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AN ANALYSIS OF INTEREST AND PRINCIPAL PAYMENTS, INTEREST RATES AND TIME IN COMMON AND UNCOMMON LOANS USING PRESENT VALUE TOOLS

By
Lindon J. Robison
Steven R. Koenig
John R. Brake

Department of
Agricultural Economics
MICHIGAN STATE UNIVERSITY
East Lansing

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Lindon J. Robison
Steven R. Koenig
and
John R. Brake*

*Lindon J. Robison and Steven R. Koenig are, respectively, Associate Professor and Graduate Research Assistant in the Department of Agricultural Economics, Michigan State University, East Lansing, Michigan. John R. Brake is Professor in the Department of Agricultural Economics, Cornell University, Ithaca, New York.

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ABSTRACT

An Analysis of Interest and Principal Payments, Interest Rates and Time in Common and Uncommon Loans Using Present Value Tools

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This report analyzes a variety of loan programs. After introducing basic concepts, definitions, and tools, present value formulas are used to compare different types of loans including fixed payment, balloon payment, moderated payment, blended rate, fixed principal payment, and disguised interest cost loans. The application of present value tools to the analysis of each type of loan is introduced and then illustrated by example. Because the sum of a uniform series appears repeatedly in the solutions, calculated values are provided in Table 1 at the end of the report.

Acknowledgements

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An Analysis of Interest and Principal Payments, Interest Rates and Time in Common and Uncommon Loans Using Present Value Tools

I. INTRODUCTION

Present value tools can be used to analyze numerous types of loans. The goal of this report is to bring together in one place the present value formulas which are relevant to loan analysis. In laying the groundwork for meeting this goal, several important topics are addressed. First, we examine the similarities between investment and loan analysis, which both use present value tools. Although the two types of analysis are similar in many respects, they differ in the selection of an appropriate discount rate. Investment analysis discounts future values with a discount rate which reflects an opportunity cost. Present value formulas used in loan analysis discount future values with a discount rate equal to an interest rate. Because of this difference, and the wide variety of terms commonly used in discussing interest rates, the terminology to be employed in this report is carefully defined.

The discussion of interest rates is followed by a discussion of geometric series and summation of geometric series. The formula for determining the present value of a uniform series of payments is developed and symbolized for later reference and application in discussing specific types of loans. Because of the importance of this formula in analyzing loans, numerical values for the formula are given in Table 1 at the end of the report.

The concepts, definitions, and tools developed in the first part of the report are used to analyze six major classes of loans: (1) constant payment loans; (2) balloon payment loans; (3) moderated payment loans; (4) wrap around loans with blended interest rates; (5) constant principal payment loans; and (6) disguised interest cost loans. The relevant present value tools for each type of loan are introduced and then illustrated with examples.

Many of the formulas presented here are available elsewhere, some are not. We hope that by combining a basic exposition of the tools with their application in different examples, this report will provide a basic handbook for decision makers facing financial decisions.

II. LOAN VS. INVESTMENT ANALYSIS

The basic premise for this primer is that loan analysis like investment analysis is performed using present value methods. The major similarity between loan and investment analysis is that only cash transactions are included in the analysis. Noncash transactions do not appear in present value models. In loan analysis, this fact is, of course, well known; loans are received as cash and are repaid as cash. In investment analysis, a purchase of a capital asset involves the transfer of cash in exchange for the asset. So the cash outlay is reported but the cash value of the machine is not recorded. What is recorded as a receipt is the cash earned by using the asset's services; this principle applies to capital gains as well. If the asset is appreciating in value, the appreciation does not enter the model until the asset is sold and its value converted to cash (Robison and Burghardt).

The other major similarity between loan and investment models is the method used to convert future costs and benefits to a common time period. Both in analysis models and in investment models, the conversion is made by discounting future costs and benefits depending on the distance to the conversion period. The factor used to discount future benefits and costs is called the discount rate.

The essential difference between investment and loan analysis models, apart from the calculation of benefits and costs is the choice of the discount rate. In loan analysis, the discount rate is the interest rate associated with the loan. It is usually determined at the time of the loan. In investment models, the discount rate is the opportunity cost, the desired rate of return, or the rate of return the investor could earn in his next best investment alternative. Often, investment analysts suggest using appropriate interest rates as the discount rate in investment models, making the two models even more similar. However, it is important to realize that unused borrowing capacity has a value too. As a result, the interest rate which does not include the value of used-up borrowing capacity underestimates the true cost of borrowing and therefore cannot be generally used as the appropriate discount rate. Thus the interest rate and the discount rate should not be used interchangeably.

Finally, one can often distinguish between loan and investment models based on the pattern of cash flows. Loan payments are often constant amounts. Investment analysis usually imposes no pattern on the receipt of benefits and disbursement of costs. Yet in the development of analytic models, regularity is often imposed; the regularity most often required is that the series are geometrically related.

To summarize, the similarities between cash flow and investment models are that they both enter only cash payments, they are based on comparisons of cash units converted to common time periods, and they convert cash values in distant time periods to equivalent values in some other period (usually current) through use of a discount rate. The differences between investment and loan analysis are the manner in which benefits and costs are calculated and the choice of a discount rate.

The similarities between loan and investment analysis outweigh the differences. This fact allows us to develop tools and concepts for loan analysis which are applicable in most cases to investment analysis. The next section begins by discussing interest rates. Since they are often the discount rate used in loan analysis, it is important that the interest rate concepts and terminology are understood.

III. INTEREST RATE CONCEPTS AND TERMINOLOGY

To understand the concept of the time value of money, it is essential that the language of finance be understood. The concepts and terminology of finance often are difficult to grasp and are sometimes used inconsistently within different areas of the business of finance. This section outlines the major concepts and terminology that are commonly encountered.

Various interest rate definitions can be found in finance literature. To avoid confusion, definitions from the mathematics of finance will be used here.

There are three major interest rate definitions which are essential to understanding interest rate calculations and present value techniques. These rates are closely related to each other and are often referred to by different names in the literature. The rates and their commonly used synonyms are listed below, with the underlined terms being used in this report. They are:

1. Actuarial rate, compound rate, true rate, or periodic rate;
2. Annual percentage rate (APR), nominal rate, annual rate, nominal annual rate; and
3. Effective rate or effective annual rate.

Comparing the Actuarial, APR and Effective Interest Rates

In many types of financial transactions, interest is computed and charged more than once a year. For example, savings deposits placed in savings institutions usually receive monthly computing of earned interest and corporate bonds usually pay interest on a semiannual basis. The interest rate which is used in the computation for these periods of less than one year is called the actuarial rate. The actuarial rate may be best defined as the interest rate or discount rate per period of conversion or compounding. It is the rate at which the principle sum is charged during each successive conversion period. For example, a one percent actuarial rate charged monthly on $1,000 means that in the first month of the loan, one percent of $1,000 or $10 of interest is computed.

Actuarial rates are usually converted to an APR rate so that comparison between different quoted rates can be simplified. The APR rate is determined by expressing the actuarial rate on an annual basis. This is accomplished by multiplying the actuarial rate by the number of conversion periods per year. In the previous example, the one percent per month is multiplied by 12 to convert the monthly rate to an annual percentage rate (APR) of 12 percent compounded monthly. When the compound period or conversion period is one year in length, then the actuarial rate and the APR rates are equivalent.

When APR rates have different numbers of compound periods per year, the different rates can be converted to their effective interest rates for comparison. The effective rate is obtained by compounding the actuarial rate for a period of one year. As the frequency of compounding periods increases, the difference between the APR and the effective rate increases. Also, as the actuarial rate increases, so does the relative difference between the APR rate and the effective rate.

These relationships can be easily summarized using some simple notation. Let "m" be the conversion periods per year, let "r" be the APR rate, then $\frac{"r"}{m}$ is the actuarial rate. Then "r_e" is the effective rate.[1] The relationship between effective rate r_e, APR rate r, and actuarial rate $\frac{r}{m}$ is:

(1) $r_e = [(1 + \frac{r}{m})^m - 1]$

It is, of course, true that when m equals 1, the APR rate r equals the effective rate r_e. To illustrate (1) numerically, let r be .12 and m be 4, then the effective rate is:

$$.1255 = [(1 + \frac{.12}{4})^4 - 1]$$

And if m is increased to 12, the effective rate is:

$$.1268 = [(1 + \frac{.12}{12})^{12} - 1]$$

IV. TOOLS FOR SOLVING PRESENT VALUE MODELS

Two tools are essential for present value modeling:[2] discounting and geometric summation. We begin by discussing discounting principles.

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[1]Interest rates will be expressed as decimals in the body of this report. Table 1 expresses interest rates as percentages.

[2]This paper focuses on discrete time periods. Were the analysis to be converted to continuous time, the discounting tool remains fundamental to the analysis, but integration would replace the geometric summation method as the second tool.

Discounting

The comparison between the value of dollars received next period and the dollars' value in the conversion or present period depends on what opportunities exist for investing in the present (or conversion) period. If the highest rate of return available is r percent, then A_t dollars invested in the t^{th} year would be worth $A_t(1 + r)$ dollars one year later. Thus, the value of A_t in the year $(t + 1)$ is:

(2) $A_t(1 + r) = A_{t+1}$ or

(3) $A_t = A_{t+1}/(1 + r)$

Similar substitutions for earnings received in more distant years, say in year $(t + n)$, allow us to express the equivalent value in year t as:

(4) $A_t = A_{t+n}/(1 + r)^n$

provided the rate of return available between the t^{th} and the $(t + n)^{th}$ year remained at r percent per year.

To illustrate discounting, consider the situation where an investment is expected to return \$10,000 five years in the future. If the opportunity cost of money is expected to be 12 percent ($r = 12\%$) during the period, the present value of the future return is computed using equation (4) as:

$$A_t = \frac{\$10,000}{(1 + .12)^5} = \$5,674.27$$

The difference between the undiscounted \$10,000 and the discounted amount of \$5,674.27 is \$4,327.73. This figure represents the size of the discount for the five years in which the receipt \$10,000 is delayed. The \$5,674.27 figure represents the present value of the \$10,000 and is the amount one should pay for the investment if their opportunity cost is 12 percent. Annual compounding of \$5,674.27 at 12 percent interest would yield \$10,000 after five years. Note that by increasing the discount rate to 13 percent the present value decreases to \$5,427.70; a decrease to an 11 percent discount rate increases the present value to \$5,934.37.

Often we are concerned with converting a series of payments A_1, A_2, A_3, . . . received over time to their present value equivalent. Let V_o be the present value of a series of future payments discounted to the current period. This relationship is expressed as:

$$(5) \quad V_o = \frac{A_1}{(1+r)^1} + \frac{A_2}{(1+r)^2} + \ldots + \frac{A_n}{(1+r)^n}$$

As an example, consider payments of \$1,000, \$2,000, and \$3,000 to be received one, two, and three years in the future. Using an APR discount rate of 12 percent, the present value of the future payments is:

$$\$4,622.63 = \frac{\$1,000}{(1+.12)^1} + \frac{\$2,000}{(1+.12)^2} + \frac{\$3,000}{(1+.12)^3}$$

Geometric Series

The discounting discussion leads naturally to the second question: Can the sum of distant period flows converted to a common period be conveniently summed? The answer is yes if the terms are geometrically related. Fortunately, many present value problems are geometrically related.

A geometric series takes the form:

$$(6) \quad S = ab + ab^2 + ab^3 + \ldots + ab^n$$

where S is the sum of the series and where "a" and "b" are constants. It is called a geometric series because there is some constant factor of multiplication which if used to multiply the t^{th} term produces the

$(t + 1)^{st}$ term. In equation (6) the term is "b," which is called the geometric factor of the series.

A geometric series has a well defined sum in two cases: (1) when the series is of finite length and each term in the series is of finite size; and (2) when each term of the series is of finite size and the geometric series converges. The geometric series is said to converge if the sum of the series approaches a constant value as additional terms are added.[3] Since "a" is a constant this requires $-1 < b < 1$ for the series in equation (6) to converge. Assuming the series does converge it can be solved in a convenient way.

Geometric Series Summation

To sum a geometric series, several algebraic manipulations must be used. First, multiply both sides of (6) by the inverse of the geometric factor b to obtain:

$$(7)\quad \frac{S}{b} = a + ab + \ldots + ab^{n-1}$$

Next subtract equation (6) from equation (7) noticing that all but the first term of equation (7) and the last term of equation (6) cancel. The result is:

$$(8)\quad \frac{S}{b} - S = a - ab^{n}$$

or

---

[3]The mathematical definition of convergence is: an infinite series with partial sums $S_1, \ldots, S_n$ is said to converge if, and only if, the limit of S_n exists as n becomes infinitely large. In our previous example this requires that:

$$\lim_{n \to \infty} ab^{n} = 0$$

$$(9) \quad S = \frac{ab[1 - b^n]}{1 - b}$$

If n is allowed to become large, remembering that $|b| < 1$, the solution to equation (9) is:

$$(10) \quad S = \frac{ab}{1 - b}$$

To illustrate the geometric summation method, assume that an annuity A is to be received on an investment for n years and that the opportunity cost is r percent. The variables A, n, and r, of course, represent or stand in place of values which may be later substituted. For the present application let V_0 stand for the present worth of the n payments of amount A discounted at r percent. The equality is written as:

$$(11) \quad V_0 = \frac{A}{(1 + r)} + \frac{A}{(1 + r)^2} + \ . \ . \ . + \frac{A}{(1 + r)^n}$$

The three dots in equation (11) stand in place of the missing terms and serve to simplify the notation. In equation (11) the term which appears in all the right-hand side rates is A while the geometric factor is $(\frac{1}{1 + r})$. Letting A substitute for "a" and $(\frac{1}{1 + r})$ substitute for "b" in equation (9), the present value V_0 can be written as:

$$(12) \quad V_0 = A[\frac{1 - (1 + r)^{-n}}{r}]$$

For example, if 20 annual payments of \$100 are to be received which are discounted at 12 percent their present worth is:

$$\$746.94 = \$100 \ [\frac{1 - (1.12)^{-20}}{.12}]$$

The expression $[\frac{1 - (1 + r)^{-n}}{r}]$ in equation (12) is the factor which converts the uniform series of annual payments A to their present value equivalent. Because this expression will appear often, a simple expression will be used to represent it. Since it sums a uniform series of \$1 payments and converts them to their present or zero period equivalent, we write:

$$(13a)\quad US_0(r, n) = \frac{[1 - (1 + r)^{-n}]}{r}$$

In general, for any amount A, we write:

$$(13b)\quad V_0 = AUS_0(r, n).$$

Were the common time period selected to be in the n^{th} year instead of the current time, V_0 and $US_0(r, n)$ are multiplied by the compound factor $(1 + r)^n$ and:

$$(14)\quad V_n = (1 + r)^n AUS_0(r, n)$$

It is often the case that annuities are received (or payments made) at intervals shorter than one year, say m times a year. In such cases the relevant interest rate is an actuarial rate $\frac{r}{m}$ and the number of periods is mn, where the payments are received m times a year for n years.

The expression which converts mn \$1 annuities discounted at rate $\frac{r}{m}$ for n years is:

$$(15)\quad US_0(\tfrac{r}{m}, mn) = \frac{[1 - (1 + \frac{r}{m})^{-mn}]}{\frac{r}{m}}$$

Numerical values, accurate to the fourth decimal place, for $US_0(\frac{r}{m}, mn)$ are calculated in Table 1 at the end of this report. The present value of mn constant payments A can be expressed as:

$$(16)\quad V_0 = AUS_0(\tfrac{r}{m}, nm)$$

Were the common time period selected in the n^{th} year instead of the current time, V_0 and $US_0(\frac{r}{m}, mn)$ are multiplied by the compound factor $(1 + \frac{r}{m})^{mn}$ and

(17) $V_n = (1 + \frac{r}{m})^{mn} AUS_0(\frac{r}{m}, mn)$.

Having introduced the tools and notation for solving a geometric series, and in particular for solving a geometric series of uniform payments, the first and most common loan is introduced: the constant payment loan.

V. CONSTANT PAYMENT LOANS

A constant payment loan is a loan repaid by a uniform series of equal payments made at equal time intervals. The present value of the payments is obtained by discounting at the loan interest rate and summing. The discounted sum of the payments equals the original amount of the loan. The fundamental relationship between the number of payments (the term of the loan), the discount rate, the constant payment, and the loan amount is:

(18) $V_0 = AUS_0(\frac{r}{m}, mn)$

Where V_0 is the value of the loan provided in the present period, A is the uniform payment, $\frac{r}{m}$ is the actuarial interest rate, and n is the term of the loan in years.

If the loan amount, interest rate and loan term are known, the payment, A, is:

(19) $A = V_0/US_0(\frac{r}{m}, mn)$

From equation (18), if $A, \frac{r}{m} V$, and are known, one can easily solve for the number of payments mn necessary to retire a borrowed sum of money.

If A, V_0 and $\frac{r}{m}$ are known, mn can be determined using natural logarithms:

$$(20) \quad mn = \frac{-\ln(1 - \frac{r}{m}\frac{Vo}{A})}{\ln(1 + \frac{r}{m})}$$

Constant Payment Loan Examples

Suppose a $5,000 sum of money is borrowed from a bank for a period of 5 years. The loan is to be repaid with monthly installments at an actuarial rate of 1 percent. What is the payment or annuity necessary to retire the loan?

Solving by use of equation (19) obtains:

$$\begin{aligned} A &= V_0/US_0(.01, 60) \\ &= \$5{,}000/44.9550 \\ &= \$111.22 \end{aligned}$$

Notice that the number 44.9550, the $US_0(.01, 60)$ value, is from Table 1 at the end of this report (see page 48) or could be calculated directly using equation (13a).

Now suppose A is known to be $111.22, $\frac{r}{m}$ is .01, and mn is 60. The loan supported by payments at these terms can be found using equation (18):

$$\begin{aligned} V_0 &= AUS_0(.01, 60) \\ &= (\$111.22)(44.9550) \\ &= \$5{,}000 \end{aligned}$$

Finally, assume V_0 = $5,000, A = 111.22, $\frac{r}{m}$ = .01, but mn is not known. To find mn use equation (20), or divide V_0 by A and scan Table 1 under the 1% column until the $US_0(.01, mn)$ value corresponding to 44.9550 is found. The mn value at which that occurs is 60.

Total Interest Costs and the Rule of 78's

The tax deductibility of interest payments leads to another calculation of interest, total interest costs. In the case of the constant payment loan the total interest costs can be calculated by subtracting from total payments (mnA) the loan principle V_0. Denoting total interest costs as TI, this subtraction equals:

(21) $TI = (mnA - V_0)$

Total interest cost in the previous example equals:

$$TI = [60(\$111.22) - \$5,000]$$
$$= \$1673.20$$

Besides determining total interest costs, there are many situations where it is important to be able to determine the accumulated interest and principal paid as of some date. Early loan repayment and the need to calculate the tax savings associated interest payments are examples which may necessitate such calculations.

