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FIVE PRINCIPLES FOR BUILDING PRESENT VALUE MODELS AND THEIR APPLICATION TO MAXIMUM (MINIMUM) BID (SELL) PRICE MODELS FOR LAND

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ABSTRACT

Five Principles for Building Present Value Models And Their Application to Maximum (Minimum) Bid (Sell) Price Models for Land

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This paper suggests that present value models could be made consistent by adhering to five construction principles. The five principles include: (1) the homogeneity of measurements principle; (2) consistency in timing principle; (3) the opportunity cost principle; (4) the life of the asset principle; and (5) the total cost and returns principle.

The paper discusses and defends each of the five principles and then employs them in the construction of maximum bid and minimum sell models. The models are developed to include such features as concessionary interest rates, property taxes, income taxes, inflation, capital gains and capital gains tax, and transactions cost. Finally, base model results are reported along with sensitivity analysis which indicates the relative importance of each of the variables included in the models.

Five Principles for Building Present Value Models And Their Application to Maximum (Minimum) Bid (Sell) Price Models for Land

Introduction

Investment decisions frequently require the comparison of an asset's current asking or offer price with its expected revenue and cost streams adjusted for the influence of time. Present value (PV) models have been designed to rationalize the influences of time on the specific parameters involved in investment decisions. Other influences, however, from such factors as taxes, inflation and transactions cost have not always been properly rationalized nor well understood. This lack of understanding continues to cause disagreements about the proper formulation of more sophisticated PV models. As a result of these problems, existing models do not always produce theoretically consistent results.

Some questions about the proper construction of PV models include the following. How should taxes enter the model? Should capital gains be allowed to accumulate independent of cash income? When should capital gains be credited? Does the method of financing influence the asset's present value? How should the life of the asset be determined? Should models be constructed using continuous or discrete time? And finally, should there be differences between the maximum bid price of a potential owner and the minimum sell price required by an asset's owner?

A number of authors have attempted to answer these questions. With regard to the question of continuous or discrete time, Fama and Miller agree that there is "little basis other than taste or convenience in choosing between a discrete time formulation ... and a continuous time formulation of the kind often found in economist's expositions of capital theory." This position was also confirmed by Hirshleifer.

The influence of income taxes in PV models has already been debated in our literature. Adams, for example, recognized the importance of discounting after-tax income with an after-tax discount rate, and points to work by Harris and Nehring and Lee and Rask as examples of lack of consistency in the application of taxes. Inconsistency also exists in the treatment of financing arrangement when calculating an asset's present value. Baker does not allow them to influence an asset's present value. Lee and Rask do. And regarding the life of the asset, Baker lets it be infinite while Lee and Rask impose a finite life.

Several issues surround the influence of capital gains in PV models. Lee and Rask allow it to accumulate at a rate independent of the rate of income in the income stream. Baker does not. And also at issue is: When should capital gains be credited? Plaxico and Kletke suggest they should influence the asset's value at the time they are earned. Both Baker and Lee and Rask count them at the time they are converted to cash.

Finally, all PV models reviewed in this paper have one feature in common. As a result of their perfect market assumptions, they solve only for the maximum bid price and ignore the question of what is the minimum sell price. This paper asks: Should maximum bid and minimum sell prices necessarily be equal? If not, then our models have been constructed for only one-half of the market participants.

Considering the importance of PV models in asset valuation, and the disagreement which appears to exist about their construction, this paper asks: Are there principles which should guide the construction of PV models? We believe there are and suggest five. They are: (1) the homogeneity of measurements principle; (2) consistency in timing principle; (3) the life of the asset principle; (4) the opportunity cost principle; and (5) the total cash returns and cost principles.

In the remainder of this paper each principle is discussed and defended. Then the principles are employed to construct PV models for land including both maximum bid price models and minimum sell price models. In the final section of the paper we provide empirical results using a base model and sensitivity analysis to indicate the importance of each variable in the model. The first principle, "the homogeneity of measurements principle" is discussed in the next section.

