	0	1213.013	1237.867	97.567	0	0.0130	0.0229	-0.4925
9	1051.793	332.400	0	63.535	0.1196	0.0069	0	-0.2107
10	3474.851	952.360	774.838	0	0.0338	-0.0168	0.0127	0
11	0	764.721	502.282	59.699	0.0000	0	0.0627	0.2731
12	3276.956	1099.887	0	178.124	0.0248	-0.0255	0	0.0838
13	877.867	380.518	564.786	76.283	0.0877	-0.1048	0.0314	0.0924
14	1432.042	0	1309.303	0	0.0766	0	-0.0068	0

Table 12. Deviations of $\hat{\mathbf{y}}$ from $\mathbf{y}_{obs}$

Farm	Absolute Deviation $\hat{\mathbf{y}}$	Observed Land Prices $\mathbf{y}_{obs}$	Percent Deviation %
1	4.41852	4.42	-0.0334
2	4.37996	4.38	-0.0008
3	6.98055	6.98	0.0079
4	5.73005	5.73	0.0009
5	4.39923	4.40	-0.0175
6	1.86001	1.86	0.0003
7	3.65025	3.65	0.0069
8	3.35966	3.36	-0.0103
9	2.75022	2.75	0.0081
10	4.28005	4.28	0.0013
11	3.27977	3.28	-0.0071
12	1.92991	1.93	-0.0049
13	2.31972	2.32	-0.0120
14	4.03018	4.03	0.0044

Table 13 presents the estimates of the parameters θ and γ of the MS utility function.

Table 13. Estimates of θ and γ

Farm	Parameter θ	Parameter γ
1	1.0992	1.3653
2	1.0464	1.2245
3	1.3575	1.5864
4	1.0820	1.2364
5	1.0608	1.1706
6	1.0310	1.1896
7	1.0651	1.2684
8	1.1711	1.3813
9	1.0643	1.3022
10	1.0575	1.2174
11	1.1608	1.3608
12	1.0294	1.1842
13	1.0223	1.1553
14	1.0567	1.2496

Again, all farmers exhibit decreasing absolute risk aversion, $\theta > 1$, and increasing relative risk aversion, $\theta < \gamma$. This combination of risk behavior is admissible by the MS utility.

The estimated parameters of the cost function are reported in Tables 14 and 15. For reasons of space, only three Q matrices are reported.

Table 14. Intercepts $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$ and $\hat{G}$ Matrix of the Marginal Cost and Input Demand Functions

Farm	$\hat{\mathbf{f}}$ Sugar Beet	Soft Wheat	Corn	Barley		$\hat{\mathbf{g}}$	$\hat{G}$	$\hat{\mathbf{f}}'\hat{\mathbf{x}}$	$\hat{\mathbf{g}}'\hat{\mathbf{y}}$
1	0.00061	0	0.01153	-0.00684		0.00277	-32.849	4.509	0.01224
2	-0.00321	0.02451	0	0.00474		0.00246	-34.265	11.427	0.01079
3	0	0.00330	-0.00095	0		0.00094	-34.475	0.647	0.00653
4	0.02039	-0.01058	-0.00838	-0.02655		0.00060	-7.848	48.583	0.00342
5	-0.00042	0.00517	0	-0.00474		0.00181	-39.007	1.879	0.00796
6	-0.00396	0.06207	0.00579	0.06711		0.01011	-34.077	63.651	0.01881
7	0.01396	0	-0.01048	0.00479		0.00150	-30.598	13.254	0.00546
8	0	0.02050	-0.00726	0.00076		0.00721	-60.936	15.953	0.02421
9	0.00045	0.01563	0	0.04656		0.01866	-31.051	8.621	0.05131
10	-0.00046	0.02169	-0.00607	0		0.00091	-35.512	14.341	0.00390
11	0	0.03886	-0.01947	-0.00459		0.01219	-21.589	19.664	0.03998
12	0.00162	0.01237	0	0.02565		0.00208	-72.909	23.497	0.00402
13	0.01012	0.02340	0.00460	-0.00015		0.01054	-30.544	20.373	0.02444
14	-0.00004	0	0.00136	0		0.00011	-269.391	1.725	0.00045

Table 14. Matrices $\hat{Q}$ and $\hat{D}$ for Three Farms

	Matrix $\hat{Q}$					Matrix $\hat{D}$			
Farm 1	Sugar Beet	Soft Wheat	Corn	Barley		Sugar Beet	Soft Wheat	Corn	Barley
S. Beet	0.0820	-0.0423	-0.1543	-0.0219		0.0820			
S.Wheat	-0.0423	1.0998	0.4039	-4.3839			1.0780		
Corn	-0.1543	0.4039	1.5045	-1.8866				1.1165	
Barley	-0.0219	-4.3839	-1.8866	41.9220					23.6662
Farm 2									
S. Beet	0.1146	-0.4099	0.2076	-0.0464		0.1146			
S.Wheat	-0.4099	2.1313	-1.1157	0.0512			0.6650		
Corn	0.2076	-1.1157	1.1823	-0.1771				0.5969	
Barley	-0.0464	0.0512	-0.1771	6.3717					6.2915
Farm 3									
S. Beet	0.8919	-0.1384	0.0735	-0.4673		0.8919			
S.Wheat	-0.1384	1.9747	-0.4868	0.0297			1.9533		
Corn	0.0735	-0.4868	0.5378	-0.0324				0.4160	
Barley	0.8919	-0.1384	0.0735	-0.4673					0.7450

Regional, exogenous own-price supply elasticities were available in the magnitude of 0.5 for sugar beet, 0.4 for soft wheat, 0.6 for corn and 0.3 for barley. The endogenous own-price elasticities of all farms were aggregated to be consistent with the regional exogenous elasticities according to relation (44). Table 15 presents the farms' own-price supply elasticities and the revenue weights used in the aggregation relation.