The Rule of 78's provides an approximation for calculating how much interest and principal is paid in each period. The approximation recognizes the fact that more interest is paid at the beginning of a loan than at the end of the loan because as principal is reduced, so are interest costs. This allows a larger portion of each fixed payment to be allocated to repaying principal at the end of the loan than at the beginning of the loan repayment.

The approximation takes its name from the calculations associated with a one-year loan consisting of 12 payments. The sum of the digits associated with 12 payments are: $1 + 2 + . . . + 12 = 78$. The formula allocates $\frac{12}{78}$ of the interest to be repaid in the first payment; $\frac{11}{78}$ of the interest to be repaid in the 2nd and so on. Thus total interest costs are allocated to periods according to the "sum of the year's digits" depreciation schedule.

The formula for calculating the sum of the digits for mn periods, SSD(mn) is:

$$(22)\quad SSD(mn) = \frac{mn(mn + 1)}{2}$$

And the interest paid on the t^{th} payment I(t) equals:

$$(23)\quad I(t) = TI[(mn - t + 1)/SSD(mn)]$$

In our example, SSD (60) is 1,830 with total interest of \$1,673.20 to be repaid on the loan. Approximate interest costs included in the 16th payment would be \$1,673.20[(60 - 15)/1830] = \$41.14.

The formula for accumulated interest paid after t payments, AI(t), is:

$$(24)\quad AI(t) = TI[1 - \frac{(nm - t)(nm - t + 1)}{nm(nm + 1)}]$$

$$AI(16) = \$1673.20[\frac{1 - (60 - 16)(60 - 15)}{60(60 + 1)}]$$

$$= \$768.03$$

Accumulated principal paid after t payments AP(t) is calculated as:

$$(25)\quad AP(t) = tA - AI(t)$$

Substituting in the formula yields:

$$AP(16) = (16)(\$111.22) - \$768.03$$

$$= \$1,011.49$$

Finally, if after t payments, the loan is to be retired or repaid, principal due PD(t) is simply:

$$(26)\quad PD(t) = V_0 - AP(t)$$

After the 16th payment, the outstanding principal is:

$$PD(16) = \$5,000 - \$1,011.49$$

$$= \$3,988.51$$

Therefore, the principal necessary to retire the remaining loan balance is approximately \$3,988.51.

The rule of 78's is a convenient method for approximating interest and principal paid and due. It was especially helpful in earlier times when widespread computing capacity was not available. It is important to keep in mind it is only an approximation; if the interest and principal calculation were computed using the $US_0(\frac{r}{m}, mn)$ formula from which payments were calculated, an exact interest and principal payment schedule could be obtained. The difference between the two methods is depicted in Figure 1. Panel a describes the differences in interest paid per period while Panel b describes the differences in accumulated interest paid. Notice that the Rule of 78's generates linear interest payment schedules while the US_0 formula does not.

For example, consider a loan for \$1125.51 which is repaid in 12 monthly installments of \$100 amount at a 1 percent actuarial rate. Since total payments equals \$1200, total interest paid must equal:

$$\begin{aligned} TI &= \$1200.00 - \$1125.51 \\ &= \$74.49 \end{aligned}$$

Notice that in the first period, interest costs calculated according to the $US_0(\frac{r}{m}, mn)$ formula equal \$11.25; calculated according to the rule of 78's, interest costs equal \$11.46. And in the sixth period, interest costs become \$6.69 with the rule of 78's and \$5.80 with the $US_0(\frac{r}{m}, mn)$ method. In the final or 12th period the interest cost calculation results in \$0.96 with the rule of 78's, compared with \$0.99 using $US_0(\frac{r}{m}, mn)$.

The formula for calculating principal paid PP(t) in the t^{th} period using the $US_0(\frac{r}{m}, mn)$ formula is:

Panel a

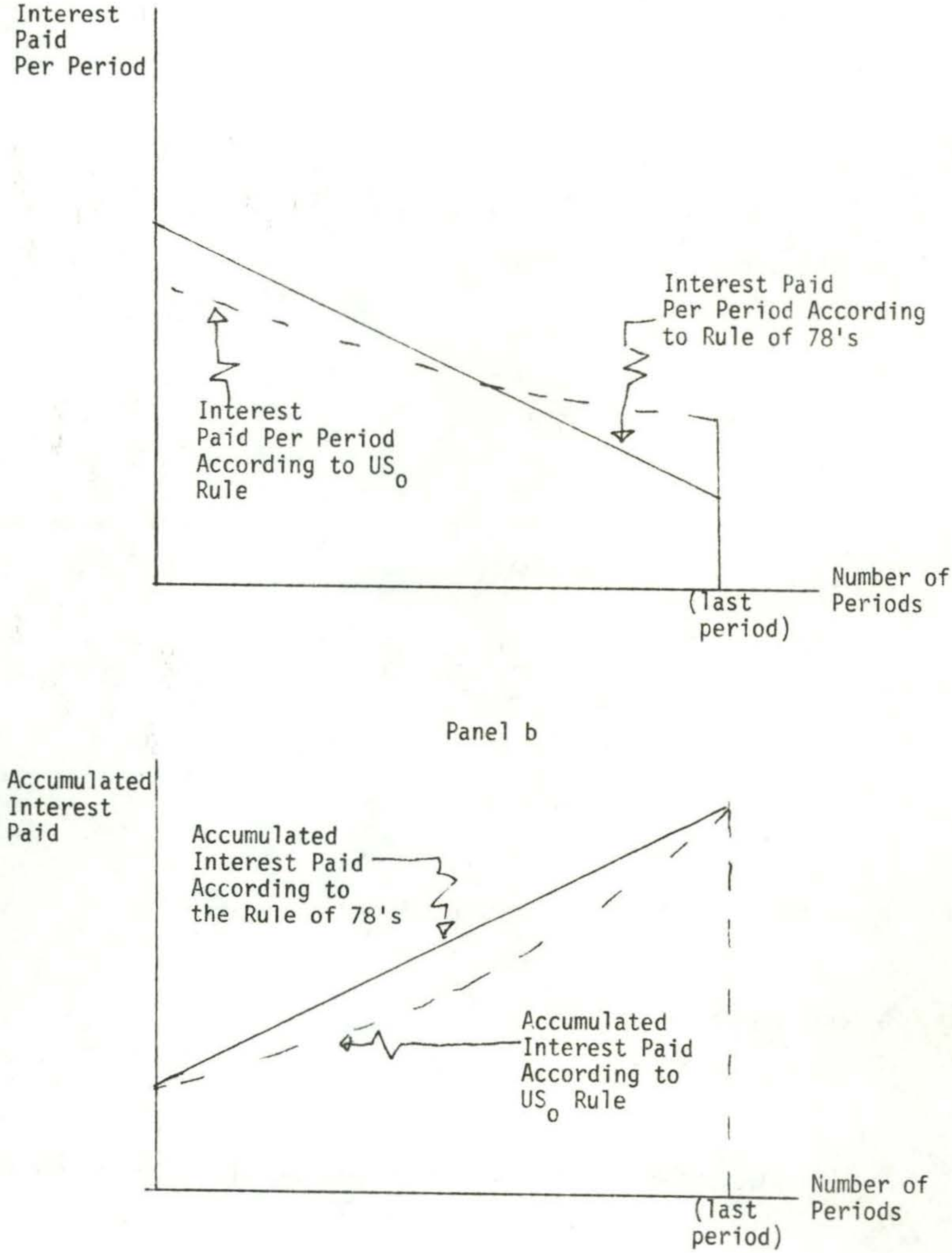

Figure 1: Comparison of Interest Costs Calculated Using the Rule of 78's and the US_0 Formula.

$$(27)\quad PP(t) = \frac{A}{(1 + r)^{n-(t-1)}}$$

In our example, of the \$100 payment, (where $\frac{r}{12} = .01$, and $mn = 12$), PP(1) equals:

$$(28)\quad PP(1) = \frac{\$100}{(1.01)^{12}} = \frac{\$100}{1.1268} = \$88.75$$

Obviously, the interest portion of the t payments, I(t) calculated using the $US_0(\frac{r}{m}, mn)$ formula is simply:

$$(29)\quad I(t) = A - PP(t)$$

In the previous example I (1) is:

$$I(1) = \$100 - \$88.75$$
$$= \$11.25$$

Accumulated principal AP(t) paid after t payments under the $US_0(\frac{r}{m}, mn)$ equation is:

$$(30)\quad AP(t) = A[US_0(\frac{r}{m}, mn) - US_0(\frac{r}{m}, mn - t)]$$

After six payments the result is:

$$AP(6) = \$100[US_0(.01, 12) - US_0(.01, 12 - 6)]$$
$$= \$100[11.2551 - 5.7955]$$
$$= \$545.96$$

Accumulated interest AI(t) payments formula follows:

$$(31)\quad AI(t) = tA - AP(t)$$

After six payments the result is:

$$AI(6) = \$600 - \$545.96$$
$$= \$54.04$$

Finally, principal due PD(t) to retire the loan after t payments is:

$$(32)\quad PD(t) = V_0 - AP(t)$$

After six payments the result is:

$$PD(6) = \$1125.51 - \$545.96 = \$579.55$$

Comparing Term and Payment Size

Both borrowers and lenders are concerned with the relationship between the term of the loan and the payment size. To calculate the reduction in loan payment, ΔA, associated with a one period increase in the term of a loan from mn to mn + 1 periods we write:

$$(33) \quad \Delta A(mn) = \frac{V_o}{US_o(\frac{r}{m}, mn + 1)} - \frac{V_o}{US_o(\frac{r}{m}, mn)}$$

It can be shown that as mn becomes large, the payment A approaches the interest cost; i.e., the smallest payment possible equals the interest charged on the outstanding loan balance.[4]

One might ask: Is there an optimal term for loans? The answer is yes, but it depends on the goals and needs of the borrower and lender. If the borrower wished to minimize his payment, the optimal term is the one which permits the borrower to repay only interest. The shortest repayment period, on the other hand, is one. We now provide a formula which provides a value which is between these extremes which economists find useful in other settings.

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[4]To see this take the limit of A after substituting in for $US_o(\frac{r}{m}, mn)$ the uniform series formula the result is:

$$\lim_{mn\to\infty} A = \frac{\frac{r}{m}V_o}{[1 - \frac{1}{(1 + \frac{r}{m})} - mn]}$$

$$= \frac{r}{m}V_o$$

We are interested in the sensitivity of the reduction in loan payment as the term increases. Obviously the absolute magnitude of the loan reduction decreases as the term increases; thus early increases in the term are more important than later. The percentage change in the loan payment size in response to a one percent increase in the term of the loan, however, is not a monotonic relationship. The term "n" at which the maximum percentage change in loan payment occurs in response to a one percent change in the term of the loan can be expressed as:[5]

$$(34)\quad mn = \frac{1}{\ln(1 + \frac{r}{m})}$$

It is interesting to note that as the interest rate in (34) goes up, the term which maximizes the elasticity of loan size reduction goes down:

$$(35)\quad \frac{d(mn)}{dr} = -(1 + \frac{r}{m})^{-1}(\ln(1 + \frac{r}{m}))^2(\frac{1}{m}) < 0$$

This result suggests as expected, that early repayment with high interest rates is more important than when interest rates are low.

Important to note is that equation (35) does not necessarily represent an optimum term. It may, however, provide a useful reference point for borrowers and lenders in deciding on an optimal loan term.

Borrowers and lenders may also wish to examine how total interest payments respond to increases in the term of a loan. Letting $\Delta TI(n)$ represent the increase in total interest payments as the term of a loan increases from mn to mn + 1 periods and A be the payment on an mn period loan we write:

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[5]This expression is obtained by maximizing the elasticity of the loan payment A with respect to the term of the loan.

$$(36)\quad \Delta TI(mn) = (mn + 1)(A + \Delta A) - mnA$$

$$= A + (mn + 1)\Delta A$$

$$= \frac{V_o}{US_o(\frac{r}{m}, mn)} + (mn + 1)V_o[\frac{1}{US_o(\frac{r}{m}, mn + 1)}$$

$$- \frac{1}{US_o(\frac{r}{m}, mn)}]$$

An Example of Term, Interest Cost and Loan Payment Trade-Offs

A landlord is financing the renovation of a property. The loan amount needed is \$28,000. The current interest rate is 15 percent. He asks: If the term of the loan is increased from 10 years to 11 years, what will be the reduction in the annual loan payment?

Using equation (33) he calculates:

$$\Delta A(10) = \frac{\$28,000}{US(.15, 11)} - \frac{\$28,000}{U.S.(.15, 10)}$$

$$= \$5349.94 - \$5579.02$$

$$= \$229.08$$

Thus the 10-year term payment of \$5579.02 is reduced by \$229.08 with a one-year increase in the term. Using equation (34) the term which finds the largest percentage change in the payment with respect to a one percent increase in term is:

$$n = \frac{1}{\ln(1.15)}$$

$$= 7.15 \text{ years}$$

Finally, the increase in total interest associated with a one-year increase in the term of the loan is found using equation (36).

$$\Delta TI(10) = \$5579.02 - (11)\$229.08$$

$$= \$3059.14$$

VI. BALLOON PAYMENT LOANS

While the constant payment loan, or the installment loan, is by far the most frequently used, many other types of loans have been developed. One reason for their development is volatile interest rates and the need to fit repayment terms to situations. When interest rates are volatile, lenders assume greater risk by offering fixed rate loans, particularly on longer term loans. The risk they incur with fixed rate lending is from a future increase in interest rates above the existing rate. In this situation the lender is caught with fixed rate assets while his cost of funds are rising. Borrowers may also be at risk from volatile interest rates if leaders change fees to renegotiate loan terms.

One response to higher interest rate risk has been to offer balloon payment loans. The balloon payment loan is like the constant payment loan except for the last payment. It is used to pay off a loan before its normal term is complete and the last payment need not equal the amount of the other payments. The advantage to the lender (and borrower) is that it provides an opportunity to renegotiate the interest rate on the remaining balance should the borrower decide to refinance his balloon payment.

The questions to answer in connection with a balloon payment loan are the same questions answered for the constant payment loan: What payments (constant and balloon) are required to repay the loan? What loan amount can be supported by known (constant and balloon) payments? And how many constant payments and a known balloon are required to retire a balloon payment loan?

Consider the relationship between the amount of a loan V_0 to be retired using a series of payments A made at regular intervals over (mn-1) periods at an interest rate of $\frac{r}{m}$ percent, with a final balloon payment B made in the mn^{th} period. The equality between the borrower loan (V_0) and the present cost (discounted series of payments A plus the balloon) is:

$$(37)\quad V_0 = \frac{A}{(1 + \frac{r}{m})} + \ldots + \frac{A}{(1 + \frac{r}{m})^{mn-1}} + \frac{B}{(1 + \frac{r}{m})^{mn}}$$

Since the n payments A constitute a uniform series, equation (37) can be written as:

$$(38)\quad V_0 = AUS_0(\frac{r}{m}, mn-1) + \frac{B}{(1 + \frac{r}{m})^{mn}}$$

To illustrate the formula, suppose a borrower could afford payments of \$150 per month for 9 years and 11 months at an actuarial rate of 1 percent. After 119 payments the borrower could then make a balloon payment of \$5,000. What loan would such a payment arrangement support? The answer is obtained by substituting the appropriate values into equation (38). Substitute 150 for A, 69.3975 for $US_0(.01, 119)$, \$5000 for B, and .3030 for $\frac{1}{(1 + .01)^{120}}$. The answer is:

$$\$11,924.63 = (\$150)(69.3975) + (\$5,000)(.3030)$$

If the amount of the loan is known as well as the size of the balloon payment, then the regular payment to be made over n periods at interest rate r can be found by solving for A in equation (38). It equals:

$$(39)\quad A = \frac{V_0 - B(1 + \frac{r}{m})^{-mn}}{US_0(\frac{r}{m}, mn-1)}$$

And if A and V_0 are known, the balloon payment can be easily found. It equals:

$$(40)\quad B = [V_0 - AUS_0(\tfrac{r}{m}, mn-1)](1 + \tfrac{r}{m})^{mn}$$

Finally suppose V_0, A, B, and $\frac{r}{m}$ are known. Can we find mn? The answer is yes! It equals:

$$(41)\quad mn = \ln\left[\frac{B - A(\frac{r}{m})^{-1}(1 + (\frac{r}{m})}{V_0 - A\,(\frac{r}{m})^{-1}}\right] / \ln(1 + \tfrac{r}{m})$$

Thus total interest paid under the terms of a balloon loan is:

$$(42)\quad TI = mnP + B - V_0$$

Balloon Payment Loan Examples

A loan is taken out for \$75,000 which has a balloon payment of \$25,000 due in 11 years. The loan terms include quarterly payments for 10 years and 9 months (43 payments) at an actuarial rate of 3 percent or 12 percent APR. What is the payment necessary to meet these restrictions? The payment is found using equation (39) and Table 1 and is equal to:

$$A = \frac{\$75,000 - \$25,000\ (1 + .03)^{-44}}{US_0(.03,43)} = \frac{\$75,000 - \$25,000\ (.2724)}{23.9819}$$

$$= \$2,843.40$$

A loan agreement calls for a \$50,000 loan to be repaid with \$900 payments every month for four years and 11 months (59 payments) at an actuarial rate of 1 percent plus a balloon payment at the end of five years. What will be the balloon payment or principal remaining after five years?

Solving using equation (40) yields:

$$B = [\$50,000 - \$900\ US_0(.01,\ 59)](1 + .01)^{60}$$

$$= \$18,232.11$$

If one wanted to determine total interest paid after five years, equation (42) is used to give:

$$TI = (59)(\$900) + \$18{,}232.11 - \$50{,}000$$

$$= \$21{,}332.11$$

VII. MODERATED PAYMENT LOANS

Another category of loans which is designed for the cash flow problems of borrowers is the moderated payment loans. Such loans are designed to reduce cash flow requirements of the borrower in early periods of the loan in anticipation of increased payment ability later on in the life of the loan. Inflationary expectations tend to add to the cost of borrowing money by increasing interest rates. This relationship between inflation and interest rates on loans creates somewhat of a problem for borrowers. With a constant payment loan, interest costs are a large percentage of the loan, especially during the early payment periods. But if the loan is for the purchase of an investment whose earnings will also benefit from inflation, cash flows later on may be more than adequate for the constant payment.