The Homogeneity of Measurements Principle

The homogeneity of measurements principle requires that units of income or cost being discounted be consistent with the unit of measure reflected by the discount rate. To acquire consistency between units in the discount rate and costs and benefits being discounted we convert asset's returns and costs received in different forms and at different times to a common measure for comparison. Unless asset returns and costs are converted to a common unit of measure, PV models cannot provide meaningful results or useful comparisons.

There are at least 7 factors affecting the way in which asset returns and costs are measured. These are: (1) the timing of returns and costs; (2) taxes; (3) the certainty of incurring costs and receiving benefits; (4) inflation; (5) the liquidity or "nearness to cash" of the returns and costs; (6) the term of the investment; and (7) the size of the investment.

The first feature, and the one most important to convert to consistent units is the timing of returns and costs. The need for reconciliation arises because a dollar received today does not have the same value as a dollar received in the future. This is partly true because a dollar owned today can be invested; moreover, today's investment opportunities are lost if the dollar is not available until some future time period.

The comparison between the value of dollars received next period and the dollars' value today depends on what opportunities exist for an investment of R today. If the R dollars could be invested at a return of r percent, and the return of r percent represents the next best use of the money, then R dollars received today would be worth R(1+r) next period. Adding time subscripts we denote R as R_t and $R_t(1+r)$ as R_{t+1}. From this we express the familiar relationships:

(1a) $R_t(1+r) = R_{t+1}$ and

(1b) $R_t = R_{t+1}/(1+r)$

Similar substitutions for earnings received in more distant time periods, say in the n^{th} period, allows us to express the present value equivalents between R received n periods from now and its present value equivalent R_{t+n} as:

(2) $R_t = R_{t+n}/(1+r)^n$

This, of course, suggests that throughout the period, the highest and best use of returns are investments earning r percent.

The need to adjust costs and returns to a common time period is the one feature of PV models in which there is unanimous agreement. The need to reconcile other aspects of an asset's returns and costs have not obtained such a consensus. For example, should capital gains which are not as liquid as cash income be adjusted? Should cash income be adjusted to either nominal or real basis? Should uncertain income be adjusted to its certainty equivalent or its expected value? Or should income be adjusted to its after-tax equivalent? All of these reconciliation questions must be answered to construct PV models. Consider first the adjustment for taxes.

Suppose a decision maker receives net earnings R, but the returns need to be adjusted for taxes. That is, R_t and R_{t+1} are worth respectively $R_t(1-T)$ and $R_{t+1}(1-T)$ after paying taxes at the rate of T percent. What is the present value of the after-tax worth of return R? The answer again depends on investment alternatives. Suppose the best alternative is an investment earning r percent. Are the returns from best alternative investment also taxed? If not, the discount rate should be r. If so, then the after-tax return is $r(1-T)$. As a result, after-tax income or $R_t(1-T)$ times one plus the after-tax rate of return equals what R received today would equal tomorrow:

$$\text{(3a)} \quad R_t(1-T)(1+r(1-T)) = R_{t+1}(1-T)$$

or the present value of after-tax dollars received next period is:

$$\text{(3b)} \quad R_t(1-T) = R_{t+1}(1-T)/(1+r(1-T))$$

Similar substitutions would allow us to express the present value of after-tax returns received in future periods as:

$$\text{(4)} \quad R_t(1-T) = R_{t+n}(1-T)/(1+r(1-T))^n$$

Now suppose returns next period were risky and that the expected return is R. Then assume that the probability of obtaining alternative possible values of R is described by the probability density function f(R). How much would the distribution of possible returns f(R), be worth today if that amount, a certainty equivalent, were received with certainty? This issue is complicated and we oversimplify it for discussion here.[1]

Assume that in any time period, a stable relationship exists such that for each decision maker we can identify a von Neumann-Morgenstern utility function U(R). Having identified a utility function U(R) such that a

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[1]The resolution of this problem requires, of course, the decision maker's utility function U(R) be known. For a more complete discussion of intertemporal problems under uncertainty see Markowitz.

certainty equivalent income level R_{CE} is equal in preference to the expected utility of the distribution of uncertain outcomes we can form the expression:

$$(5) \quad U(R_{CE}) = \sum_{R} U(R)f(R)$$

Then in our PV formulas we can use R_{CE} (a certainty equivalent return) as the return to be received in the future and proceed with the analysis as before. The rate of return, r, will then be equivalent to the rate of return received on risk-free investments since the returns have been computed on a certainty equivalent basis.