Table 15. Disaggregation/Aggregation of the Regional, Exogenous Supply Elasticities with zero observations of some output levels

	Exogenous Own-Supply Elasticities					Revenue Weights			
Farms	Sugar Beet:0.5	Soft Wheat:0.4	Corn: 0.6	Barley: 0.3		Sugar Beet	Soft Wheat	Corn	Barley
1	0.449	0	0.447	0.385		0.0524	0	0.0368	0.0193
2	0.404	0.537	0	0.571		0.1719	0.1139	0	0.0748
3	0	0.494	0.790	0		0	0.0543	0.0871	0
4	0.161	0.556	0.553	0.952		0.1288	0.1086	0.1726	0.0641
5	0.261	0.566	0	0.761		0.0421	0.0502	0	0.0307
6	0.061	0.216	0.339	0.103		0.0479	0.1007	0.1437	0.1909
7	0.157	0	1.075	0.518		0.0647	0	0.0960	0.0721
8	0	0.337	0.330	0.228		0	0.1571	0.1275	0.1104
9	0.234	0.441	0	0.129		0.0485	0.0407	0	0.0685
10	0.360	0.333	0.436	0		0.1323	0.1130	0.0810	0
11	0	0.410	0.278	0.216		0	0.0895	0.0520	0.0644
12	0.356	0.358	0	0.170		0.2003	0.1311	0	0.2227
13	0.097	0.226	0.351	0.276		0.0385	0.0408	0.0567	0.0822
14	3.141	0	1.061	0		0.0727	0.0000	0.1465	0

10. Conclusion

This paper accomplished several objectives. First, it extended the treatment of risk in a mathematical programming framework to include, in principle, any combination of risk preferences represented by absolute risk aversion and relative risk aversion. Second, it modified the traditional PMP approach to deal with calibration constraints regarding observed output levels and observed input prices by eliminating the user-determined vector of perturbation parameters. The combination of these two approaches provides suitable models for agricultural policy analysis that take into consideration farmers' risk preferences associated with the randomness of output prices. Third, this paper integrated the use of exogenous supply elasticities observed for, say, an entire region with the endogenous elasticities derived from the supply functions of the sample farms. This objective is achieved by specifying a complete and flexible total cost function that fulfills all the theoretical requirements. Fourth, the calibrating model resulting from the PMP-MS framework described here allows the analysis of policy scenarios dealing with farm subsidies that are decoupled from the current crop production. Consider the parameter $\overline{w}$ in the measure of wealth that may represent exogenous income subsidy. With a Freund approach to risk based upon a constant absolute risk aversion utility function, the wealth parameter disappears from the model. On the contrary, one version of the calibrating equilibrium model presented in this paper allows the analysis of decoupled farm subsidies that are more frequently the target of policy makers. This general model has been tested on different farm samples with satisfactory results including a data sample where not all farms produce all the commodities.

References

Brooke, A., Kendrick, D., Meeraus, A. (1988). GAMS, A User's Guide, The Scientific Press, Redwood City, California

Fisher, I. (1906). *The Nature of Capital and Income*. MacMillan Company, London

Freund, R.J. (1956). The Introduction of Risk Into a Programming Model. *Econometrica* 24:253-263.

Hazell, P. B. R. (1971). A Linear Alternative to Quadratic and Semivariance Programming for Farm Planning under Uncertainty. *American Journal of Agricultural Economics* 53:53-62.

Hicks, J.R. (1933). The Application of Mathematical Methods in the Theory of Risk, Lecture presented at The Meeting of the Econometric Society in Leyden, September-October 1933, summarized by J. Marschak, *Econometrica* 1934, 2:187-203.

Howitt, R. E. (1995a). "A Calibration Method for Agricultural Economic Production Models," *Journal of Agricultural Economics*, 46, 2:147–159.

Howitt, R. E. (1995b). "Positive Mathematical Programming, "*American Journal of Agricultural Economics*, 77, 2:329-342.

Markowitz, H. (1952). Portfolio Selection. *Journal of Finance* 7:77-91.

Meyer, J. (1987). Two-Moment Decision Models and Expected Utility Maximization. *American Economic Review,* 77, 3:421-430.

Saha, A. (1997). Risk Preference Estimation in the Nonlinear Mean Standard Deviation Approach. *Economic Inquiry* 35:770-782.

Tawarmalani, M., Sahinidis, N. V. (2005). A Polyhedral Branch-and-Cut Approach to Global Optimization. *Mathematical Programming*, 103:225-249.

Tintner, G. (1941). The Theory of Choice Under Subjective Risk and Uncertainty. *Econometrica* 9:298-304.

Tobin, J. (1958). Liquidity Preference as Behavior Toward Risk. *Review of Economic Studies* 67:65-86.

Sahinidis, N. V. (20015). BARON User Manual v. 15.2. The Optimization Firm, LLC. http://www.minlp.com.