Skip Payment Loan

One type of moderated payment loan is the "skip payment" loan. As its name implies, payments are not made during the first t periods of the loan. The relationship between the loan amount V_o, the payments A, the actuarial interest rate on the loan $\frac{r}{m}$, the term of the loan mn, and the $(t + 1)^{st}$ period in which the first payment occurs is:

$$(43) \quad V_o = \frac{A}{(1 + \frac{r}{m})^{t+1}} + \ . \ . \ . + \frac{A}{(1 + \frac{r}{m})^{mn}}$$

$$= \frac{1}{(1 + \frac{r}{m})^t} [\frac{A}{(1 + \frac{r}{m})} + . . . + \frac{A}{(1 + \frac{r}{m})^{mn-t}}]$$

Since the bracketed series is a uniform series equal to $US_o(\frac{r}{m}, mn - t)$, (43) can be written as:

$$(44) \quad V_o = \frac{A}{(1 + \frac{r}{m})^t} US_o(\frac{r}{m}, mn - t)$$

From this relationship the payment A can be found. It equals:

$$(45) \quad A = \frac{V_o(1 + \frac{r}{m})^t}{US_o(\frac{r}{m}, mn - t)}$$

The term of the loan given V_o, A, r, and t can easily be found. It equals:

$$(46) \quad mn = \frac{-\ln[1 - (\frac{r}{m})(\frac{V_o}{A})(1 + \frac{r}{m})^t]}{\ln(1 + \frac{r}{m})} + t$$

Total interest costs are calculated much as before, namely, total payments (mn - t)A less principal.

$$(47) \quad TI = (mn - t)A - V_o$$

Examples of Skip Payment Loans

The purchasers of a home could afford monthly payments of \$500.00 a month but will not be able to begin payments for one year. If the lender agrees to forego the first year's payments and loan terms include an interest rate of 14 percent APR for 20 years, how large a loan can the purchaser afford?

Solving using equation (44) yields:

$$V_o = [\frac{\$500}{(1 + .14/12)^{12}}] [US_o(.14/12, 240 - 12)]$$

$$= [434.8582][79.6257]$$

$$= \$34,625.89$$

The purchaser of a car is not required to make the first six monthly payments on the loan. If the loan is for \$10,000 at 18 percent APR for four years, what is the monthly payment amount necessary to retire the loan?

Solving using equation (45) yields:

$$A = \frac{\$10{,}000(1 + .015)^6}{US_0(.015, 42)}$$

$$= \frac{\$10{,}000(1.0934)}{30.9941}$$

$$= \$352.79$$

Skip Principal Loan

A loan similar to the skip payment loan is the skip principal loan. This loan requires interest payments for the first few t periods but does not require principal payments. After t periods, payments include both principal and interest. The relationship between the original loan V_0, the payment A, the term of the loan mn, the interest rate on the loan $\frac{r}{m}$ and the period of the first principal and interest payment t + 1 is:

$$(48) \quad V_0 = \frac{\frac{r}{m}V_0}{(1 + \frac{r}{m})} + \cdots + \frac{\frac{r}{m}V_0}{(1 + \frac{r}{m})^t}$$

$$+ (1 + \frac{r}{m})^{-t} \quad \frac{A}{(1 + \frac{r}{m})} + \cdots + \frac{A}{(1 + \frac{r}{m})^{mn-t}}$$

Since the first braced sum is a uniform series of payments $\frac{r}{m}V_0$ made for t periods at interest rate $\frac{r}{m}$, and the second braced series is the one associated with the skip payment loan, V_0 can be written as:

$$(49) \quad V_0 = \frac{rV_0}{m} US_0(\frac{r}{m}, t) + \frac{AUS_0(\frac{r}{m}, mn - t)}{(1 + \frac{r}{m})^t}$$

Solving for V_o in equation (49) obtains:

(50) $V_o = AUS_o(\frac{r}{m}, mn - t)$

From the above equation A can be easily found, equal to:

(51) $A = V_o/US_o(\frac{r}{m}, mn - t)$

The term of the loan can best be solved for using equation (51). Assuming V_o, r, A and t are all known, mn can be expressed as:

(52) $mn = \dfrac{-\ln(1 - \frac{r}{m}\frac{V_o}{A})}{\ln(1 + \frac{r}{m})} + t$

or using the $US_o(\frac{r}{m}, mn)$ approach as:

(53) $US(\frac{r}{m}, mn - t) = \dfrac{V_o}{A}$

When mn - t is found from the above expression t is added to obtain mn.

Finally, total interest costs can be calculated as:

(54) $TI = t\frac{r}{m}V_o + (mn - t)A - V_o$

Examples of Skip Principal Loans

In order to moderate cash flow problems in the early part of a five-year loan, the principal portion of the loan payment is skipped for two years. If the terms of the \$20,000 loan include monthly payments at 13 percent APR, or an actuarial rate of 1.08 percent, what is the payment necessary to retire the loan?

Solving using equation (51) gives:

$$A = \frac{\$20,000}{US_o(.0108, 36)}$$

$$= \$673.88$$

For 24 months of the loan, payments are \$216.00 ($\frac{r}{m} \times V_o$). Payments for the next 36 periods are \$673.88. A payment of \$673.88 is, of course, larger than \$455.06, the payment necessary to retire the loan if no skips were made in the five-year amortization.

The purchaser of a new home can afford monthly payments of \$500 <u>after</u> the first year of the loan. Suppose the lender agrees to permit the borrower to pay only interest costs the first year of his loan and principal and interest payments of \$500 per month for the next 19 years. At an APR rate of 14 percent, how large a loan can the borrower obtain?

The question is solved using equation (50) so that:

$$\begin{aligned} V_0 &= \$500\ US_0(.0117,\ 228) \\ &= \$39{,}812.85 \end{aligned}$$

Buy Down Loan

A third loan in the moderated payment class is the "buy down loan." This type of loan moderates or reduces the interest charged on early payments. For example, assume the APR rate of interest is r percent. The lender offering a "buy down" loan agrees to place in escrow a fund which pays part of the first t payments sufficient to reduce the actual interest rate charged during the first t periods to an APR rate of $(r - \Delta r)$ percent.

The relationship between the principal amount borrowed V_0, the APR interest rate r, the subsidized rate $(\frac{r - \Delta r}{m})$, the terms of the loan mn, the subsidized payment A_s and the regular payment A is:

$$(55)\quad V_0 = \frac{A_s}{(1 + \frac{r - \Delta r}{m})} + \cdot\cdot\cdot + \frac{A_s}{(1 + \frac{r - \Delta r}{m})^t} + \frac{A}{(1 + \frac{r}{m})^{t+1}} + \cdot\cdot\cdot + \frac{A}{(1 + \frac{r}{m})^{mn}}$$

Since the sum obviously contains two uniform series, it can be rewritten as:

$$(56)\quad V_o = A_s US_o(\frac{r - \Delta r}{m}, t) + AUS_o(\frac{r}{m}, mn - t)/(1 + \frac{r}{m})^t$$

If the interest rate over mn periods remained at $(r - \Delta r)$, a constant payment of A_s would fully amortize the loan. Similarly, if the interest rate remained at r, a constant payment of A would fully amortize the loan. Thus A_s and A can be calculated using equation (19):

$$(57)\quad A_s = V_o/US_o((\frac{r - \Delta r}{m}), mn)$$

and

$$(58)\quad A = V_o/US_o(\frac{r}{m}, mn)$$

One may also be interested in the present value of the subsidy required to reduce the APR rate from r to $r - \Delta r$. The answer can be easily found by comparing the present value of annuity payments equal to the difference between A and A_s. The present value of that subsidy, V_s, is:

$$(59)\quad V_s = (A - A_s)US_o(\frac{r}{m}, t)$$

The term of the loan can be found using either equation (57) or (58) above. But for the latter, mn can be found equal to:

$$(60)\quad mn = \frac{-\ln(1 - \frac{r}{m}\frac{V_o}{A})}{\ln(1 + \frac{r}{m})}$$

or using the $US_o(\frac{r}{m}, mn)$ formula where

$$US_o(\frac{r}{m}, mn) = \frac{V_o}{A}$$

Finally, total interest costs can be calculated as:

$$(61)\quad TI = tA_s + (mn - t)A - V_o$$

Examples of Buy Down Loans

A lender offers a buy down loan to moderate cash flow problems on a new business. Suppose the lender deposits sufficient funds in an escrow account to lower the stated APR interest rate of 12 percent to 6 percent for a period of two years. If the new loan is for $100,000 and is to be repaid with monthly payments for 10 years, what will be the payments necessary to fully amortize the loan?

Solving using equations (57) and (58) gives:

$$A_s = \$100{,}000/US_0(.005,\ 120)$$
$$= \$1{,}110.20$$

and

$$A = \$100{,}000/US_0(.01,\ 120)$$
$$= \$1{,}434.71$$

The present value of the subsidy provided by the lender can be found using equation (59):

$$V_s = (\$1{,}434.71 - \$1{,}110.20)US_0(.01,\ 24)$$
$$= \$6{,}893.70$$

Now consider a similar situation in which the subsidized payment amount is $2,500 and the unsubsidized amount is $3.000. Using the same payment schedule and interest rates as described in the previous example, what is the loan amount these payments will support?

Solving using equation (56) gives:

$$V_0 = \$2{,}500\ US_0(.005,\ 24) + \$3{,}000\ US_0(.01,\ 96)/(1 + .01)^{24}$$
$$= \$201{,}782.75$$

Graduated Payment Loans

Still another loan which moderates early payments is the graduated payment loan (GPL). GPL's have payments increasing at the end of each year or each payment period. Unlike regular loans, GPL's early payments do not necessarily cover interest costs. If early payments do not cover interest costs, the loan balance increases until payments exceed interest costs.

Several different versions of GLM's can be considered. The most general GPL form however, has payments increasing by g percent every m periods or every year until the loan is repaid after nm periods. The relationship between V_o, the present value of the loan, r, the interest rate, m, payments per year, mn, the term of the loan and A, the initial payment is:

$$(62)\quad V_o = \frac{A}{(1+\frac{r}{m})} + \ldots + \frac{A}{(1+\frac{r}{m})^m} + \frac{A(1+g)}{(1+\frac{r}{m})^{m+1}} + \ldots + \frac{A(1+g)}{(1+\frac{r}{m})^{2m}} + \ldots + \frac{A(1+g)^{n-1}}{(1+\frac{r}{m})^{(n-1)m+1}} + \ldots + \frac{A(1+g)^{n-1}}{(1+\frac{r}{m})^{nm}}$$

To solve for V_o as a function of A, r, g, n, and m consider the first braced expression. It is simply the uniform series $AUS_o(\frac{r}{m}, m)$. The second braced series is also a uniform series when the $(1 + g)/(1 + \frac{r}{m})^m$ term is factored out. The same relationships hold for all other braced expressions. So equation (62) is simplified as:

$$(63)\quad V_o = AUS_o(\tfrac{r}{m}, m) + \frac{A(1+g)}{(1+\frac{r}{m})^m} US_o(\tfrac{r}{m}, m) + \ldots + \frac{A(1+g)^{n-1}}{(1+\frac{r}{m})^{(n-1)m}} US_o(\tfrac{r}{m}, m)$$

$$= AUS_o(\tfrac{r}{m}, m)\left[1 + \frac{(1+g)}{(1+\frac{r}{m})^m} + \ldots + \frac{(1+g)^{n-1}}{(1+\frac{r}{m})^{(n-1)m}}\right]$$

Equation (63) is simplified by finding the geometric sum of the bracketed expression in equation (58). While it is not the same geometric series solved in most loan formulas, it is a geometric formula and can be solved using equation (9) using $(1 + g)/(1 + \frac{r}{m})^m$ as the geometric factor. Letting W equal the bracketed expression in equation (63), it can be expressed as:

$$(64) \quad W = \frac{(1 + r)^m}{(1 + r)^m - (1 + g)} [1 - \frac{(1 + g)^n}{(1 + r)^{nm}}]$$

which finally allows us to write V_0 as:

$$(65) \quad V_0 = AUS_0(\frac{r}{m}, m)W$$

Meanwhile, the first m payments equal:

$$(66) \quad A = V_0/US_0(\frac{r}{m}, m)W$$

Payments in the 2nd m periods are $A(1 + g)$, etc.; payments in the last m periods are $A(1 + g)^{n-1}$. Total interest costs can be found equal to:

$$(67) \quad TI = mA + mA(1 + g) + . . . + mA(1 + g)^{n-1} - V_0$$

which is another geometric sum equal to:

$$(68) \quad TI = \frac{mA}{g} [(1 + g)^n - 1] - V_0$$

For the special case where each payment is subject to growth, the formula for V_0 collapses to:

$$(69) \quad V_0 = \frac{A}{\frac{r}{m} - g} [1 - \frac{(1 + g)^{mn}}{(1 + \frac{r}{m})^{mn}}]$$

and

$$(70) \quad A = \frac{V_0(\frac{r}{m} - g)}{[1 - \frac{(1 + g)^{mn}}{(1 + \frac{r}{m})^{mn}}]}$$

An Example of a Graduated Payment Loan

Suppose a business project expects a future increase in repayment capacity, but currently is encountering cash flow difficulties. To moderate the current cash flow requirements, a bank agrees to make an increasing payment loan. The lender provides a loan amount of \$100,000, with semiannual payments amortized for five years at an APR of 14 percent with 3 percent increases in the payment amount every six months. What is the amount of the first and last payments of the loan plan? What is the total interest paid?

Solving using equation (70) gives:

$$A = \frac{\$100{,}000(.07 - .03)}{[1 - \frac{(1 + .03)^{10}}{(1 + .07)^{10}}]}$$

$$= \$12{,}626.26$$

The first payment is \$12,626.26 while the last payment equals \$12,626.26 times $(1 + .03)^{10-1}$ or \$16,474.74.

Total interest paid is solved using equation (68) giving:

$$TI = \frac{\$12{,}626.26}{.03}[(1 + .03)^{10} - 1] - \$100{,}000$$

$$= \$44{,}739.03$$

Concessionary Interest Rate Loans

A concessionary interest rate loan is one whose stated interest rate is below the market rate of interest. Often concessionary interest rates are exchanged for other considerations. For example, a seller may offer a concessionary rate to a buyer in exchange for a higher purchase price. In the case of land, this converts interest income to capital gains which are taxed at a lower interest rate. There are, however, legal restrictions limiting the extent to which this conversion can occur.

In connection with a concessionary interest rate loan, there are several questions to consider. First, what is the present value of a loan amount V_0 with a term of mn periods issued at a contract rate of $\frac{r^*}{m}$ when the current interest rate is $\frac{r}{m}$ percent?

To answer this question, consider the payment A* associated with the constant payment concessionary rate loan. Using equation (18), it can be written as:

$$(71) \quad A^* = V_0 / US_0(\frac{r^*}{m}, mn)$$

The term, interest costs and other calculations associated with A* and V_0 are the same as for the constant payment loan.

The present value of mn payments of A*, discounted at the market rate of interest $\frac{r}{m}$ percent can be written as $V_0{}^*$ equal to:

$$(72) \quad V_0{}^* = \frac{A^*}{(1 + \frac{r}{m})} + \ . \ . \ . + \frac{A^*}{(1 + \frac{r}{m})^{mn}}$$

$$= A^* \ US_0(r, mn)$$

and substituting for A* using equation (71) obtains:

$$(73) \quad V_0{}^* = V_0 US_0(\frac{r}{m}, mn)/US \ (\frac{r^*}{m}, mn)$$

An Example of a Concessionary Rate Loan

A parcel of land is offered for sale with two different selling terms. The first includes a \$50,000 purchase price with \$30,000 being financed for 10 years with monthly payments at the market rate of interest, an APR of 12 percent. The second includes a \$55,000 purchase price with \$35,000 being financed for 10 years with monthly payments and an APR of 11 percent, a concessionary interest rate. Assuming the market rate of interest continues at 12 percent during the life of the loan, what is the present value of both offers?

The present value of the loan discounted at the market interest rate is \$50,000. The present value of the concessionary rate loan is found using equation (73) plus the \$20,000 down payment:

$$V_0^* = \$35{,}000\ US_0(.01,\ 120)/US_0(.0092,\ 120) + \$20{,}000$$
$$= (\$35{,}000)(69.7005)/72.5953 + \$20{,}000$$
$$= \$53{,}604.35$$

The present value of the concessionary loan is greater than \$50,000, the present value of the first option discounted at the market interest rate. Thus, ignoring taxes, the first option is preferred.

VIII. WRAP AROUND LOANS WITH BLENDED INTEREST RATES

A blended or consolidated rate loan may be offered when a borrower holds an existing mortgage but desires additional financing. If the interest rates have not changed, the additional loan request may be granted at the same interest rate. But if the new interest rate is different, a new loan may be made to the borrower that blends the lower interest rate on the existing mortgage with the higher market rate on new loans. With a wrap around loan, the borrower makes one payment to the new lender and the new lender assumes responsibility for repaying the old loan. In many instances, both the old and the new loan may be to the same institution.

The wrap around mortgage in essence consolidates the borrowers debt requirements under a single loan. The question is: What is a "fair" interest rate to charge on the new loan? A fair rate would be one which allows the borrower to retain the lower interest rate benefit associated with his old loan while allowing the lender to earn the current return on the new funds provided.

To calculate, let the (now) concessionary interest rate on the old loan balance be V_o with a remaining term of mn* at a concessionary rate of $\frac{r^*}{m}$. The present value of this loan was calculated as V_o^* in equation (73). Let the new extension be $\hat{V}_o$ which is extended for mn periods at the market interest rate of $\frac{r}{m}$ on a constant payment loan.

The lender now desires to find a blended rate $r_{b/m}$ to offer on a new loan balance of $V_o + \hat{V}_o$ for mn periods. One requirement is that the present value of the blended rate payments, discounted at the market rate $\frac{r}{m}$, equal the present value of the concessionary rate loan plus the extension. This is written as:

$$(74)\quad A_b US_o(\tfrac{r}{m}, mn) = V_o^* + \hat{V}_o$$

where A_b is the blended interest rate loan payment.

The second requirement is that the present value of the blended rate payments discounted at the blended rate $r_{b/m}$ equal the sum of the concessionary interest rate loan balance plus the extension. This second requirement is expressed as:

$$(75)\quad A_b US_o(r_{b/m}, mn) = V_o + \hat{V}_o$$

Dividing (75) by (74) and rearranging terms produces the result:

$$(76)\quad US_o(r_{b/m}, mn) = \left[\frac{V_o + \hat{V}_o}{V_o^* + \hat{V}_o}\right] US_o(\tfrac{r}{m}, mn)$$

From this expression the blended rate can be determined by interpolating values from Table 1. Once the blended rate is determined, the blended rate payment can be found using equation (75):

$$(77)\quad A_b = (V_o + \hat{V}_o)/US_o(r_{b/m}, mn)$$

Solving for the blended rate allows the lender to find the "break even," blended rate. This rate earns a return on the additional loan $\hat{V}_o$ equal to the rate of return on a separate new loan. If the lender wanted to earn a return more than an APR rate of r_b on the new loan amount $\hat{V}_o$, a rate higher than r_b must be charged. Suppose that rate is $r_b + \Delta r_b$. The question is: What would the higher payment A_b equal? The answer can be obtained easily from equation (72). It equals:

$$(78)\quad A_b = (V_o + \hat{V}_o)/US_o(r_b + \Delta r_b)/m, mn)$$

Having once calculated A_b and r_b and knowing mn, all of the interest and principal calculations performed for the constant payment loan can be repeated.