Another feature of an asset's return requiring the application of the homogeneity of measurement principle is accounting for inflation. To illustrate this point, assume the asset's returns, R, over time are constant in real terms but are received in an inflating economy. What should the discount rate be to determine the present value of the return? If the next best investment opportunity is also affected by inflation at the rate of i percent, then the appropriate discount factor is represented as the product $(1+r)(1+i)$ and the discount rate is $r+i+ir$. Thus an income of R_t in period t is equivalent to a future after-tax return in the $(t+n)^{th}$ period of $R_t(1+(r+i+ir)(1-T))^n$ (see equation 5a). After applying the tax rate T, the discount rate becomes $(r+i+ir)(1-T)$. To convert inflating after-tax returns to their purchasing power in the current time period t, we divide R_{t+n} by the inflated after-tax discount factor raised to the n^{th} power to convert returns to a common measure. (See equation 5b below).

$$(5a) \quad R_t(1+i)^n(1-T) = R_{t+n}(1-T) \text{ and}$$

$$(5b) \quad R_t(1-T) = R_{t+n}(1-T)/(1+(r+i+ir)(1-T))^n$$

Our last application of the homogeneity of measurements principle applies to capital gains. Is there a need to adjust capital gains to their cash equivalent value? The answer to this question depends in part on the assumptions underlying the model. If the market is assumed to be perfect--that is, the price of the asset does not depend on whether it is being purchased or sold--then no adjustment is required. In this frictionless market, transactions costs are zero. On the other hand, if the purchase price is different from the sale price, capital gains should be adjusted to reflect this difference--a difference sometimes referred to as a transactions cost.

In this paper we do not assume a perfect capital market. As a result, we are required to build into our model those transactions costs which distinguish between the sale and the acquisition price of land. In so doing we explicitly adjust capital gains to their cash equivalent.

Finally, all present value models involve comparisons between the asset under review and an alternative. The alternative asset's return dictates the choice of the discount rate. Inasmuch as an asset's return may vary with the size of the investment and the term or length of time the funds are committed, both features should be chosen to be consistent with the primary asset under review. Thus the homogeneity of measurements principle requires that the discount rate be reflective of an investment equal in amount and terms to the committment made to the primary investment.

Consistency in Timing Principle

The next principle asserts that returns and costs should enter the PV model in the period they are received or paid for by the firm. The obviousness of the principle aside, some questions naturally arise. For example, should capital gains enter the model in the period in which they are earned or in the period in which they are converted to cash? The answer is they should enter in the period in which they are converted to cash because cash is the common unit of measure and inflating income assumes capital gains are investments retained by the firm.

The importance of the timing principle is that the date a cost is incurred or a return is received alters the value of an asset. For this reason, one of the important assumptions underlying PV models is the length of time a prospective buyer intends to keep an asset, since there are specific costs associated with an asset's transfer. For example, consider an asset purchased at price V which inflates annually for n periods. At the end of the n^{th} period the asset is sold for price $V(1+i)^n$. From the proceeds of the sale, the seller may likely pay a realtor's commission of s percent if the asset is land. In addition, on the net difference between the acquisition price which accounts for initial closing costs of c percent and sale price adjusted for realtor fees, the seller can expect to pay capital gains tax of .4T, forty percent of his marginal income tax rate.

If V_n is the asset's sale price in the n^{th} period, where n is now a critical variable, capital gains CG adjusted for time, taxes, certainty, liquidity and inflation in the n^{th} period equal:

$$(6) \quad CG = \frac{(V_n(1-s)-V(1+c))(1-.4T)}{(1+(r+i+ir)(1-T))^n}$$

The question may arise: What tax rate should be used to adjust capital gains to their present value, after-tax equivalent: The income tax rate T or the capital gains tax rate .4T? In (6) we used the income tax rate T. Are there times when the capital gains tax rate of .4T should be used? The answer depends in part on what opportunity is being reflected by the discount rate. The discussion of the opportunity cost reflected by the discount rate continues in the section under the heading: "The Opportunity Cost Principle."

The Life of the Asset Principle

The third principle for evaluating PV models answers the question: "What is the proper length of time period for measuring an asset's returns and costs?" A related question might be: Does the length of the time period really matter? An example will help illustrate the basis for this second question. Consider an asset with a potentially infinite useful life such as land which earns a constant return of R in each time period. If an alternative investment earning r percent is available, then the present value of this infinite stream of earnings is:

$$(7) \quad V = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \ldots = \frac{R}{r}$$

Now suppose the decision maker intended to hold the asset for only a finite n periods, after which the asset would be exchanged for its cash equivalent. The acquisition value, V, can now be related to r, R, n, and the sale price V_n as:

$$(8) \quad V = \frac{R}{(1+r)} + \ldots + \frac{R}{(1+r)^n} + \frac{V_n}{(1+r)^n}$$

But how do we arrive at a value for V_n? The asset's future value, V_n, should be equal to the present value of all income received after period n, discounted back to period n. So we can write:

$$(9) \quad V_n = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \ldots$$

But if we made the substitution for V_n in (8) using equation (9) we would obtain the first model described in (7).

Thus, the appropriate number of periods to use will always depend on the asset's total expected useful life. Sometimes an asset's future value, V_n, is assumed to equal V. This results in the expression:

$$(10) \quad V = \frac{R}{(1+r)} + \ldots + \frac{R}{(1+r)^n} + \frac{V}{(1+r)^n} = \frac{R}{r}$$

as we obtained earlier.

In this special case, V is equal to V_n because R was a nondepreciating income series. Making the substitution of V for V_n would generally not be an acceptable adjustment to make, especially when transactions costs are paid.

The "Life of the Asset" principle is important because it eliminates the necessity of reassessing the value of the asset in future time periods. It correctly links the present value of the asset to the asset's expected returns and costs. In some instances, however, we are required to make an assessment of an asset's value at regular future intervals. For example, annual property tax payments depend on the asset's value in the current period. But every time an asset is traded and sales commission and capital gain taxes are paid this taxable basis should be adjusted.

We believe that when a valuation is required, such as for the calculation of property taxes, it should be made in such a way as to assure consistency with the stream of future returns received after the revaluation date. In the absence of periodic costs, our earlier example illustrated that with a series of constant returns over time, the future value of the asset, V_n, was equal to V. Now consider the case where incremental returns, R, are inflating at inflation rate i. The infinite series which equates the asset's future returns back to period t can be written as:

$$(11) \quad V_t = \frac{(1+i)R_t}{(1+r)(1+i)} + \frac{(1+i)^2 R_t}{(1+r)(1+i)^2} + \ldots = \frac{R_t}{r}$$

Since the (1+i)'s in the numerator and discount factor cancel out, the result is the familiar constant value model.

Now suppose we wish to determine the value of the asset in the j^{th} period, where period j occurs after period t. Our model would then be:

$$(12)\quad V_{t+j} = \frac{(1+i)^{j+1}R_t}{(1+r)(1+i)} + \ldots = \frac{R_t(1+i)^j}{r}$$

Then substituting V_t for R_t/r we can write:

$$(13)\quad V_{t+j} = V_t(1+i)^j$$

A similar result would be obtained for an asset with a series of returns which depreciates over time. For example, suppose the asset under consideration was a depreciating durable whose returns declined at the rate of d percent per period. Then in the t^{th} period, the asset's value is:

$$(14)\quad V_t = \frac{R_t(1-d)}{(1+r)} + \frac{R_t(1-d)^2}{(1+r)^2} + \ldots$$

and

$$(15)\quad V_t = \frac{R_t(1-d)}{(r+d)}$$

Now consider the same asset's value j periods later. We express the relationship between the asset and its returns as:

$$(16)\quad V_{t+j} = \frac{R_t(1-d)^{j+1}}{(1+r)} + \frac{R_t(1-d)^{j+2}}{(1+r)^2} + \ldots$$

This we can write as:

$$(17)\quad V_{t+j} = \frac{R_t(1-d)^{j+1}}{r-d}$$

and after substituting V_t for $R_t(1-d)/(r-d)$, we can write:

$$(18)\quad V_{t+j} = V_t(1-d)^j$$

It is not clear to us how the pattern would be altered by periodic payments such as capital gains and sales commissions. Thus, this consistency between the revaluation of the asset and the return streams only applies to one ownership period. Each transfer of the asset would create a new base value for revaluating the asset over time.