Suppose the borrower in the wrap around loan just described is offered a blend rate $r_{b/m}$. He asks: What interest rate $\frac{r}{m}$ applied to the extension and term adjustment is implied by the blend rate $r_{b/m}$? This implied interest rate can be found using (76) and the right-hand side of (73) substituted for V_o^*. It equals:

$$(79)\quad US_o\ (\tfrac{r}{m}, mn) = \frac{\hat{V}_o US\ \ (r_{b/m}, mn)}{\left[V_o + \hat{V}_o - \frac{V_o US_o(r_{b/m}, mn)}{US_o(\frac{r^*}{m}, mn)}\right]}$$

<u>An Example of a Blended Rate Loan</u>

Due to an increase in sales volume, company A decides to increase production capacity. The expansion would require additional borrowings of \$100,000 for five years. The company already has \$71,000 borrowed at 8 percent APR with 10 semiannual payments remaining. New loans terms offered by the lender include semiannual payments and a five-year amortization at 14 percent APR. What will be the blended interest rate?

To find the blended interest rate requires the present value of the concessionary interest rate loan be calculated. Using equation (73),

V_o* is calculated to equal \$61,482. Then equation (76) allows us to express $US_o(r_b/2, 10)$ as:

$$US_o(r_b/2, 10) = \left[\frac{\$71,000 + \$100,000}{\$61,482 + \$100,000}\right] US_o(.07, 10)$$

$$= (1.0589)(7.0236)$$

$$= 7.4373$$

Scanning Table 1 along the 10 period row, 7.4373 is found to be between 5 percent and 6 percent. By interpolation the blended actuarial rate equals 5.79 percent or an APR of 11.58 percent.[6]

Suppose from the previous example the lender was not satisfied with the 5.79 percent actuarial blended rate and required a blended actuarial rate one percent higher. What would the higher blended payment be?

Solving using equation (73) gives:

$$A_b = (\$71,000 + \$100,000)/US_o(.0579 + .01, 10)$$

$$= \$171,000/7.0923$$

$$= \$24,110.65$$

Suppose the borrower in the previous example asks: For the blended rate of 6.79 percent, what acturial rate is implied on the extension and firm adjustment on the concessionary rate loan?

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[6]Interpolation procedure can be demonstrated for this example. Using the uniform series present value table find the mn = 10 column and then look across the table until a value near 7.4373 is found. This value lies between 7.7217 and 7.3601 which correspond to 5 percent and 6 percent. Using the difference between $US_o(.05, 10)$ and $US_o(.06, 10)$ to weight the difference between $US_o(5, 10)$ and $US_o(r_b/m, 10)$, the following interpreted value is obtained:

$$= 5\% + [(7.7217 - 7.4373)/(7.7217 - 7.3601)](6\% - 5\%)$$

$$= 5.7865\%$$

Using equation (79) obtain the result:

$$US_o(\frac{r}{2}, 10) = \frac{(\$100,000)(7.0923)}{[\$71,000 + \$100,000 - \frac{(\$71,000)(7.0923)}{8.1109}]}$$

$$= 6.5117$$

Scanning the 10 period row in Table 1 locates a US ($\frac{r}{2}$ 10) equal to 6.5117 between 8 and 9 percent. Extrapolating and multiplying by 2 gives an implied interest rate of 17.4 percent.

IX. CONSTANT PRINCIPAL PAYMENT LOANS

Another type of loan is the constant principal payment loan. Like its name implies, it requires a constant principal payment each period. As a result, interest costs and total loan payment decrease over time.

The principal payment in each period is V_o/mn; the interest cost in the t^{th} payment is $r(V_o - \frac{(t - 1)V_o}{mn})$. Thus the payment in the t^{th} period is:

$$(80) \quad A_t = \frac{V_o}{mn} + rV_o \quad [1 - \frac{(t - 1)}{mn}]$$

To illustrate, suppose a \$50,000 loan offers annual constant principal payments for 20 years at 14 percent APR. The principal payment each year is:

$$V_o/n = \$50,000/20 = \$2,500$$

To determine the interest cost in the 8th period use:

$$r[V_o - \frac{(t - 1)(V_o)}{n}] = .14[\$50,000 - \frac{(7)(\$50,000)}{20}]$$

$$= \$4,550$$

so that the payment in the eighth period equals:

$$A_8 = \$2,500 + \$4,550$$
$$= \$7,050$$

X. DISGUISED INTEREST RATE LOANS

A final category of loans we call disguised interest rate loans. We call them disguished because interest rates are increased by methods other than increasing the interest rate associated with the loan. Various methods can be employed to increase the interest rate on the loan. For example, interest costs can be subtracted in the initial period, reducing the actual loan amount received by the borrower (a discounted loan). Interest can be charged as though the original loan balances were outstanding throughout the life of the loan (an add-on loan). Alternatively, the lender can charge "points" to close a loan, reducing the actual loan balance received by the borrower. Or, the interest can compound more frequently than loan payments occur. Each of these methods will increase the interest rate above the stated interest rate. Several types of disguised interest cost loans are now discussed.

The Discount Loan

A borrower approaches his lender for a loan of V_0 for mn periods. The borrower learns that the stated interest rate is "i" percent. When the borrower picks up the check for his loan, the amount he receives equals only:

$$(81) \quad V_d = V_0(1 - in)$$

The amount of loan actually received has had the interest cost subtracted in advance. The discount was the stated interest times the number of years for which the loan will be outstanding.

The periodic loan payments, meanwhile, are calculated as:

(82) $A = V_o/mn$

To calculate the APR interest rate r associated with this loan, treat payments A as if they were associated with a constant payment loan which retires a principal amount of V_d equal to $V_o(1 - in)$. The relationship is expressed as:

(83) $V_d = AUS_o(\frac{r}{m}, mn)$

Next, substitute for A using equation (82) to obtain the expression:

(84) $mnV_d/V_o = US_o(\frac{r}{m}, mn)$

The APR interest rate r in equation (84) will always be higher than the stated interest rate i because: one, the interest costs are subtracted at the beginning of the loan; and two, the interest cost for the loan term is calculated on the original loan balance.

An Example of a Discount Loan

A consumer obtains an installment loan for \$10,000 from which \$2,500 is deducted for interest costs. The loan is to be repaid over two years with monthly payments equal to \$416.67 (\$10,000/24). Using equation (84) $US_o(\frac{r}{12}, 24)$ associated with this loan is calculated to equal:

$$US_o(\frac{r}{m}, mn) = (24)(\$7{,}500)/\$10{,}000$$
$$= 18.0000$$

Using Table 1, and scanning across the row when mn equals 24, the $US_o(\frac{r}{m}, 24)$ value of 18.0000 lies between the values 18.9139 and 16.9355 which are associated with actuarial interest rates of 2 percent and 3 percent respectively. The actuarial rate after interpolation equals:

$$.0246 = \frac{18.9139 - 18.0000}{18.9139 - 16.9355}(.03 - .02) + .02$$

The actuarial rate is converted to an APR rate by multiplying by 12 which equals:

$$r = .295$$

or 29.5 percent.

The Add-On Loan

The "add-on loan" like the discount loan, calculates interest costs in an inappropriate manner. It adds interest costs as though the entire loan was to be outstanding for the life of the loan. To illustrate, the borrower applies for a loan of V_0 for n periods at a stated interest rate of i percent. In this case, the borrower actually receives the loan amount of V_0, but loan payments are calculated as:

$$(85) \quad A = \frac{(1 + in)V_0}{mn}$$

The actuarial rate $\frac{r}{m}$, however, is calculated from the relationship:

$$(86) \quad V_0 = AUS_0(\tfrac{r}{m}, mn)$$

from which one can find $US_0(\frac{r}{m}, mn)$ equal to:

$$(87) \quad US_0(\tfrac{r}{m}, mn) = V_0/A$$

The interest rate corresponding to $US_0(\frac{r}{m}, mn)$ can be found from Table 1 by finding a value corresponding to the value calculated in (87) along the row mn.

An Example of an Add-On Loan

A bank offers a personal loan that uses an add-on method of interest calculation with a stated interest rate of 10 percent. On a $2,000 loan the borrower repays the loan in 24 monthly installments. What is the APR for this loan plan?

The payment is determined by equation (85) and equals:

$$A = \frac{[1 + in]V_0}{mn} = \frac{[1 + (.10)(2)] \ \$2,000}{24}$$

$$= \$100.00$$

The $US_0(\frac{r}{12}, 24)$ value is calculated using equation (87):

$$US_0(\frac{r}{12}, 24) = \$2,000/\$100$$

$$= 20.000$$

By interpolating values in Table 1 along the row equal to 24, 20.00 yields an actuarial rate of .0151. The APR rate then equals .0151 x 12 or 18.2 percent.

Points Added Loan

Sometimes lenders charge "points" to close a loan. For example let "p" be the percent of the loan charged as a closing fee. This has the effect of increasing the interest rate on the loan since the borrower earns more than the stated rate suggests. The APR rate for such a loan can be calculated by first computing the payment which retires the loan plus points at the stated interest rate i. It equals:

$$(88) \quad A = \frac{(1 + p)V_0}{US_0(\frac{r}{m}, mn)}$$

Next express the relationship between the payment A, the APR rate r and the actual amount of the loan received as:

$$(89) \quad US_0(\frac{r}{m}, mn) = V_0/A$$

$$= \frac{US_0(\frac{i}{m}, mn)}{(1 + p)}$$

An Example of a Points Added Loan

A bank offers a loan rate of 12 percent with monthly payments for three years with a 3 percent loan closing fee. What is the APR?

Using equation (88), $US_0(\frac{r}{m}, mn)$ is found to equal:

$$US_0(\frac{r}{m}, mn) = \frac{30.1075}{1.03}$$

$$= 29.2306$$

Using Table 1, a value of 29.2306 corresponds approximately to an actuarial interest rate of 1.17 percent, an APR rate of (12 x 1.17) equals 14.04 percent.

XI. SUMMARY

This report has introduced and analyzed major types of loans using present value techniques. Loan types included in the analysis were: constant payment loans, moderated payment loans, blended rate loans, constant principal loans, and disguised interest rate loans. Because of the importance of constant payment loans, this type of loan was discussed in the most detail.

Moderated payment loans are designed to reduce the size of loan payments during the early periods of the loan. Different loan plans achieve this objective. Several plans, including skip payment and skip principal loans, graduated payment loans, concessionary interest rate loans, and pay down loans, were analyzed. Other unique types of loans analyzed using present value tools were wrap-around loans and constant principal loans.

Another class of loans are disguised interest cost loans. Discounted, add-on, and points added loans all have one feature in common: they increase the APR rate of interest above the stated interest rate.

Fortunately, federal truth in lending laws require reporting of the APR interest rate.

One loan plan not reviewed is the adjustable rate mortgage (ARM). The essential feature of ARMs is that the interest rate changes according to some previously agreed upon signal. Unfortunately, there is such a wide variety of ARMs that they are difficult to typify.

Hopefully, the formulas, examples, and accompanying tables provided in this report will provide a useful reference of up-to-date tools which will help those facing financial decisions to analyze the implications of selecting one of the various types of loans discussed.

Table 1

Present Value of a Uniform Series of Payments

Interest Rate Per Period $\frac{r}{m}$

mn	0.50%	0.58%	0.67%	0.75%	0.83%	0.92%	1.00%	1.08%	1.17%	1.25%	1.33%	1.42%	1.50%
1	0.9950	0.9942	0.9934	0.9926	0.9917	0.9909	0.9901	0.9893	0.9885	0.9877	0.9868	0.9860	0.9852
2	1.9851	1.9826	1.9802	1.9777	1.9753	1.9728	1.9704	1.9680	1.9655	1.9631	1.9607	1.9583	1.9559
3	2.9702	2.9653	2.9604	2.9556	2.9507	2.9458	2.9410	2.9362	2.9313	2.9265	2.9217	2.9170	2.9122
4	3.9505	3.9423	3.9342	3.9261	3.9180	3.9100	3.9020	3.8940	3.8860	3.8781	3.8701	3.8622	3.8544
5	4.9259	4.9137	4.9015	4.8894	4.8774	4.8654	4.8534	4.8415	4.8297	4.8178	4.8061	4.7943	4.7826
6	5.8964	5.8794	5.8625	5.8456	5.8288	5.8121	5.7955	5.7789	5.7624	5.7460	5.7297	5.7134	5.6972
7	6.8621	6.8395	6.8170	6.7946	6.7724	6.7502	6.7282	6.7063	6.6844	6.6627	6.6411	6.6196	6.5982
8	7.8230	7.7940	7.7652	7.7366	7.7081	7.6798	7.6517	7.6237	7.5958	7.5681	7.5406	7.5132	7.4859
9	8.7791	8.7430	8.7072	8.6716	8.6362	8.6010	8.5660	8.5313	8.4967	8.4623	8.4282	8.3943	8.3605
10	9.7304	9.6865	9.6429	9.5996	9.5565	9.5138	9.4713	9.4291	9.3872	9.3455	9.3041	9.2630	9.2222
11	10.6770	10.6245	10.5724	10.5207	10.4693	10.4183	10.3676	10.3173	10.2674	10.2178	10.1686	10.1197	10.0711
12	11.6189	11.5571	11.4958	11.4349	11.3745	11.3146	11.2551	11.1960	11.1375	11.0793	11.0216	10.9643	10.9075
13	12.5562	12.4843	12.4130	12.3423	12.2722	12.2027	12.1337	12.0653	11.9975	11.9302	11.8634	11.7972	11.7315
14	13.4887	13.4061	13.3242	13.2430	13.1626	13.0828	13.0037	12.9253	12.8476	12.7706	12.6942	12.6185	12.5434
15	14.4166	14.3225	14.2293	14.1370	14.0455	13.9549	13.8651	13.7761	13.6879	13.6005	13.5140	13.4282	13.3432
16	15.3399	15.2337	15.1285	15.0243	14.9212	14.8190	14.7179	14.6177	14.5185	14.4203	14.3230	14.2267	14.1313
17	16.2586	16.1395	16.0217	15.9050	15.7896	15.6753	15.5623	15.4503	15.3396	15.2299	15.1214	15.0140	14.9076
18	17.1728	17.0401	16.9089	16.7792	16.6508	16.5239	16.3983	16.2740	16.1511	16.0295	15.9093	15.7903	15.6726
19	18.0824	17.9355	17.7903	17.6468	17.5050	17.3647	17.2260	17.0889	16.9533	16.8193	16.6868	16.5557	16.4262
20	18.9874	18.8257	18.6659	18.5080	18.3520	18.1979	18.0456	17.8950	17.7463	17.5993	17.4541	17.3105	17.1686
21	19.8880	19.7107	19.5357	19.3628	19.1921	19.0235	18.8570	18.6925	18.5301	18.3697	18.2112	18.0547	17.9001
22	20.7841	20.5906	20.3997	20.2112	20.0252	19.8416	19.6604	19.4815	19.3049	19.1306	18.9585	18.7886	18.6208
23	21.6757	21.4654	21.2579	21.0533	20.8514	20.6523	20.4558	20.2620	20.0707	19.8820	19.6959	19.5121	19.3309
24	22.5629	22.3351	22.1105	21.8891	21.6709	21.4556	21.2434	21.0341	20.8277	20.6242	20.4235	20.2256	20.0304
25	23.4456	23.1998	22.9575	22.7188	22.4835	22.2516	22.0232	21.7980	21.5760	21.3573	21.1417	20.9291	20.7196
26	24.3240	24.0594	23.7988	23.5422	23.2894	23.0404	22.7952	22.5536	22.3157	22.0813	21.8503	21.6228	21.3986
27	25.1980	24.9141	24.6346	24.3595	24.0887	23.8221	23.5596	23.3012	23.0468	22.7963	22.5497	22.3068	22.0676
28	26.0677	25.7638	25.4648	25.1707	24.8813	24.5966	24.3164	24.0408	23.7695	23.5025	23.2398	22.9812	22.7267
29	26.9330	26.6086	26.2896	25.9759	25.6674	25.3641	25.0658	24.7724	24.4838	24.2000	23.9208	23.6462	23.3761
30	27.7941	27.4485	27.1088	26.7751	26.4470	26.1246	25.8077	25.4962	25.1900	24.8889	24.5929	24.3019	24.0158
31	28.6508	28.2835	27.9227	27.5683	27.2202	26.8782	26.5423	26.2122	25.8879	25.5693	25.2562	24.9485	24.6461
32	29.5033	29.1137	28.7312	28.3557	27.9870	27.6250	27.2696	26.9206	26.5779	26.2413	25.9107	25.5860	25.2671
33	30.3515	29.9390	29.5343	29.1371	28.7474	28.3650	27.9897	27.6214	27.2598	26.9050	26.5566	26.2147	25.8790
34	31.1955	30.7596	30.3320	29.9128	29.5016	29.0983	28.7027	28.3146	27.9339	27.5605	27.1940	26.8345	26.4817
35	32.0354	31.5754	31.1246	30.6827	30.2495	29.8249	29.4086	29.0004	28.6003	28.2079	27.8231	27.4457	27.0756
36	32.8710	32.3865	31.9118	31.4468	30.9912	30.5449	30.1075	29.6789	29.2589	28.8473	28.4438	28.0483	27.6607
37	33.7025	33.1928	32.6938	32.2053	31.7268	31.2583	30.7995	30.3501	29.9100	29.4788	29.0564	28.6426	28.2371
38	34.5299	33.9945	33.4707	32.9581	32.4564	31.9653	31.4847	31.0141	30.5535	30.1025	29.6609	29.2285	28.8051
39	35.3531	34.7916	34.2424	33.7053	33.1799	32.6659	32.1630	31.6710	31.1896	30.7185	30.2575	29.8062	29.3646
40	36.1722	35.5840	35.0090	34.4469	33.8974	33.3601	32.8347	32.3209	31.8184	31.3269	30.8462	30.3759	29.9158
41	36.9873	36.3718	35.7706	35.1831	34.6090	34.0480	33.4997	32.9638	32.4399	31.9278	31.4272	30.9376	30.4590
42	37.7983	37.1551	36.5270	35.9137	35.3147	34.7296	34.1581	33.5998	33.0543	32.5213	32.0005	31.4915	30.9941
43	38.6053	37.9338	37.2785	36.6389	36.0146	35.4051	34.8100	34.2290	33.6616	33.1075	32.5663	32.0376	31.5212
44	39.4082	38.7080	38.0250	37.3587	36.7087	36.0744	35.4555	34.8514	34.2619	33.6864	33.1246	32.5761	32.0406
45	40.2072	39.4777	38.7666	38.0732	37.3970	36.7376	36.0945	35.4672	34.8552	34.2582	33.6756	33.1071	32.5523
46	41.0022	40.2430	39.5032	38.7823	38.0797	37.3948	36.7272	36.0764	35.4417	34.8229	34.2194	33.6307	33.0565
47	41.7932	41.0038	40.2350	39.4862	38.7567	38.0461	37.3537	36.6790	36.0215	35.3806	34.7559	34.1469	33.5532
48	42.5803	41.7602	40.9619	40.1848	39.4282	38.6914	37.9740	37.2752	36.5945	35.9315	35.2855	34.6560	34.0426
49	43.3635	42.5122	41.6840	40.8782	40.0940	39.3309	38.5881	37.8650	37.1610	36.4755	35.8080	35.1579	34.5247
50	44.1428	43.2599	42.4013	41.5664	40.7544	39.9645	39.1961	38.4485	37.7209	37.0129	36.3237	35.6526	34.9997

Table 1 (cont.)

m\n	0.50%	0.58%	0.67%	0.75%	0.83%	0.92%	1.00%	1.08%	1.17%	1.25%	1.33%	1.42%	1.50%
51	44.9182	44.0032	43.1139	42.2496	41.4093	40.5924	39.7981	39.0257	38.2744	37.5436	36.8326	36.1408	35.4677
52	45.6897	44.7422	43.8218	42.9276	42.0589	41.2146	40.3942	39.5967	38.8215	38.0677	37.3348	36.6220	35.9287
53	46.4575	45.4769	44.5249	43.6006	42.7030	41.8312	40.9844	40.1616	39.3622	38.5854	37.8304	37.0965	36.3830
54	47.2214	46.2074	45.2235	44.2686	43.3418	42.4421	41.5687	40.7205	39.8968	39.0967	38.3195	37.5643	36.8305
55	47.9814	46.9336	45.9173	44.9316	43.9754	43.0475	42.1472	41.2734	40.4252	39.6017	38.8021	38.0256	37.2715
56	48.7378	47.6556	46.6066	45.5897	44.6037	43.6474	42.7200	41.8203	40.9474	40.1004	39.2784	38.4805	37.7059
57	49.4903	48.3734	47.2913	46.2429	45.2268	44.2419	43.2871	42.3614	41.4637	40.5930	39.7484	38.9290	38.1339
58	50.2391	49.0871	47.9715	46.8912	45.8447	44.8309	43.8486	42.8967	41.9740	41.0795	40.2123	39.3712	38.5555
59	50.9842	49.7966	48.6472	47.5347	46.4576	45.4146	44.4046	43.4262	42.4784	41.5600	40.6700	39.8073	38.9710
60	51.7256	50.5020	49.3184	48.1734	47.0654	45.9930	44.9550	43.9501	42.9770	42.0346	41.1217	40.2373	39.3803
61	52.4632	51.2033	49.9852	48.8073	47.6681	46.5662	45.5000	44.4684	43.4699	42.5033	41.5675	40.6612	39.7835
62	53.1973	51.9006	50.6475	49.4365	48.2659	47.1341	46.0396	44.9811	43.9570	42.9662	42.0074	41.0793	40.1808
63	53.9276	52.5938	51.3055	50.0611	48.8588	47.6969	46.5739	45.4883	44.4386	43.4234	42.4415	41.4915	40.5722
64	54.6543	53.2829	51.9591	50.6810	49.4467	48.2546	47.1029	45.9901	44.9146	43.8750	42.8699	41.8979	40.9579
65	55.3775	53.9681	52.6084	51.2963	50.0298	48.8072	47.6266	46.4865	45.3851	44.3210	43.2927	42.2987	41.3378
66	56.0970	54.6493	53.2534	51.9070	50.6081	49.3547	48.1452	46.9775	45.8502	44.7615	43.7099	42.6939	41.7121
67	56.8129	55.3266	53.8941	52.5131	51.1815	49.8973	48.6586	47.4633	46.3099	45.1965	44.1216	43.0835	42.0809
68	57.5253	55.9999	54.5305	53.1147	51.7503	50.4350	49.1669	47.9440	46.7643	45.6262	44.5279	43.4677	42.4442
69	58.2341	56.6694	55.1628	53.7119	52.3143	50.9678	49.6702	48.4194	47.2135	46.0505	44.9288	43.8466	42.8022
70	58.9394	57.3349	55.7909	54.3046	52.8737	51.4958	50.1685	48.8898	47.6575	46.4697	45.3245	44.2201	43.1549
71	59.6412	57.9966	56.4148	54.8929	53.4285	52.0189	50.6619	49.3551	48.0964	46.8836	45.7150	44.5885	43.5023
72	60.3395	58.6544	57.0345	55.4768	53.9787	52.5373	51.1504	49.8154	48.5302	47.2925	46.1003	44.9516	43.8447
73	61.0343	59.3085	57.6502	56.0564	54.5243	53.0510	51.6341	50.2708	48.9590	47.6963	46.4805	45.3097	44.1819
74	61.7257	59.9587	58.2618	56.6317	55.0654	53.5601	52.1129	50.7213	49.3828	48.0951	46.8558	45.6629	44.5142
75	62.4136	60.6052	58.8693	57.2027	55.6021	54.0645	52.5871	51.1670	49.8018	48.4890	47.2261	46.0110	44.8416
76	63.0982	61.2479	59.4728	57.7694	56.1343	54.5643	53.0565	51.6079	50.2160	48.8780	47.5916	46.3543	45.1641
77	63.7793	61.8869	60.0723	58.3319	56.6621	55.0596	53.5213	52.0441	50.6253	49.2622	47.9522	46.6929	45.4819
78	64.4570	62.5222	60.6679	58.8902	57.1856	55.5504	53.9815	52.4756	51.0300	49.6417	48.3081	47.0267	45.7950
79	65.1313	63.1538	61.2595	59.4444	57.7047	56.0367	54.4371	52.9025	51.4300	50.0165	48.6593	47.3558	46.1034
80	65.8023	63.7817	61.8472	59.9944	58.2195	56.5186	54.8882	53.3248	51.8253	50.3867	49.0059	47.6803	46.4073
81	66.4700	64.4060	62.4310	60.5404	58.7301	56.9962	55.3349	53.7426	52.2162	50.7523	49.3479	48.0003	46.7067
82	67.1343	65.0267	63.0109	61.0823	59.2365	57.4694	55.7771	54.1559	52.6025	51.1133	49.6854	48.3158	47.0017
83	67.7953	65.6438	63.5870	61.6201	59.7386	57.9383	56.2149	54.5648	52.9843	51.4700	50.0185	48.6270	47.2923
84	68.4530	66.2573	64.1593	62.1540	60.2367	58.4029	56.6485	54.9693	53.3618	51.8222	50.3472	48.9337	47.5786
85	69.1075	66.8672	64.7277	62.6838	60.7306	58.8633	57.0777	55.3695	53.7349	52.1701	50.6716	49.2362	47.8607
86	69.7587	67.4736	65.2925	63.2098	61.2204	59.3196	57.5026	55.7654	54.1036	52.5136	50.9917	49.5345	48.1386
87	70.4067	68.0765	65.8534	63.7318	61.7062	59.7717	57.9234	56.1570	54.4682	52.8530	51.3076	49.8286	48.4125
88	71.0514	68.6759	66.4107	64.2499	62.1880	60.2196	58.3400	56.5444	54.8285	53.1881	51.6194	50.1186	48.6822
89	71.6930	69.2718	66.9643	64.7642	62.6657	60.6636	58.7525	56.9277	55.1847	53.5191	51.9270	50.4045	48.9480
90	72.3313	69.8643	67.5142	65.2746	63.1396	61.1034	59.1609	57.3069	55.5368	53.8461	52.2306	50.6864	49.2099
91	72.9665	70.4533	68.0604	65.7812	63.6095	61.5393	59.5652	57.6820	55.8848	54.1689	52.5302	50.9644	49.4678
92	73.5985	71.0389	68.6031	66.2841	64.0755	61.9713	59.9656	58.0531	56.2288	54.4879	52.8258	51.2386	49.7220
93	74.2273	71.6211	69.1421	66.7832	64.5377	62.3993	60.3620	58.4202	56.5688	54.8028	53.1176	51.5088	49.9724
94	74.8531	72.2000	69.6776	67.2787	64.9961	62.8234	60.7544	58.7834	56.9049	55.1139	53.4055	51.7754	50.2191
95	75.4757	72.7754	70.2096	67.7704	65.4507	63.2437	61.1430	59.1427	57.2371	55.4211	53.6897	52.0382	50.4622
96	76.0952	73.3476	70.7380	68.2584	65.9015	63.6601	61.5277	59.4981	57.5655	55.7246	53.9701	52.2973	50.7017
97	76.7117	73.9164	71.2629	68.7429	66.3486	64.0728	61.9086	59.8497	57.8902	56.0243	54.2468	52.5528	50.9376
98	77.3250	74.4819	71.7843	69.2237	66.7920	64.4817	62.2858	60.1976	58.2110	56.3203	54.5199	52.8047	51.1701
99	77.9354	75.0442	72.3023	69.7009	67.2317	64.8869	62.6592	60.5417	58.5282	56.6126	54.7893	53.0531	51.3991
100	78.5426	75.6031	72.8169	70.1746	67.6678	65.2884	63.0289	60.8822	58.8417	56.9013	55.0553	53.2981	51.6247

Table 1 (cont.)

mn	0.50%	0.58%	0.67%	0.75%	0.83%	0.92%	1.00%	1.08%	1.17%	1.25%	1.33%	1.42%	1.50%
101	79.1469	76.1589	73.3280	70.6448	68.1003	65.6863	63.3949	61.2190	59.1516	57.1865	55.3177	53.5396	51.8470
102	79.7482	76.7114	73.8358	71.1115	68.5292	66.0806	63.7574	61.5522	59.4579	57.4682	55.5767	53.7777	52.0660
103	80.3464	77.2607	74.3402	71.5746	68.9546	66.4712	64.1162	61.8818	59.7607	57.7463	55.8322	54.0126	52.2818
104	80.9417	77.8068	74.8412	72.0344	69.3765	66.8584	64.4715	62.2079	60.0600	58.0211	56.0844	54.2441	52.4944
105	81.5341	78.3498	75.3390	72.4907	69.7949	67.2420	64.8232	62.5304	60.3559	58.2924	56.3333	54.4724	52.7038
106	82.1234	78.8896	75.8334	72.9436	70.2098	67.6221	65.1715	62.8496	60.6483	58.5604	56.5789	54.6975	52.9102
107	82.7099	79.4263	76.3246	73.3932	70.6213	67.9988	65.5164	63.1653	60.9374	58.8251	56.8213	54.9195	53.1135
108	83.2934	79.9598	76.8125	73.8394	71.0294	68.3720	65.8578	63.4776	61.2231	59.0865	57.0605	55.1384	53.3137
109	83.8741	80.4903	77.2972	74.2823	71.4341	68.7419	66.1958	63.7866	61.5055	59.3447	57.2966	55.3542	53.5111
110	84.4518	81.0177	77.7787	74.7219	71.8354	69.1084	66.5305	64.0922	61.7847	59.5997	57.5295	55.5670	53.7055
111	85.0267	81.5421	78.2569	75.1582	72.2335	69.4716	66.8619	64.3946	62.0607	59.8516	57.7594	55.7768	53.8970
112	85.5987	82.0634	78.7321	75.5912	72.6283	69.8315	67.1900	64.6938	62.3335	60.1003	57.9862	55.9837	54.0858
113	86.1678	82.5816	79.2040	76.0211	73.0198	70.1881	67.5149	64.9897	62.6031	60.3460	58.2101	56.1877	54.2717
114	86.7342	83.0969	79.6729	76.4477	73.4080	70.5414	67.8365	65.2825	62.8696	60.5886	58.4310	56.3889	54.4549
115	87.2977	83.6092	80.1386	76.8712	73.7931	70.8916	68.1549	65.5721	63.1331	60.8283	58.6490	56.5872	54.6353
116	87.8584	84.1185	80.6013	77.2915	74.1750	71.2386	68.4702	65.8587	63.3935	61.0650	58.8642	56.7828	54.8131
117	88.4163	84.6248	81.0609	77.7087	74.5537	71.5824	68.7824	66.1421	63.6509	61.2987	59.0765	56.9757	54.9883
118	88.9714	85.1283	81.5174	78.1228	74.9293	71.9231	69.0915	66.4226	63.9053	61.5296	59.2860	57.1658	55.1609
119	89.5238	85.6288	81.9710	78.5338	75.3018	72.2607	69.3975	66.7000	64.1568	61.7576	59.4928	57.3533	55.3309
120	90.0735	86.1264	82.4215	78.9417	75.6712	72.5953	69.7005	66.9744	64.4054	61.9828	59.6968	57.5382	55.4985
121	90.6204	86.6211	82.8690	79.3466	76.0375	72.9268	70.0005	67.2459	64.6512	62.2053	59.8982	57.7205	55.6635
122	91.1645	87.1129	83.3136	79.7485	76.4008	73.2553	70.2975	67.5145	64.8941	62.4250	60.0969	57.9002	55.8261
123	91.7060	87.6019	83.7552	80.1474	76.7612	73.5808	70.5916	67.7802	65.1342	62.6419	60.2930	58.0775	55.9863
124	92.2448	88.0880	84.1939	80.5433	77.1185	73.9033	70.8828	68.0431	65.3715	62.8562	60.4865	58.2522	56.1442
125	92.7809	88.5714	84.6297	80.9363	77.4729	74.2230	71.1711	68.3031	65.6061	63.0679	60.6775	58.4245	56.2997
126	93.3143	89.0519	85.0627	81.3263	77.8244	74.5397	71.4565	68.5604	65.8380	63.2769	60.8659	58.5944	56.4529
127	93.8451	89.5297	85.4927	81.7135	78.1729	74.8535	71.7391	68.8149	66.0672	63.4834	61.0519	58.7620	56.6038
128	94.3732	90.0046	85.9199	82.0977	78.5186	75.1645	72.0189	69.0667	66.2938	63.6873	61.2354	58.9272	56.7525
129	94.8987	90.4768	86.3443	82.4792	78.8614	75.4727	72.2960	69.3158	66.5177	63.8887	61.4165	59.0901	56.8990
130	95.4216	90.9463	86.7658	82.8577	79.2014	75.7780	72.5703	69.5622	66.7391	64.0876	61.5953	59.2507	57.0434
131	95.9419	91.4131	87.1846	83.2335	79.5386	76.0806	72.8419	69.8059	66.9579	64.2840	61.7716	59.4091	57.1856
132	96.4596	91.8771	87.6006	83.6064	79.8730	76.3805	73.1108	70.0471	67.1742	64.4781	61.9457	59.5652	57.3257
133	96.9747	92.3385	88.0138	83.9766	80.2046	76.6776	73.3770	70.2857	67.3880	64.6697	62.1175	59.7192	57.4638
134	97.4873	92.7972	88.4243	84.3440	80.5335	76.9720	73.6406	70.5217	67.5994	64.8590	62.2870	59.8710	57.5998
135	97.9973	93.2532	88.8321	84.7087	80.8597	77.2638	73.9016	70.7552	67.8083	65.0459	62.4542	60.0207	57.7338
136	98.5048	93.7066	89.2372	85.0707	81.1831	77.5529	74.1600	70.9862	68.0148	65.2305	62.6193	60.1683	57.8658
137	99.0097	94.1573	89.6396	85.4299	81.5039	77.8394	74.4158	71.2147	68.2189	65.4128	62.7822	60.3139	57.9958
138	99.5122	94.6055	90.0394	85.7865	81.8221	78.1232	74.6691	71.4407	68.4206	65.5929	62.9430	60.4574	58.1240
139	100.0121	95.0510	90.4364	86.1405	82.1376	78.4045	74.9199	71.6644	68.6201	65.7708	63.1016	60.5989	58.2502
140	100.5096	95.4939	90.8309	86.4918	82.4505	78.6833	75.1682	71.8856	68.8172	65.9465	63.2582	60.7385	58.3746
141	101.0045	95.9343	91.2228	86.8405	82.7609	78.9595	75.4141	72.1045	69.0121	66.1200	63.4127	60.8761	58.4971
142	101.4971	96.3722	91.6120	87.1866	83.0686	79.2332	75.6575	72.3210	69.2047	66.2913	63.5651	61.0117	58.6179
143	101.9871	96.8075	91.9987	87.5301	83.3738	79.5044	75.8985	72.5352	69.3951	66.4606	63.7156	61.1455	58.7368
144	102.4747	97.2402	92.3828	87.8711	83.6765	79.7731	76.1372	72.7471	69.5833	66.6277	63.8641	61.2774	58.8540
145	102.9599	97.6705	92.7644	88.2095	83.9767	80.0394	76.3734	72.9567	69.7693	66.7928	64.0106	61.4075	58.9695
146	103.4427	98.0982	93.1434	88.5454	84.2744	80.3033	76.6073	73.1641	69.9532	66.9559	64.1552	61.5357	59.0832
147	103.9231	98.5235	93.5199	88.8788	84.5697	80.5648	76.8390	73.3693	70.1349	67.1169	64.2979	61.6622	59.1953
148	104.4011	98.9463	93.8940	89.2098	84.8625	80.8239	77.0683	73.5723	70.3146	67.2760	64.4387	61.7868	59.3057
149	104.8767	99.3667	94.2656	89.5382	85.1529	81.0807	77.2953	73.7730	70.4922	67.4330	64.5777	61.9098	59.4145
150	105.3500	99.7846	94.6347	89.8642	85.4409	81.3351	77.5201	73.9717	70.6677	67.5882	64.7148	62.0310	59.5217

Table 1 (cont.)