The Opportunity Cost Principle

The PV models discussed in this paper are designed to answer a very specific question: What is the value of the asset under review? The question is answered by comparing the asset under review to another asset, presumably the next best investment opportunity which we call the secondary asset. The existence of a next best investment creates a cost for the firm--an opportunity cost. It is the return foregone by investing in the primary rather than the secondary asset.

The next best opportunity in maximum bid and minimum sell price models is a secondary asset equal in present value terms to the primary asset. Moreover, since whatever discount rate is chosen equates the returns from the primary and secondary asset the discount rate is also an "internal rate of return," because it is the rate which equates the present value of returns to be present value of opportunity costs.

An example may illustrate the relationship between returns from the primary and secondary asset. Consider equation (12); it represents the relationship between a constant stream of income R, discounted at rate r, and a present cash value equal to V. If the primary asset was the asset producing the constant income stream R, then the opportunity cost is described by V and the rate of return foregone on the secondary investment equals r. On the other hand suppose the primary investment was the ownership of the investment base equal to V. Then the opportunity cost, that which would be given up to acquire V was the present value of the income stream. Thus within our PV models, opportunity costs and benefits from the primary assets may be arbitrarily switched depending on whether we are solving for a maximum bid price or a minimum sell price.

The next best investment may be one of several alternatives. It may be the repayment of the loan used to finance the primary asset, or it may be another investment if the firm's equity is invested. In any case, the rate of return on the next best investment is reflected in the discount rate used in the PV model.

Sometimes questions arise as to what the discount rate should be if the next best investment opportunity includes returns both in the form of capital gains as well as cash income. In this instance taxes may be paid at two different rates depending on whether returns are cash (taxed at rate T) or capital gains (taxed at rate .4T). The customary approach is to avoid answering the question and treat the next best investment opportunity as one with only a cash return. This approach avoids the issue of how to blend tax rates. One need not, however, adopt such a restrictive approach. One could easily form a weighted tax equal to $(Z\,T+(1-Z).4T)$ for $0 \leq Z \leq 1$ to reflect a next best investment earning both cash and capital gains.

One may also reflect differences in the primary asset (described by units being discounted) and the secondary asset (whose rate of return is described by the discount rate) by differentiating inflation rates. For example, suppose the rate of inflation embedded in the secondary asset is i percent while returns from the primary asset were inflating a g percent. Then discounted after-tax returns received in the n^{th} period equals:

$$(19) \quad R_t = R_t(1+g)^n(1-T)/(1+(r+i+ir)(1-T))^n$$

<u>The Total Costs and Returns Principle</u>

Whenever financial concessions such as low interest loans or preferential tax treatments are tied to the ownership of an asset, the economic concessions will influence the value of the asset just as services from the asset influence its value. In the case of real estate, while tax assessors and other interested parties may have an interest in attempting to determine an asset's value independent of its financing arrangements, there are others whose interest in the maximum bid (minimum sell) price is dependent on the details of the financing arrangements. The question is whether or not all returns and costs associated with an asset's control should be included in a PV model? Principle five argues that all cash costs and benefits linked to the ownership or control of an asset should be included when determining its present value.

Historically, the difficulty with including financial considerations in the analytic expressions of present value models was the computational complexity they introduced. However, once the analytic relationships have been determined, computers can significantly reduce the cost of calculating the effects of financial arrangements on asset prices. Lee and Rask included financial considerations in their model by assuming that the loan payments were constant while the principle portion of the loan repayment increased geometrically. Later on in this paper, we calculate simple analytic formulas for measuring the after-tax effects of financial arrangements of an asset's bid price.

Other considerations as well as financial instruments tied to the ownership of the asset affect PV models. Sometimes an asset has more than one source of return. Mineral deposits and timber or recreational use potential may reflect returns in addition to those associated with agricultural use. Urbanization pressures may create capital gains over and above those associated with influencing pressures in the economy. Pollution standards may impose costs in addition to those normally experienced. Whatever the source of the costs on returns we believe they should be included in the PV model.