mn	0.50%	0.58%	0.67%	0.75%	0.83%	0.92%	1.00%	1.08%	1.17%	1.25%	1.33%	1.42%	1.50%
151	105.8209	100.2001	95.0013	90.1878	85.7265	81.5872	77.7427	74.1682	70.8412	67.7414	64.8502	62.1506	59.6273
152	106.2894	100.6132	95.3655	90.5090	86.0098	81.8370	77.9631	74.3626	71.0128	67.8928	64.9837	62.2684	59.7313
153	106.7556	101.0239	95.7274	90.8278	86.2907	82.0846	78.1813	74.5549	71.1823	68.0422	65.1155	62.3846	59.8338
154	107.2195	101.4322	96.0868	91.1442	86.5693	82.3299	78.3973	74.7452	71.3499	68.1899	65.2456	62.4992	59.9348
155	107.6811	101.8381	96.4438	91.4583	86.8455	82.5730	78.6112	74.9334	71.5155	68.3357	65.3739	62.6122	60.0342
156	108.1404	102.2417	96.7985	91.7700	87.1195	82.8139	78.8229	75.1196	71.6793	68.4797	65.5006	62.7236	60.1323
157	108.5975	102.6430	97.1508	92.0794	87.3913	83.0525	79.0326	75.3038	71.8411	68.6219	65.6256	62.8335	60.2288
158	109.0522	103.0419	97.5008	92.3865	87.6608	83.2891	79.2402	75.4861	72.0011	68.7624	65.7489	62.9418	60.3240
159	109.5047	103.4385	97.8485	92.6913	87.9280	83.5234	79.4458	75.6663	72.1593	68.9011	65.8706	63.0486	60.4177
160	109.9549	103.8328	98.1939	92.9939	88.1931	83.7557	79.6493	75.8447	72.3156	69.0381	65.9907	63.1540	60.5101
161	110.4029	104.2248	98.5370	93.2942	88.4560	83.9858	79.8508	76.0211	72.4701	69.1735	66.1093	63.2578	60.6010
162	110.8486	104.6146	98.8778	93.5922	88.7167	84.2138	80.0503	76.1957	72.6228	69.3071	66.2263	63.3602	60.6907
163	111.2922	105.0021	99.2163	93.8881	88.9752	84.4398	80.2478	76.3683	72.7738	69.4391	66.3417	63.4612	60.7790
164	111.7335	105.3873	99.5526	94.1817	89.2316	84.6637	80.4433	76.5392	72.9230	69.5695	66.4556	63.5607	60.8660
165	112.1726	105.7703	99.8867	94.4732	89.4859	84.8856	80.6370	76.7082	73.0705	69.6983	66.5681	63.6589	60.9517
166	112.6096	106.1511	100.2186	94.7624	89.7381	85.1055	80.8287	76.8754	73.2164	69.8255	66.6790	63.7557	61.0362
167	113.0444	106.5297	100.5483	95.0496	89.9882	85.3233	81.0185	77.0407	73.3605	69.9511	66.7885	63.8511	61.1194
168	113.4770	106.9061	100.8758	95.3346	90.2362	85.5392	81.2064	77.2044	73.5029	70.0751	66.8965	63.9452	61.2014
169	113.9075	107.2803	101.2011	95.6174	90.4822	85.7532	81.3925	77.3662	73.6438	70.1977	67.0032	64.0380	61.2821
170	114.3358	107.6523	101.5243	95.8982	90.7261	85.9651	81.5767	77.5264	73.7830	70.3187	67.1084	64.1295	61.3617
171	114.7620	108.0222	101.8453	96.1769	90.9681	86.1752	81.7591	77.6848	73.9206	70.4382	67.2122	64.2197	61.4401
172	115.1860	108.3899	102.1642	96.4535	91.2080	86.3834	81.9398	77.8415	74.0566	70.5562	67.3147	64.3087	61.5174
173	115.6080	108.7555	102.4810	96.7280	91.4459	86.5896	82.1186	77.9965	74.1910	70.6728	67.4158	64.3964	61.5934
174	116.0279	109.1190	102.7957	97.0005	91.6819	86.7940	82.2956	78.1499	74.3239	70.7880	67.5156	64.4829	61.6684
175	116.4456	109.4803	103.1083	97.2710	91.9160	86.9965	82.4709	78.3016	74.4553	70.9017	67.6141	64.5682	61.7423
176	116.8613	109.8396	103.4189	97.5394	92.1481	87.1972	82.6445	78.4517	74.5851	71.0140	67.7113	64.6523	61.8151
177	117.2749	110.1968	103.7273	97.8059	92.3782	87.3961	82.8163	78.6002	74.7134	71.1250	67.8072	64.7352	61.8868
178	117.6865	110.5519	104.0338	98.0704	92.6065	87.5932	82.9864	78.7471	74.8403	71.2345	67.9018	64.8170	61.9574
179	118.0960	110.9050	104.3382	98.3329	92.8329	87.7884	83.1549	78.8925	74.9657	71.3428	67.9952	64.8976	62.0270
180	118.5035	111.2560	104.6406	98.5934	93.0574	87.9819	83.3217	79.0363	75.0897	71.4496	68.0874	64.9771	62.0956
181	118.9090	111.6049	104.9410	98.8520	93.2801	88.1737	83.4868	79.1785	75.2122	71.5552	68.1783	65.0555	62.1631
182	119.3124	111.9519	105.2394	99.1087	93.5009	88.3637	83.6503	79.3192	75.3333	71.6595	68.2681	65.1327	62.2297
183	119.7138	112.2968	105.5358	99.3635	93.7199	88.5520	83.8122	79.4584	75.4530	71.7624	68.3567	65.2090	62.2952
184	120.1133	112.6397	105.8303	99.6164	93.9371	88.7385	83.9724	79.5961	75.5713	71.8641	68.4441	65.2841	62.3598
185	120.5107	112.9807	106.1228	99.8673	94.1525	88.9234	84.1311	79.7323	75.6883	71.9646	68.5304	65.3582	62.4235
186	120.9062	113.3197	106.4134	100.1165	94.3661	89.1066	84.2883	79.8671	75.8039	72.0638	68.6155	65.4312	62.4862
187	121.2997	113.6567	106.7020	100.3637	94.5780	89.2881	84.4438	80.0004	75.9182	72.1618	68.6995	65.5033	62.5480
188	121.6912	113.9917	106.9888	100.6092	94.7881	89.4680	84.5978	80.1323	76.0312	72.2585	68.7824	65.5743	62.6088
189	122.0808	114.3248	107.2736	100.8528	94.9964	89.6462	84.7503	80.2628	76.1429	72.3541	68.8642	65.6443	62.6688
190	122.4685	114.6560	107.5566	101.0946	95.2031	89.8228	84.9013	80.3919	76.2532	72.4485	68.9449	65.7134	62.7279
191	122.85[illegible]	[illegible].9852	107.8377	101.3346	95.4080	89.9979	85.0508	80.5196	76.3623	72.5417	69.0246	65.7815	62.7861
192	123.2380	115.3126	108.1169	101.5728	95.6113	90.1713	85.1988	80.6460	76.4702	72.6338	69.1032	65.8486	62.8435
193	123.6199	115.6380	108.3942	101.8092	95.8128	90.3431	85.3454	80.7709	76.5768	72.7247	69.1808	65.9149	62.9000
194	123.9999	115.9616	108.6698	102.0439	96.0127	90.5134	85.4905	80.8946	76.6822	72.8146	69.2574	65.9801	62.9556
195	124.3780	116.2833	108.9435	102.2768	96.2110	90.6822	85.6341	81.0169	76.7863	72.9033	69.3329	66.0445	63.0105
196	124.7543	116.6031	109.2154	102.5080	96.4076	90.8494	85.7764	81.1379	76.8893	72.9909	69.4075	66.1080	63.0645
197	125.1286	116.9210	109.4855	102.7375	96.6025	91.0151	85.9172	81.2576	76.9911	73.0774	69.4811	66.1706	63.1177
198	125.5011	117.2372	109.7538	102.9652	96.7959	91.1793	86.0566	81.3760	77.0917	73.1629	69.5537	66.2323	63.1702
199	125.8718	117.5514	110.0203	103.1913	96.9877	91.3420	86.1947	81.4932	77.1911	73.2473	69.6254	66.2931	63.2218
200	126.2406	117.8639	110.2851	103.4157	97.1779	91.5032	86.3314	81.6091	77.2894	73.3306	69.6961	66.3531	63.2728

Table 1 (cont.)

mn	0.50%	0.58%	0.67%	0.75%	0.83%	0.92%	1.00%	1.08%	1.17%	1.25%	1.33%	1.42%	1.50%
201	126.6075	118.1746	110.5481	103.6384	97.3665	91.6630	86.4667	81.7238	77.3865	73.4130	69.7659	66.4123	63.3229
202	126.9727	118.4834	110.8094	103.8594	97.5535	91.8213	86.6007	81.8372	77.4826	73.4943	69.8348	66.4706	63.3723
203	127.3360	118.7905	111.0689	104.0788	97.7390	91.9781	86.7334	81.9494	77.5775	73.5746	69.9027	66.5281	63.4210
204	127.6975	119.0957	111.3267	104.2966	97.9230	92.1336	86.8647	82.0604	77.6713	73.6540	69.9698	66.5848	63.4690
205	128.0572	119.3992	111.5828	104.5128	98.1055	92.2876	86.9948	82.1702	77.7641	73.7323	70.0360	66.6408	63.5162
206	128.4151	119.7010	111.8373	104.7273	98.2864	92.4402	87.1235	82.2789	77.8558	73.8097	70.1013	66.6959	63.5628
207	128.7713	120.0010	112.0900	104.9403	98.4659	92.5915	87.2510	82.3864	77.9464	73.8861	70.1657	66.7503	63.6087
208	129.1256	120.2992	112.3411	105.1516	98.6438	92.7414	87.3772	82.4927	78.0360	73.9616	70.2294	66.8039	63.6539
209	129.4782	120.5958	112.5905	105.3614	98.8203	92.8899	87.5022	82.5979	78.1245	74.0361	70.2921	66.8567	63.6984
210	129.8291	120.8906	112.8382	105.5696	98.9954	93.0370	87.6260	82.7019	78.2121	74.1098	70.3541	66.9089	63.7422
211	130.1782	121.1837	113.0843	105.7763	99.1690	93.1828	87.7485	82.8049	78.2986	74.1825	70.4152	66.9603	63.7855
212	130.5256	121.4751	113.3288	105.9815	99.3411	93.3273	87.8698	82.9067	78.3841	74.2543	70.4755	67.0109	63.8280
213	130.8712	121.7648	113.5716	106.1851	99.5118	93.4705	87.9899	83.0075	78.4686	74.3252	70.5351	67.0609	63.8700
214	131.2152	122.0528	113.8129	106.3872	99.6812	93.6124	88.1088	83.1072	78.5522	74.3953	70.5938	67.1102	63.9113
215	131.5574	122.3391	114.0525	106.5878	99.8491	93.7530	88.2265	83.2058	78.6348	74.4645	70.6518	67.1588	63.9520
216	131.8979	122.6238	114.2906	106.7869	100.0156	93.8923	88.3431	83.3033	78.7164	74.5328	70.7090	67.2067	63.9922
217	132.2367	122.9069	114.5271	106.9845	100.1808	94.0304	88.4585	83.3998	78.7971	74.6003	70.7655	67.2539	64.0317
218	132.5738	123.1883	114.7620	107.1806	100.3446	94.1672	88.5728	83.4953	78.8769	74.6670	70.8212	67.3005	64.0706
219	132.9093	123.4680	114.9954	107.3753	100.5070	94.3028	88.6859	83.5897	78.9557	74.7328	70.8762	67.3464	64.1090
220	133.2431	123.7462	115.2272	107.5685	100.6681	94.4371	88.7979	83.6832	79.0337	74.7978	70.9304	67.3917	64.1468
221	133.5752	124.0227	115.4575	107.7603	100.8279	94.5702	88.9089	83.7756	79.1107	74.8621	70.9840	67.4364	64.1840
222	133.9057	124.2977	115.6862	107.9507	100.9863	94.7021	89.0187	83.8670	79.1869	74.9255	71.0368	67.4804	64.2207
223	134.2345	124.5710	115.9135	108.1397	101.1435	94.8328	89.1274	83.9575	79.2621	74.9882	71.0890	67.5238	64.2569
224	134.5617	124.8427	116.1392	108.3272	101.2993	94.9623	89.2350	84.0470	79.3366	75.0500	71.1404	67.5666	64.2925
225	134.8872	125.1129	116.3635	108.5134	101.4539	95.0906	89.3416	84.1355	79.4101	75.1111	71.1912	67.6088	64.3276
226	135.2112	125.3815	116.5862	108.6981	101.6071	95.2178	89.4472	84.2231	79.4828	75.1715	71.2413	67.6504	64.3621
227	135.5335	125.6486	116.8075	108.8815	101.7592	95.3438	89.5516	84.3097	79.5547	75.2311	71.2908	67.6915	64.3962
228	135.8542	125.9141	117.0273	109.0635	101.9099	95.4687	89.6551	84.3955	79.6257	75.2900	71.3396	67.7319	64.4297
229	136.1734	126.1780	117.2457	109.2442	102.0594	95.5924	89.7575	84.4803	79.6959	75.3481	71.3877	67.7718	64.4628
230	136.4909	126.4405	117.4626	109.4235	102.2077	95.7150	89.8589	84.5641	79.7653	75.4056	71.4353	67.8112	64.4954
231	136.8069	126.7014	117.6781	109.6015	102.3547	95.8365	89.9593	84.6471	79.8339	75.4623	71.4822	67.8500	64.5275
232	137.1213	126.9608	117.8921	109.7782	102.5005	95.9569	90.0587	84.7292	79.9017	75.5183	71.5285	67.8882	64.5591
233	137.4341	127.2187	118.1048	109.9535	102.6452	96.0762	90.1572	84.8104	79.9688	75.5736	71.5741	67.9259	64.5902
234	137.7454	127.4751	118.3160	110.1276	102.7886	96.1944	90.2546	84.8908	80.0350	75.6283	71.6192	67.9631	64.6209
235	138.0551	127.7300	118.5258	110.3003	102.9308	96.3116	90.3511	84.9703	80.1005	75.6822	71.6637	67.9998	64.6511
236	138.3633	127.9834	118.7343	110.4718	103.0719	96.4277	90.4466	85.0489	80.1653	75.7356	71.7076	68.0359	64.6809
237	138.6699	128.2354	118.9413	110.6420	103.2118	96.5427	90.5412	85.1267	80.2292	75.7882	71.7509	68.0716	64.7103
238	138.9751	128.4859	119.1470	110.8109	103.3506	96.6567	90.6349	85.2037	80.2925	75.8402	71.7937	68.1068	64.7392
239	139.2787	128.7349	119.3513	110.9785	103.4882	96.7696	90.7276	85.2798	80.3550	75.8916	71.8359	68.1414	64.7677
240	139.5808	128.9825	119.5543	111.1450	103.6246	96.8815	90.8194	85.3551	80.4168	75.9423	71.8775	68.1756	64.7957

Table 1 (cont.)

mn	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%	11.00%	12.00%	13.00%	14.00%
1	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346	0.9259	0.9174	0.9091	0.9009	0.8929	0.8850	0.8772
2	1.9416	1.9135	1.8861	1.8594	1.8334	1.8080	1.7833	1.7591	1.7355	1.7125	1.6901	1.6681	1.6467
3	2.8839	2.8286	2.7751	2.7232	2.6730	2.6243	2.5771	2.5313	2.4869	2.4437	2.4018	2.3612	2.3216
4	3.8077	3.7171	3.6299	3.5460	3.4651	3.3872	3.3121	3.2397	3.1699	3.1024	3.0373	2.9745	2.9137
5	4.7135	4.5797	4.4518	4.3295	4.2124	4.1002	3.9927	3.8897	3.7908	3.6959	3.6048	3.5172	3.4331
6	5.6014	5.4172	5.2421	5.0757	4.9173	4.7665	4.6229	4.4859	4.3553	4.2305	4.1114	3.9975	3.8887
7	6.4720	6.2303	6.0021	5.7864	5.5824	5.3893	5.2064	5.0330	4.8684	4.7122	4.5638	4.4226	4.2883
8	7.3255	7.0197	6.7327	6.4632	6.2098	5.9713	5.7466	5.5348	5.3349	5.1461	4.9676	4.7988	4.6389
9	8.1622	7.7861	7.4353	7.1078	6.8017	6.5152	6.2469	5.9952	5.7590	5.5370	5.3282	5.1317	4.9464
10	8.9826	8.5302	8.1109	7.7217	7.3601	7.0236	6.7101	6.4177	6.1446	5.8892	5.6502	5.4262	5.2161
11	9.7868	9.2526	8.7605	8.3064	7.8869	7.4987	7.1390	6.8052	6.4951	6.2065	5.9377	5.6869	5.4527
12	10.5753	9.9540	9.3851	8.8633	8.3838	7.9427	7.5361	7.1607	6.8137	6.4924	6.1944	5.9176	5.6603
13	11.3484	10.6350	9.9856	9.3936	8.8527	8.3577	7.9038	7.4869	7.1034	6.7499	6.4235	6.1218	5.8424
14	12.1062	11.2961	10.5631	9.8986	9.2950	8.7455	8.2442	7.7862	7.3667	6.9819	6.6282	6.3025	6.0021
15	12.8493	11.9379	11.1184	10.3797	9.7122	9.1079	8.5595	8.0607	7.6061	7.1909	6.8109	6.4624	6.1422
16	13.5777	12.5611	11.6523	10.8378	10.1059	9.4466	8.8514	8.3126	7.8237	7.3792	6.9740	6.6039	6.2651
17	14.2919	13.1661	12.1657	11.2741	10.4773	9.7632	9.1216	8.5436	8.0216	7.5488	7.1196	6.7291	6.3729
18	14.9920	13.7535	12.6593	11.6896	10.8276	10.0591	9.3719	8.7556	8.2014	7.7016	7.2497	6.8399	6.4674
19	15.6785	14.3238	13.1339	12.0853	11.1581	10.3356	9.6036	8.9501	8.3649	7.8393	7.3658	6.9380	6.5504
20	16.3514	14.8775	13.5903	12.4622	11.4699	10.5940	9.8181	9.1285	8.5136	7.9633	7.4694	7.0248	6.6231
21	17.0112	15.4150	14.0292	12.8212	11.7641	10.8355	10.0168	9.2922	8.6487	8.0751	7.5620	7.1016	6.6870
22	17.6580	15.9369	14.4511	13.1630	12.0416	11.0612	10.2007	9.4424	8.7715	8.1757	7.6446	7.1695	6.7429
23	18.2922	16.4436	14.8568	13.4886	12.3034	11.2722	10.3711	9.5802	8.8832	8.2664	7.7184	7.2297	6.7921
24	18.9139	16.9355	15.2470	13.7986	12.5504	11.4693	10.5288	9.7066	8.9847	8.3481	7.7843	7.2829	6.8351
25	19.5235	17.4131	15.6221	14.0939	12.7834	11.6536	10.6748	9.8226	9.0770	8.4217	7.8431	7.3300	6.8729
26	20.1210	17.8768	15.9828	14.3752	13.0032	11.8258	10.8100	9.9290	9.1609	8.4881	7.8957	7.3717	6.9061
27	20.7069	18.3270	16.3296	14.6430	13.2105	11.9867	10.9352	10.0266	9.2372	8.5478	7.9426	7.4086	6.9352
28	21.2813	18.7641	16.6631	14.8981	13.4062	12.1371	11.0511	10.1161	9.3066	8.6016	7.9844	7.4412	6.9607
29	21.8444	19.1885	16.9837	15.1411	13.5907	12.2777	11.1584	10.1983	9.3696	8.6501	8.0218	7.4701	6.9830
30	22.3965	19.6004	17.2920	15.3725	13.7648	12.4090	11.2578	10.2737	9.4269	8.6938	8.0552	7.4957	7.0027
31	22.9377	20.0004	17.5885	15.5928	13.9291	12.5318	11.3498	10.3428	9.4790	8.7331	8.0850	7.5183	7.0199
32	23.4683	20.3888	17.8736	15.8027	14.0840	12.6466	11.4350	10.4062	9.5264	8.7686	8.1116	7.5383	7.0350
33	23.9886	20.7658	18.1476	16.0025	14.2302	12.7538	11.5139	10.4644	9.5694	8.8005	8.1354	7.5560	7.0482
34	24.4986	21.1318	18.4112	16.1929	14.3681	12.8540	11.5869	10.5178	9.6086	8.8293	8.1566	7.5717	7.0599
35	24.9986	21.4872	18.6646	16.3742	14.4982	12.9477	11.6546	10.5668	9.6442	8.8552	8.1755	7.5856	7.0700
36	25.4888	21.8323	18.9083	16.5469	14.6210	13.0352	11.7172	10.6118	9.6765	8.8786	8.1924	7.5979	7.0790
37	25.9695	22.1672	19.1426	16.7113	14.7368	13.1170	11.7752	10.6530	9.7059	8.8996	8.2075	7.6087	7.0868
38	26.4406	22.4925	19.3679	16.8679	14.8460	13.1935	11.8289	10.6908	9.7327	8.9186	8.2210	7.6183	7.0937
39	26.9026	22.8082	19.5845	17.0170	14.9491	13.2649	11.8786	10.7255	9.7570	8.9357	8.2330	7.6268	7.0997
40	27.3555	23.1148	19.7928	17.1591	15.0463	13.3317	11.9246	10.7574	9.7791	8.9511	8.2438	7.6344	7.1050
41	27.7995	23.4124	19.9931	17.2944	15.1380	13.3941	11.9672	10.7866	9.7991	8.9649	8.2534	7.6410	7.1097
42	28.2348	23.7014	20.1856	17.4232	15.2245	13.4524	12.0067	10.8134	9.8174	8.9774	8.2619	7.6469	7.1138
43	28.6616	23.9819	20.3708	17.5459	15.3062	13.5070	12.0432	10.8380	9.8340	8.9886	8.2696	7.6522	7.1173
44	29.0800	24.2543	20.5488	17.6628	15.3832	13.5579	12.0771	10.8605	9.8491	8.9988	8.2764	7.6568	7.1205
45	29.4902	24.5187	20.7200	17.7741	15.4558	13.6055	12.1084	10.8812	9.8628	9.0079	8.2825	7.6609	7.1232
46	29.8923	24.7754	20.8847	17.8801	15.5244	13.6500	12.1374	10.9002	9.8753	9.0161	8.2880	7.6645	7.1256
47	30.2866	25.0247	21.0429	17.9810	15.5890	13.6916	12.1643	10.9176	9.8866	9.0235	8.2928	7.6677	7.1277
48	30.6731	25.2667	21.1951	18.0772	15.6500	13.7305	12.1891	10.9336	9.8969	9.0302	8.2972	7.6705	7.1296
49	31.0521	25.5017	21.3415	18.1687	15.7076	13.7668	12.2122	10.9482	9.9063	9.0362	8.3010	7.6730	7.1312
50	31.4236	25.7298	21.4822	18.2559	15.7619	13.8007	12.2335	10.9617	9.9148	9.0417	8.3045	7.6752	7.1327