Including total cash returns and costs over the life of the asset in the period experienced discounted by the rate of return available on the next best opportunity and measured in homogeneous units summarizes our PV model construction principles. We now discuss how these principles influence the construction of a practical investment model for evaluating land purchases.

A Practical Model for Finding the Maximum Bid and Minimum Sell Prices for Land

Under the certainty assumption, PV models can be easily solved to determine the present value of benefits generated by an asset. In this basic valuation model we equate an asset's price to the present value of its net returns. From this equality between the present value of net returns and the asset's price, a break-even price can be found. This break-even price is often referred to as the maximum bid or minimum sell price depending on whose perspective the analysis is being conducted. Procedures for determining the buyer's maximum bid price for an asset are similar to those used for finding minimum sell price. In a perfect capital market, the maximum bid price equals the minimum sell price, and the two PV models are identical except that the returns to the seller are a cost to the buyer and vice versa. When the perfect market assumption is relaxed, the maximum bid price is not necessarily equal to the minimum sell price because some costs incurred by one of the market participants, either the buyer or the seller, are not experienced by the other. The modeling features required for determining land values include property taxes, capital gains and capital gains taxes, sales commission (transactions cost), and, initially, a constant stream of real income each time period. We also assume an inflation rate of g percent which will affect returns and an inflation rate of "i" percent reflected in the discount rate.

The model notation, some of which has already been introduced, consists of the following:

r = a real rate of return available to the firm on its primary and secondary assets,

T = a constant proportional income tax rate paid by the firm,

$.4T$ = the capital gains tax rate,

T_p = the property tax rate paid on the nominal property value,

i = the general rate of inflation implicit in the discount rate,

g = the inflation rate applied to the returns from the asset and the asset's value within an ownership period,

R = the constant real cash return from the land using the first period as the base year,

n = the length of time the asset will be owned by the current owner and the m subsequent owners,

s = the percentage of the asset's sale price paid as realtor's fee,

c = the percent of the acquisition price paid as a loan closing fee which includes fees for title searches and points charged to close a loan,

q = the length of term on the loan used to finance the control of the land,

q_s = the length of term on the loan held by the seller at the time of the sale,

D = the percentage of the asset's price the buyer paid as a downpayment,

r^* = the nominal interest rate paid on funds borrowed to acquire the asset,

r_s^* = the nominal interest rate on a loan held by the seller at the time of the land sale,

L = the seller's outstanding loan balance at the time of the sale,

V = the maximum bid price for land without financing consideration,

V^* = the maximum bid price for land with financing considerations,

V_0 = price paid for the land by the seller when he acquires it,

V_s = the minimum sell price without financial consideration,

V_s^c = the minimum sell price with a "due on sale" clause,

V_s^* = the minimum sell price with seller financing,

V_s^{1*} = the minimum sell price with the seller providing financing at interest rate r^* for q periods on a land contract while retaining an earlier loan with an interest rate r_s^* for q_s periods,

V_s^{c*} = the minimum sell price with the seller providing financing at interest rate r^* for q periods while giving up a loan at interest rate r_s^* for q_s periods because of a "due on sale" clause.

The Maximum Bid Price Model (V)

To facilitate the calculation of the maximum bid price V in the current period, we divide our calculations into two components: the opportunity cost component and the total returns component.

The opportunity cost component is the present value of the investment required to control the asset. This opportunity cost is compared to the primary asset. The return on the primary asset includes returns in the form of cash and capital gains less property taxes, sales commission and capital gains tax.

The opportunity cost principle guides our choice of the discount rate depending on which alternative is being considered. The usual approach is to select the opportunity rate of return associated with borrowed funds. If the cost of debt capital is utilized, then the tax rate which adjusts the discount rate is the marginal income tax rate. The life of the asset principle dictates that the land model be of infinite length to correspond with the planning horizon of the asset's returns. However, each period's assessment of property taxes and the periodic assessment of transactions costs and capital gains taxes requires a time dated pattern of land values to be consistent with the timing principle. In this case, the value of land is constant in real terms.