Table 1 (cont.)

mn	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%	11.00%	12.00%	13.00%	14.00%
51	31.7878	25.9512	21.6175	18.3390	15.8131	13.8325	12.2532	10.9740	9.9226	9.0465	8.3076	7.6772	7.1339
52	32.1449	26.1662	21.7476	18.4181	15.8614	13.8621	12.2715	10.9853	9.9296	9.0509	8.3103	7.6789	7.1350
53	32.4950	26.3750	21.8727	18.4934	15.9070	13.8898	12.2884	10.9957	9.9360	9.0549	8.3128	7.6805	7.1360
54	32.8383	26.5777	21.9930	18.5651	15.9500	13.9157	12.3041	11.0053	9.9418	9.0585	8.3150	7.6818	7.1368
55	33.1748	26.7744	22.1086	18.6335	15.9905	13.9399	12.3186	11.0140	9.9471	9.0617	8.3170	7.6830	7.1376
56	33.5047	26.9655	22.2198	18.6985	16.0288	13.9626	12.3321	11.0220	9.9519	9.0646	8.3187	7.6841	7.1382
57	33.8281	27.1509	22.3267	18.7605	16.0649	13.9837	12.3445	11.0294	9.9563	9.0672	8.3203	7.6851	7.1388
58	34.1452	27.3310	22.4296	18.8195	16.0990	14.0035	12.3560	11.0361	9.9603	9.0695	8.3217	7.6859	7.1393
59	34.4561	27.5058	22.5284	18.8758	16.1311	14.0219	12.3667	11.0423	9.9639	9.0717	8.3229	7.6866	7.1397
60	34.7609	27.6756	22.6235	18.9293	16.1614	14.0392	12.3766	11.0480	9.9672	9.0736	8.3240	7.6873	7.1401
61	35.0597	27.8404	22.7149	18.9803	16.1900	14.0553	12.3857	11.0532	9.9701	9.0753	8.3250	7.6879	7.1404
62	35.3526	28.0003	22.8028	19.0288	16.2170	14.0704	12.3942	11.0580	9.9729	9.0768	8.3259	7.6884	7.1407
63	35.6398	28.1557	22.8873	19.0751	16.2425	14.0845	12.4020	11.0624	9.9753	9.0782	8.3267	7.6888	7.1410
64	35.9214	28.3065	22.9685	19.1191	16.2665	14.0976	12.4093	11.0664	9.9776	9.0795	8.3274	7.6892	7.1412
65	36.1975	28.4529	23.0467	19.1611	16.2891	14.1099	12.4160	11.0701	9.9796	9.0806	8.3281	7.6896	7.1414
66	36.4681	28.5950	23.1218	19.2010	16.3105	14.1214	12.4222	11.0735	9.9815	9.0816	8.3286	7.6899	7.1416
67	36.7334	28.7330	23.1940	19.2391	16.3307	14.1322	12.4280	11.0766	9.9831	9.0826	8.3291	7.6902	7.1418
68	36.9936	28.8670	23.2635	19.2753	16.3497	14.1422	12.4333	11.0794	9.9847	9.0834	8.3296	7.6904	7.1419
69	37.2486	28.9971	23.3303	19.3098	16.3676	14.1516	12.4382	11.0820	9.9861	9.0841	8.3300	7.6906	7.1420
70	37.4986	29.1234	23.3945	19.3427	16.3845	14.1604	12.4428	11.0844	9.9873	9.0848	8.3303	7.6908	7.1421
71	37.7437	29.2460	23.4563	19.3740	16.4005	14.1686	12.4471	11.0867	9.9885	9.0854	8.3307	7.6910	7.1422
72	37.9841	29.3651	23.5156	19.4038	16.4156	14.1763	12.4510	11.0887	9.9895	9.0860	8.3310	7.6911	7.1423
73	38.2197	29.4807	23.5727	19.4322	16.4298	14.1834	12.4546	11.0905	9.9905	9.0864	8.3312	7.6913	7.1424
74	38.4507	29.5929	23.6276	19.4592	16.4432	14.1901	12.4580	11.0922	9.9914	9.0869	8.3314	7.6914	7.1424
75	38.6771	29.7018	23.6804	19.4850	16.4558	14.1964	12.4611	11.0938	9.9921	9.0873	8.3316	7.6915	7.1425
76	38.8991	29.8076	23.7312	19.5095	16.4678	14.2022	12.4640	11.0952	9.9929	9.0876	8.3318	7.6916	7.1425
77	39.1168	29.9103	23.7800	19.5329	16.4790	14.2077	12.4666	11.0965	9.9935	9.0880	8.3320	7.6917	7.1426
78	39.3302	30.0100	23.8269	19.5551	16.4897	14.2128	12.4691	11.0977	9.9941	9.0883	8.3321	7.6918	7.1426
79	39.5394	30.1068	23.8720	19.5763	16.4997	14.2175	12.4714	11.0988	9.9946	9.0885	8.3323	7.6918	7.1426
80	39.7445	30.2008	23.9154	19.5965	16.5091	14.2220	12.4735	11.0998	9.9951	9.0888	8.3324	7.6919	7.1427
81	39.9456	30.2920	23.9571	19.6157	16.5180	14.2262	12.4755	11.1008	9.9956	9.0890	8.3325	7.6919	7.1427
82	40.1427	30.3806	23.9972	19.6340	16.5265	14.2301	12.4773	11.1016	9.9960	9.0892	8.3326	7.6920	7.1427
83	40.3360	30.4666	24.0358	19.6514	16.5344	14.2337	12.4790	11.1024	9.9963	9.0893	8.3326	7.6920	7.1427
84	40.5255	30.5501	24.0729	19.6680	16.5419	14.2371	12.4805	11.1031	9.9967	9.0895	8.3327	7.6920	7.1427
85	40.7113	30.6312	24.1085	19.6838	16.5489	14.2403	12.4820	11.1038	9.9970	9.0896	8.33[illegible]	[illegible].6921	7.1428
86	40.8934	30.7099	24.1428	19.6989	16.5556	14.2433	12.4833	11.1044	9.9972	9.0898	8.3328	7.6921	7.1428
87	41.0720	30.7863	24.1758	19.7132	16.5619	14.2460	12.4845	11.1050	9.9975	9.0899	8.3329	7.6921	7.1428
88	41.2470	30.8605	24.2075	19.7269	16.5678	14.2486	12.4857	11.1055	9.9977	9.0900	8.3329	7.6921	7.1428
89	41.4187	30.9325	24.2380	19.7399	16.5734	14.2511	12.4868	11.1059	9.9979	9.0901	8.3330	7.6922	7.1428
90	41.5869	31.0024	24.2673	19.7523	16.5787	14.2533	12.4877	11.1064	9.9981	9.0902	8.3330	7.6922	7.1428
91	41.7519	31.0703	24.2955	19.7641	16.5837	14.2554	12.4886	11.1067	9.9983	9.0902	8.3331	7.6922	7.1428
92	41.9136	31.1362	24.3226	19.7753	16.5884	14.2574	12.4895	11.1071	9.9984	9.0903	8.3331	7.6922	7.1428
93	42.0722	31.2002	24.3486	19.7860	16.5928	14.2593	12.4903	11.1074	9.9986	9.0904	8.3331	7.6922	7.1428
94	42.2276	31.2623	24.3737	19.7962	16.5970	14.2610	12.4910	11.1077	9.9987	9.0904	8.3331	7.6922	7.1428
95	42.3800	31.3227	24.3978	19.8059	16.6009	14.2626	12.4917	11.1080	9.9988	9.0905	8.3332	7.6922	7.1428
96	42.5294	31.3812	24.4209	19.8151	16.6047	14.2641	12.4923	11.1083	9.9989	9.0905	8.3332	7.6922	7.1428
97	42.6759	31.4381	24.4432	19.8239	16.6082	14.2655	12.4928	11.1085	9.9990	9.0905	8.3332	7.6923	7.1428
98	42.8195	31.4933	24.4646	19.8323	16.6115	14.2669	12.4934	11.1087	9.9991	9.0906	8.3332	7.6923	7.1428
99	42.9603	31.5469	24.4852	19.8403	16.6146	14.2681	12.4939	11.1089	9.9992	9.0906	8.3332	7.6923	7.1428
100	43.0984	31.5989	24.5050	19.8479	16.6175	14.2693	12.4943	11.1091	9.9993	9.0906	8.3332	7.6923	7.1428

Table 1 (cont.)

mn	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%	11.00%	12.00%	13.00%	14.00%
101	43.2337	31.6494	24.5240	19.8552	16.6203	14.2703	12.4947	11.1093	9.9993	9.0907	8.3332	7.6923	7.1428
102	43.3664	31.6985	24.5423	19.8621	16.6229	14.2713	12.4951	11.1094	9.9994	9.0907	8.3333	7.6923	7.1428
103	43.4964	31.7461	24.5599	19.8686	16.6254	14.2723	12.4955	11.1096	9.9995	9.0907	8.3333	7.6923	7.1428
104	43.6239	31.7923	24.5769	19.8749	16.6278	14.2732	12.4958	11.1097	9.9995	9.0907	8.3333	7.6923	7.1428
105	43.7490	31.8372	24.5931	19.8808	16.6300	14.2740	12.4961	11.1098	9.9995	9.0908	8.3333	7.6923	7.1428
106	43.8715	31.8808	24.6088	19.8865	16.6320	14.2747	12.4964	11.1099	9.9996	9.0908	8.3333	7.6923	7.1429
107	43.9917	31.9231	24.6238	19.8919	16.6340	14.2755	12.4967	11.1100	9.9996	9.0908	8.3333	7.6923	7.1429
108	44.1095	31.9642	24.6383	19.8971	16.6358	14.2761	12.4969	11.1101	9.9997	9.0908	8.3333	7.6923	7.1429
109	44.2250	32.0040	24.6522	19.9020	16.6376	14.2768	12.4972	11.1102	9.9997	9.0908	8.3333	7.6923	7.1429
110	44.3382	32.0428	24.6656	19.9066	16.6392	14.2773	12.4974	11.1103	9.9997	9.0908	8.3333	7.6923	7.1429
111	44.4493	32.0803	24.6785	19.9111	16.6408	14.2779	12.4976	11.1103	9.9997	9.0908	8.3333	7.6923	7.1429
112	44.5581	32.1168	24.6908	19.9153	16.6423	14.2784	12.4977	11.1104	9.9998	9.0908	8.3333	7.6923	7.1429
113	44.6648	32.1523	24.7027	19.9193	16.6436	14.2789	12.4979	11.1105	9.9998	9.0908	8.3333	7.6923	7.1429
114	44.7694	32.1867	24.7141	19.9232	16.6449	14.2793	12.4981	11.1105	9.9998	9.0908	8.3333	7.6923	7.1429
115	44.8720	32.2201	24.7251	19.9268	16.6462	14.2797	12.4982	11.1106	9.9998	9.0909	8.3333	7.6923	7.1429
116	44.9725	32.2525	24.7357	19.9303	16.6473	14.2801	12.4983	11.1106	9.9998	9.0909	8.3333	7.6923	7.1429
117	45.0711	32.2840	24.7459	19.9336	16.6484	14.2805	12.4985	11.1106	9.9999	9.0909	8.3333	7.6923	7.1429
118	45.1677	32.3145	24.7557	19.9368	16.6495	14.2808	12.4986	11.1107	9.9999	9.0909	8.3333	7.6923	7.1429
119	45.2625	32.3442	24.7651	19.9398	16.6504	14.2812	12.4987	11.1107	9.9999	9.0909	8.3333	7.6923	7.1429
120	45.3554	32.3730	24.7741	19.9427	16.6514	14.2815	12.4988	11.1108	9.9999	9.0909	8.3333	7.6923	7.1429
121	45.4465	32.4010	24.7828	19.9454	16.6522	14.2817	12.4989	11.1108	9.9999	9.0909	8.3333	7.6923	7.1429
122	45.5357	32.4281	24.7911	19.9480	16.6530	14.2820	12.4990	11.1108	9.9999	9.0909	8.3333	7.6923	7.1429
123	45.6233	32.4545	24.7992	19.9505	16.6538	14.2822	12.4990	11.1108	9.9999	9.0909	8.3333	7.6923	7.1429
124	45.7091	32.4801	24.8069	19.9528	16.6545	14.2825	12.4991	11.1109	9.9999	9.0909	8.3333	7.6923	7.1429
125	45.7932	32.5050	24.8143	19.9551	16.6552	14.2827	12.4992	11.1109	9.9999	9.0909	8.3333	7.6923	7.1429
126	45.8757	32.5291	24.8215	19.9572	16.6559	14.2829	12.4992	11.1109	9.9999	9.0909	8.3333	7.6923	7.1429
127	45.9566	32.5525	24.8283	19.9593	16.6565	14.2831	12.4993	11.1109	9.9999	9.0909	8.3333	7.6923	7.1429
128	46.0359	32.5753	24.8349	19.9612	16.6571	14.2832	12.4993	11.1109	9.9999	9.0909	8.3333	7.6923	7.1429
129	46.1136	32.5973	24.8413	19.9631	16.6576	14.2834	12.4994	11.1109	10.0000	9.0909	8.3333	7.6923	7.1429
130	46.1898	32.6188	24.8474	19.9648	16.6581	14.2836	12.4994	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
131	46.2645	32.6396	24.8533	19.9665	16.6586	14.2837	12.4995	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
132	46.3378	32.6598	24.8589	19.9681	16.6591	14.2838	12.4995	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
133	46.4096	32.6794	24.8643	19.9696	16.6595	14.2839	12.4996	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
134	46.4800	32.6985	24.8695	19.9710	16.6599	14.2841	12.4996	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
135	46.5490	32.7169	24.8746	19.9724	16.6603	14.2842	12.4996	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
136	46.6167	32.7349	24.8794	19.9737	16.6606	14.2843	12.4996	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
137	46.6830	32.7523	24.8840	19.9750	16.6610	14.2844	12.4997	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
138	46.7480	32.7693	24.8885	19.9762	16.6613	14.2845	12.4997	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
139	46.8118	32.7857	24.8928	19.9773	16.6616	14.2845	12.4997	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
140	46.8743	32.8016	24.8969	19.9784	16.6619	14.2846	12.4997	11.1110	10.0000	9.0909	8.3333	7.6923	7.1429
141	46.9356	32.8171	24.9009	19.9794	16.6622	14.2847	12.4998	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
142	46.9957	32.8322	24.9047	19.9804	16.6624	14.2848	12.4998	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
143	47.0546	32.8468	24.9083	19.9813	16.6627	14.2848	12.4998	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
144	47.1123	32.8609	24.9119	19.9822	16.6629	14.2849	12.4998	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
145	47.1690	32.8747	24.9153	19.9831	16.6631	14.2849	12.4998	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
146	47.2245	32.8880	24.9185	19.9839	16.6633	14.2850	12.4998	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
147	47.2789	32.9010	24.9216	19.9846	16.6635	14.2850	12.4998	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
148	47.3323	32.9136	24.9247	19.9854	16.6637	14.2851	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
149	47.3846	32.9258	24.9276	19.9861	16.6638	14.2851	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
150	47.4358	32.9377	24.9303	19.9867	16.6640	14.2852	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429

Table 1 (cont.)

mn	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%	11.00%	12.00%	13.00%	14.00%
151	47.4861	32.9492	24.9330	19.9874	16.6642	14.2852	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
152	47.5354	32.9604	24.9356	19.9880	16.6643	14.2852	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
153	47.5837	32.9713	24.9381	19.9885	16.6644	14.2853	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
154	47.6311	32.9818	24.9405	19.9891	16.6646	14.2853	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
155	47.6776	32.9921	24.9428	19.9896	16.6647	14.2853	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
156	47.7231	33.0020	24.9450	19.9901	16.6648	14.2853	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
157	47.7677	33.0116	24.9471	19.9906	16.6649	14.2854	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
158	47.8115	33.0210	24.9491	19.9910	16.6650	14.2854	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
159	47.8544	33.0301	24.9511	19.9915	16.6651	14.2854	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
160	47.8965	33.0389	24.9529	19.9919	16.6652	14.2854	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
161	47.9377	33.0475	24.9548	19.9922	16.6653	14.2854	12.4999	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
162	47.9782	33.0558	24.9565	19.9926	16.6653	14.2855	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
163	48.0178	33.0639	24.9582	19.9930	16.6654	14.2855	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
164	48.0567	33.0718	24.9598	19.9933	16.6655	14.2855	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
165	48.0948	33.0794	24.9613	19.9936	16.6656	14.2855	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
166	48.1322	33.0868	24.9628	19.9939	16.6656	14.2855	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
167	48.1688	33.0940	24.9642	19.9942	16.6657	14.2855	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
168	48.2047	33.1009	24.9656	19.9945	16.6657	14.2855	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
169	48.2399	33.1077	24.9669	19.9948	16.6658	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
170	48.2744	33.1143	24.9682	19.9950	16.6658	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
171	48.3082	33.1207	24.9694	19.9952	16.6659	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
172	48.3414	33.1269	24.9706	19.9955	16.6659	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
173	48.3739	33.1329	24.9717	19.9957	16.6660	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
174	48.4058	33.1387	24.9728	19.9959	16.6660	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
175	48.4371	33.1444	24.9739	19.9961	16.6660	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
176	48.4677	33.1499	24.9749	19.9963	16.6661	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
177	48.4978	33.1552	24.9758	19.9964	16.6661	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
178	48.5272	33.1604	24.9768	19.9966	16.6661	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
179	48.5561	33.1654	24.9777	19.9968	16.6662	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
180	48.5844	33.1703	24.9785	19.9969	16.6662	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
181	48.6122	33.1751	24.9794	19.9971	16.6662	14.2856	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
182	48.6394	33.1797	24.9801	19.9972	16.6663	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
183	48.6661	33.1842	24.9809	19.9973	16.6663	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
184	48.6922	33.1885	24.9816	19.9975	16.6663	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
185	48.7179	33.1927	24.9823	19.9976	16.6663	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
186	48.7430	33.1968	24.9830	19.9977	16.6663	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
187	48.7676	33.2008	24.9837	19.9978	16.6664	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
188	48.7918	33.2047	24.9843	19.9979	16.6664	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
189	48.8155	33.2084	24.9849	19.9980	16.6664	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
190	48.8387	33.2120	24.9855	19.9981	16.6664	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
191	48.8615	33.2156	24.9861	19.9982	16.6664	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
192	48.8838	33.2190	24.9866	19.9983	16.6664	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
193	48.9057	33.2223	24.9871	19.9984	16.6664	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
194	48.9272	33.2256	24.9876	19.9985	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
195	48.9482	33.2287	24.9881	19.9985	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
196	48.9688	33.2318	24.9885	19.9986	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
197	48.9890	33.2347	24.9890	19.9987	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
198	49.0089	33.2376	24.9894	19.9987	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
199	49.0283	33.2404	24.9898	19.9988	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
200	49.0473	33.2431	24.9902	19.9988	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429