The opportunity cost component of the maximum bid price model, B_1, calculates the cost of controlling of the asset. If the acquisition price is V and closing fees of c percent are paid by the buyer, S_1 can be written as:

$$(20) \quad B_1 = (1+c)V$$

Since the firm's equity is used to acquire the asset, we assume taxes have already been paid and V is measured in after-tax cash units.

The present value of property taxes paid over n periods by the first owner which are a cost associated with the primary asset can be written as B_2 equal to:

$$(21) \quad B_2 = \frac{VT_p(1-T)}{(1+(r+i+ir))(1-T)} + \ldots + \frac{V(1+g)^{n-1}T_p(1-T)}{(1+(r+i+ir)(1-T))^n}$$

$$= \frac{VT_p(1-T)}{((r+i+ir)(1-T)-g)} \left(1 - \frac{(1+g)^n}{(1+(r+i+ir)(1-T))^n}\right)$$

It should be noted that property taxes are tax deductible and, therefore, are reduced by T percent. The value of the land to which the property tax rate is applied is assumed to inflate at an annual rate of g percent.

While we assume V is inflating at g percent to maintain consistency, we do so only within the current ownership period. The sale price V_n may not be equal to $V(1+g)^n$ and, therefore, requires a different base for tax purposes.

The first returns component is cash returns. If returns are inflating at g percent, then the cash return series B_3 for the n periods controlled by the first owner can be written as:

$$(22) \quad B_3 = \frac{R(1-T)}{(1+(r+i+ir)(1-T))} + \ldots + \frac{R(1+g)^{n-1}(1-T)}{(1+(r+i+ir)(1-T))^n}$$

$$= \frac{R(1-T)}{((r+i+ir)(1-T)-g)} \left(1 - \frac{(1+g)^n}{(1+(r+i+ir)(1-T))^n}\right)$$

The second component of returns to the primary asset is the present value of after-tax cash equivalent of capital gains less transactions costs plus the present value of the recovered purchase price. The expression for adjusted capital gains was given in (6). To that we add the present value of V(1+c) and write it as B_4 equal to:

$$(23) \quad B_4 = \frac{(V_n(1-s)-V(1+c))(1-.4T)}{(1+(r+i+ir)(1-T))^n} + \frac{V(1+c)}{(1+(r+i+ir)(1-T))^n}$$

where V_n, the price paid by the second buyer n, depends on the second buyer's returns, costs, and capital gains net of transactions cost.

We summarize the maximum bid price model by equating returns $B_3 + B_4$ less B_2 to opportunity costs B_1:

$$(24) \quad B_1 = B_3 - B_2 + B_4$$

Next substituting for B_1, B_2, B_3 and B_4 the right-hand sides of equations (20), (21), (22) and (23) and solving for V we obtain:

$$(25) \quad V = \frac{R(1-T)(..)}{(.)k_1} + \frac{V_n(1-s)(1-.4T)}{(\ldots)k_1}$$

where

$$(.) = ((r+i+ir)(1-T)-g)$$

$$(..) = (1 - \frac{(1+g)^n}{(1+(r+i+ir)(1-T))^n})$$

$$(...) = (1+(r+i+ir)(1-T))^n \text{ and}$$

$$k_1 = (1+c) + \frac{T_p(1-T)(..)}{(.)} - \frac{.4T(1+c)}{(...)}$$

The problem we have solving (25) is that we don't know V_n. We cannot merely assume it is $V(1+g)^n$ because of transactions cost. As a result, we write V_n as the worth of the asset as the second buyer's maximum bid price:

$$(26) \qquad V_n = \frac{R(1-T)(1+g)^n(..)}{(.)k_1} + \frac{V_{2n}(1-s)(1-.4T)}{(...)k_1}$$

Similarly, we could write V_{2n}, the third buyer's maximum bid price in period 2n, as:

$$(27) \qquad V_{2n} = \frac{R(1-T)(1+g)^{2n}(..)}{(.)k_1} + \frac{V_{3n}(1-s)(1-.4T)}{(...)k_1}$$

And we could continue expressing such maximum bid prices up to the m^{th} buyer in the mn^{th} period, which would depend on a terminal value for land equal to $V_{n(m+1)}$.