Table 1 (cont.)

mn	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%	11.00%	12.00%	13.00%	14.00%
201	49.0660	33.2457	24.9906	19.9989	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
202	49.0843	33.2483	24.9909	19.9990	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
203	49.1023	33.2507	24.9913	19.9990	16.6665	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
204	49.1199	33.2531	24.9916	19.9990	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
205	49.1372	33.2555	24.9919	19.9991	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
206	49.1541	33.2578	24.9923	19.9991	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
207	49.1707	33.2600	24.9926	19.9992	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
208	49.1869	33.2621	24.9928	19.9992	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
209	49.2029	33.2642	24.9931	19.9993	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
210	49.2185	33.2662	24.9934	19.9993	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
211	49.2338	33.2681	24.9936	19.9993	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
212	49.2488	33.2700	24.9939	19.9994	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
213	49.2636	33.2719	24.9941	19.9994	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
214	49.2780	33.2737	24.9943	19.9994	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
215	49.2922	33.2754	24.9946	19.9994	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
216	49.3060	33.2771	24.9948	19.9995	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
217	49.3196	33.2787	24.9950	19.9995	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
218	49.3330	33.2803	24.9952	19.9995	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
219	49.3461	33.2819	24.9953	19.9995	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
220	49.3589	33.2834	24.9955	19.9996	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
221	49.3715	33.2848	24.9957	19.9996	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
222	49.3838	33.2862	24.9959	19.9996	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
223	49.3959	33.2876	24.9960	19.9996	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
224	49.4077	33.2889	24.9962	19.9996	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
225	49.4193	33.2902	24.9963	19.9997	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
226	49.4307	33.2915	24.9965	19.9997	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
227	49.4419	33.2927	24.9966	19.9997	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
228	49.4528	33.2939	24.9967	19.9997	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
229	49.4635	33.2950	24.9969	19.9997	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
230	49.4741	33.2962	24.9970	19.9997	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
231	49.4844	33.2972	24.9971	19.9997	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
232	49.4945	33.2983	24.9972	19.9998	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
233	49.5044	33.2993	24.9973	19.9998	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
234	49.5141	33.3003	24.9974	19.9998	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
235	49.5236	33.3013	24.9975	19.9998	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
236	49.5330	33.3022	24.9976	19.9998	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
237	49.5421	33.3031	24.9977	19.9998	16.6666	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
238	49.5511	33.3040	24.9978	19.9998	16.6667	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
239	49.5599	33.3048	24.9979	19.9998	16.6667	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429
240	49.5686	33.3057	24.9980	19.9998	16.6667	14.2857	12.5000	11.1111	10.0000	9.0909	8.3333	7.6923	7.1429

Table 1 (cont.)

mn	15.00%	16.00%	17.00%	18.00%	19.00%	20.00%	21.00%	22.00%	23.00%	24.00%	25.00%	26.00%	27.00
1	0.8696	0.8621	0.8547	0.8475	0.8403	0.8333	0.8264	0.8197	0.8130	0.8065	0.8000	0.7937	0.7874
2	1.6257	1.6052	1.5852	1.5656	1.5465	1.5278	1.5095	1.4915	1.4740	1.4568	1.4400	1.4235	1.4074
3	2.2832	2.2459	2.2096	2.1743	2.1399	2.1065	2.0739	2.0422	2.0114	1.9813	1.9520	1.9234	1.8956
4	2.8550	2.7982	2.7432	2.6901	2.6386	2.5887	2.5404	2.4936	2.4483	2.4043	2.3616	2.3202	2.2800
5	3.3522	3.2743	3.1993	3.1272	3.0576	2.9906	2.9260	2.8636	2.8035	2.7454	2.6893	2.6351	2.5827
6	3.7845	3.6847	3.5892	3.4976	3.4098	3.3255	3.2446	3.1669	3.0923	3.0205	2.9514	2.8850	2.8210
7	4.1604	4.0386	3.9224	3.8115	3.7057	3.6046	3.5079	3.4155	3.3270	3.2423	3.1611	3.0833	3.0087
8	4.4873	4.3436	4.2072	4.0776	3.9544	3.8372	3.7256	3.6193	3.5179	3.4212	3.3289	3.2407	3.1564
9	4.7716	4.6065	4.4506	4.3030	4.1633	4.0310	3.9054	3.7863	3.6731	3.5655	3.4631	3.3657	3.2728
10	5.0188	4.8332	4.6586	4.4941	4.3389	4.1925	4.0541	3.9232	3.7993	3.6819	3.5705	3.4648	3.3644
11	5.2337	5.0286	4.8364	4.6560	4.4865	4.3271	4.1769	4.0354	3.9018	3.7757	3.6564	3.5435	3.4365
12	5.4206	5.1971	4.9884	4.7932	4.6105	4.4392	4.2784	4.1274	3.9852	3.8514	3.7251	3.6059	3.4933
13	5.5831	5.3423	5.1183	4.9095	4.7147	4.5327	4.3624	4.2028	4.0530	3.9124	3.7801	3.6555	3.5381
14	5.7245	5.4675	5.2293	5.0081	4.8023	4.6106	4.4317	4.2646	4.1082	3.9616	3.8241	3.6949	3.5733
15	5.8474	5.5755	5.3242	5.0916	4.8759	4.6755	4.4890	4.3152	4.1530	4.0013	3.8593	3.7261	3.6010
16	5.9542	5.6685	5.4053	5.1624	4.9377	4.7296	4.5364	4.3567	4.1894	4.0333	3.8874	3.7509	3.6228
17	6.0472	5.7487	5.4746	5.2223	4.9897	4.7746	4.5755	4.3908	4.2190	4.0591	3.9099	3.7705	3.6400
18	6.1280	5.8178	5.5339	5.2732	5.0333	4.8122	4.6079	4.4187	4.2431	4.0799	3.9279	3.7861	3.6536
19	6.1982	5.8775	5.5845	5.3162	5.0700	4.8435	4.6346	4.4415	4.2627	4.0967	3.9424	3.7985	3.6642
20	6.2593	5.9288	5.6278	5.3527	5.1009	4.8696	4.6567	4.4603	4.2786	4.1103	3.9539	3.8083	3.6726
21	6.3125	5.9731	5.6648	5.3837	5.1268	4.8913	4.6750	4.4756	4.2916	4.1212	3.9631	3.8161	3.6792
22	6.3587	6.0113	5.6964	5.4099	5.1486	4.9094	4.6900	4.4882	4.3021	4.1300	3.9705	3.8223	3.6844
23	6.3988	6.0442	5.7234	5.4321	5.1668	4.9245	4.7025	4.4985	4.3106	4.1371	3.9764	3.8273	3.6885
24	6.4338	6.0726	5.7465	5.4509	5.1822	4.9371	4.7128	4.5070	4.3176	4.1428	3.9811	3.8312	3.6918
25	6.4641	6.0971	5.7662	5.4669	5.1951	4.9476	4.7213	4.5139	4.3232	4.1474	3.9849	3.8342	3.6943
26	6.4906	6.1182	5.7831	5.4804	5.2060	4.9563	4.7284	4.5196	4.3278	4.1511	3.9879	3.8367	3.6963
27	6.5135	6.1364	5.7975	5.4919	5.2151	4.9636	4.7342	4.5243	4.3316	4.1542	3.9903	3.8387	3.6979
28	6.5335	6.1520	5.8099	5.5016	5.2228	4.9697	4.7390	4.5281	4.3346	4.1566	3.9923	3.8402	3.6991
29	6.5509	6.1656	5.8204	5.5098	5.2292	4.9747	4.7430	4.5312	4.3371	4.1585	3.9938	3.8414	3.7001
30	6.5660	6.1772	5.8294	5.5168	5.2347	4.9789	4.7463	4.5338	4.3391	4.1601	3.9950	3.8424	3.7009
31	6.5791	6.1872	5.8371	5.5227	5.2392	4.9824	4.7490	4.5359	4.3407	4.1614	3.9960	3.8432	3.7015
32	6.5905	6.1959	5.8437	5.5277	5.2430	4.9854	4.7512	4.5376	4.3421	4.1624	3.9968	3.8438	3.7019
33	6.6005	6.2034	5.8493	5.5320	5.2462	4.9878	4.7531	4.5390	4.3431	4.1632	3.9975	3.8443	3.7023
34	6.6091	6.2098	5.8541	5.5356	5.2489	4.9898	4.7546	4.5402	4.3440	4.1639	3.9980	3.8447	3.7026
35	6.6166	6.2153	5.8582	5.5386	5.2512	4.9915	4.7559	4.5411	4.3447	4.1644	3.9984	3.8450	3.7028
36	6.6231	6.2201	5.8617	5.5412	5.2531	4.9929	4.7569	4.5419	4.3453	4.1649	3.9987	3.8452	3.7030
37	6.6288	6.2242	5.8647	5.5434	5.2547	4.9941	4.7578	4.5426	4.3458	4.1652	3.9990	3.8454	3.7032
38	6.6338	6.2278	5.8673	5.5452	5.2561	4.9951	4.7585	4.5431	4.3462	4.1655	3.9992	3.8456	3.7033
39	6.6380	6.2309	5.8695	5.5468	5.2572	4.9959	4.7591	4.5435	4.3465	4.1657	3.9993	3.8457	3.7034
40	6.6418	6.2335	5.8713	5.5482	5.2582	4.9966	4.7596	4.5439	4.3467	4.1659	3.9995	3.8458	3.7034
41	6.6450	6.2358	5.8729	5.5493	5.2590	4.9972	4.7600	4.5441	4.3469	4.1661	3.9996	3.8459	3.7035
42	6.6478	6.2377	5.8743	5.5502	5.2596	4.9976	4.7603	4.5444	4.3471	4.1662	3.9997	3.8459	3.7035
43	6.6503	6.2394	5.8755	5.5510	5.2602	4.9980	4.7606	4.5446	4.3472	4.1663	3.9997	3.8460	3.7036
44	6.6524	6.2409	5.8765	5.5517	5.2607	4.9984	4.7608	4.5447	4.3473	4.1663	3.9998	3.8460	3.7036
45	6.6543	6.2421	5.8773	5.5523	5.2611	4.9986	4.7610	4.5449	4.3474	4.1664	3.9998	3.8460	3.7036
46	6.6559	6.2432	5.8781	5.5528	5.2614	4.9989	4.7612	4.5450	4.3475	4.1665	3.9999	3.8461	3.7036
47	6.6573	6.2442	5.8787	5.5532	5.2617	4.9991	4.7613	4.5451	4.3476	4.1665	3.9999	3.8461	3.7037
48	6.6585	6.2450	5.8792	5.5536	5.2619	4.9992	4.7614	4.5451	4.3476	4.1665	3.9999	3.8461	3.7037
49	6.6596	6.2457	5.8797	5.5539	5.2621	4.9993	4.7615	4.5452	4.3477	4.1666	3.9999	3.8461	3.7037
50	6.6605	6.2463	5.8801	5.5541	5.2623	4.9995	4.7616	4.5452	4.3477	4.1666	3.9999	3.8461	3.7037

Table 1 (cont.)

mn	15.00%	16.00%	17.00%	18.00%	19.00%	20.00%	21.00%	22.00%	23.00%	24.00%	25.00%	26.00%	27.00%
51	6.6613	6.2468	5.8804	5.5544	5.2624	4.9995	4.7616	4.5453	4.3477	4.1666	4.0000	3.8461	3.7037
52	6.6620	6.2472	5.8807	5.5545	5.2625	4.9996	4.7617	4.5453	4.3477	4.1666	4.0000	3.8461	3.7037
53	6.6626	6.2476	5.8809	5.5547	5.2626	4.9997	4.7617	4.5453	4.3478	4.1666	4.0000	3.8461	3.7037
54	6.6631	6.2479	5.8811	5.5548	5.2627	4.9997	4.7617	4.5454	4.3478	4.1666	4.0000	3.8461	3.7037
55	6.6636	6.2482	5.8813	5.5549	5.2628	4.9998	4.7618	4.5454	4.3478	4.1666	4.0000	3.8461	3.7037
56	6.6640	6.2485	5.8815	5.5550	5.2628	4.9998	4.7618	4.5454	4.3478	4.1666	4.0000	3.8461	3.7037
57	6.6644	6.2487	5.8816	5.5551	5.2629	4.9998	4.7618	4.5454	4.3478	4.1666	4.0000	3.8461	3.7037
58	6.6647	6.2489	5.8817	5.5552	5.2629	4.9999	4.7618	4.5454	4.3478	4.1667	4.0000	3.8461	3.7037
59	6.6649	6.2490	5.8818	5.5552	5.2630	4.9999	4.7618	4.5454	4.3478	4.1667	4.0000	3.8461	3.7037
60	6.6651	6.2492	5.8819	5.5553	5.2630	4.9999	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
61	6.6653	6.2493	5.8819	5.5553	5.2630	4.9999	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
62	6.6655	6.2494	5.8820	5.5554	5.2630	4.9999	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
63	6.6657	6.2495	5.8821	5.5554	5.2631	4.9999	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
64	6.6658	6.2495	5.8821	5.5554	5.2631	5.0000	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
65	6.6659	6.2496	5.8821	5.5554	5.2631	5.0000	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
66	6.6660	6.2497	5.8822	5.5555	5.2631	5.0000	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
67	6.6661	6.2497	5.8822	5.5555	5.2631	5.0000	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
68	6.6662	6.2497	5.8822	5.5555	5.2631	5.0000	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
69	6.6662	6.2498	5.8822	5.5555	5.2631	5.0000	4.7619	4.5454	4.3478	4.1667	4.0000	3.8462	3.7037
70	6.6663	6.2498	5.8823	5.5555	5.2631	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
71	6.6663	6.2498	5.8823	5.5555	5.2631	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
72	6.6664	6.2499	5.8823	5.5555	5.2631	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
73	6.6664	6.2499	5.8823	5.5555	5.2631	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
74	6.6665	6.2499	5.8823	5.5555	5.2631	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
75	6.6665	6.2499	5.8823	5.5555	5.2631	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
76	6.6665	6.2499	5.8823	5.5555	5.2631	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
77	6.6665	6.2499	5.8823	5.5555	5.2631	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
78	6.6665	6.2499	5.8823	5.5555	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
79	6.6666	6.2499	5.8823	5.5555	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
80	6.6666	6.2500	5.8823	5.5555	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
81	6.6666	6.2500	5.8823	5.5555	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
82	6.6666	6.2500	5.8823	5.5555	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
83	6.6666	6.2500	5.8823	5.5555	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
84	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
85	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
86	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
87	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
88	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
89	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
90	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
91	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
92	6.6666	6.2500	5.8823	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
93	6.6667	6.2500	5.8824	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
94	6.6667	6.2500	5.8824	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
95	6.6667	6.2500	5.8824	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
96	6.6667	6.2500	5.8824	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
97	6.6667	6.2500	5.8824	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
98	6.6667	6.2500	5.8824	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
99	6.6667	6.2500	5.8824	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037
100	6.6667	6.2500	5.8824	5.5556	5.2632	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037

XIII. APPENDIX I

DEFINITIONS OF VARIABLES USED

r = the annual percentage rate (APR) expressed as a decimal;

m = number of conversion periods within a year;

r_e = the effective (annual) interest rate expressed as a decimal;

V_o = value of the loan in the present period;

n = number of years to loan maturity;

mn = number of periods to loan maturity;

b = geometric factor;

a = a constant in a geometric series;

A_t = a single payment made in year t;

A = a constant payment or annuity;

$US_o(r,n)$ = abbreviation for the factor which converts to a present value a uniform series of \$1 payments discounted at r percent for n periods;

$US_n(r,n)$ = abbreviation for the factor which converts to period n values a uniform series of \$1 payments compounded at r percent;

TI = total interest costs paid over the life of the loan;

$SSD(mn)$ = sum of the periods digits for an mn period loan;

t = time period number or number of periods away from the present period;

$I(t)$ = interest paid on the t^{th} payment;

$AI(t)$ = accumulated interest paid after t periods;

$AP(t)$ = accumulated principal paid after t periods;

$PD(t)$ = loan principal outstanding after the 16^{th} payments;

$PP(t)$ = principal paid on the t^{th} payment;

$\Delta P(n)$ = change in loan payment associated with a term increase from n to n + 1 periods;

$\Delta TI(n)$ = change in total interest paid associated with a term increase from n to n + 1 periods;

B = a balloon payment made to retire a loan;

A_s = subsidized payment on a buy down loan;

Δr = interest rate subsidy on a buy down loan;

V_s = present value of subsidy provided on a buy down loan;

mn = term on a graduated payment loan;

g = percentage increase in the loan payment of a graduated payment mortgage which occurs every m periods;

r^* = a concessionary APR interest rate, expressed as a decimal;

V_o^* = the present value of a concessionary interest rate loan with an outstanding loan balance of V_o;

A^* = payment on a concessionary interest rate loan;

A_b = payment on a blended interest rate loan;

r_b = blended interest rate expressed as a decimal;

A_t = the t^{th} payment on a constant principal payment loan;

p = the percentage of a loan charged (points) to close a loan;

$\hat{V}_o$ = present value of a new loan extension;

i = stated interest rate on disguised interest cost loans;

V_d = discounted loan amount actually received by the borrower;

d = maximizing derivative.

XIV. REFERENCES

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