We can avoid assuming a terminal value for land for a particular buyer by making successive substitutions and taking the limit of m. To do so we first substitute equation (26) for V_n, then (27) for V_{2n}, etc. Then finding geometric sums from our equation and taking the limit of m, we can solve explicitly for V in terms of known parameters.

The geometric series we obtain after successive substitutions and factorings is:

$$(28) \qquad V = \frac{R(1-T)(..)B_5}{(.)k_1}$$

where

$$B_5 = 1 + \frac{(1+g)^n(1-s)(1-.4T)}{(\ldots)k_1} + \frac{(1+g)^{2n}(1-s)^2(1-.4T)^2}{(\ldots)^2 k_1{}^2} + \frac{(1+g)^{mn}(1-s)^m(1-.4T)^m}{k_1{}^m(\ldots)^m}$$

Let the geometric factor in the braced expression be represented by k_2 equal to:

$$(29a) \quad k_2 = \frac{(1+g)^n(1-s)(1-.4T)}{k_1(\ldots)}, \text{ and}$$

$$(29b) \quad k_2{}^m = \frac{(1+g)^{mn}(1-s)^m(1-.4T)^4}{k_1{}^m(\ldots)^m}$$

We know k_2 is less than one because $(\ldots)$ is greater than $(1+g)^n$. Thus the series B_5 converges to:

$$(30) \quad \lim_{m\to\infty} B_5 = 1/(1-k_2)$$

Now we are prepared to write the expression for V in (28) as:

$$(31) \quad V = \frac{R(1-T)(..)}{(.)k_1(1-k_2)}$$

A simplified approach for analyzing this model is to assume the asset is traded only once; that is, assume the length of time between transactions n is infinitely long. Under such an assumption V can be expressed as:[2]

$$(32) \quad \lim_{n\to\infty} V = \frac{R(1-T)}{((r+i+ir)(1-T)-g)(1+c)+T_p(1-T)}$$

In the above formulation, s does not enter since a sales commission is never paid by the first buyer. Otherwise, the derivatives have the predicted signs:

---

[2]It is helpful when establishing the limit of V to recognize that:

$$\lim_{n\to\infty} k_1 = (1+c) + \frac{T_p(1-T)}{(.)}, \text{ and}$$

$$\lim_{n\to\infty} k_2 = 0$$

$dV/dR > 0,$

$dV/dT > 0,$

$dV/dg > 0,$

$dV/dc < 0,$

$dV/T_p < 0,$

$dV/di < 0$, and

$dV/dr < 0.$

One interesting question we might ask, having solved for V, is: What is V_n? The answer should be easy to obtain since the solution procedure for finding V_n is the same procedure used to find V; except we start with equation (26), then we use equation (27) to eliminate V_{2n}, and so on.

$$V_n = \frac{R(1-T)(1+g)^n(..)}{(.)k_1} + \frac{V_{2n}(1-s)(1-.4T)}{(...)k_1}$$

$$= \frac{R(1-T)(1+g)^n(..)}{(.)k_1} + \frac{R(1-T)(1+g)^{2n}(..)(1-s)(1-.4T)}{(.)k_1^2(...)}$$

$$+ \frac{V_{3n}(1-s)^2(1-.4T)^2}{(...)^2k_1^2}$$

After m+1 substitutions, the result is:

$$(33) \quad V_n = \frac{R(1-T)(1+g)^n(..)}{(.)k_1}\left(1 + \frac{(1+g)^n(1-s)(1-.4T)}{k_1(...)}\right.$$

$$\left. + \frac{(1+g)^{2n}(1-s)^2(1-.4T)^2}{(...)^2k_1} + \ldots\right)$$

Upon examining (33) above and comparing it to equation (28) which equals V, the only difference is the term $(1+g)^n$ which multiplies the constant factor. Thus we can express V_n as a function of V equal to:

$$(34) \quad V_n = (1+g)^n V$$

and so on. This deductive fact will prove useful later on as we solve more complicated models.[3]

The Minimum Sell Price Model (V_s)

Present value models are all constructed following a similar structure. The structure allows for a comparison between two investments. In the maximum bid price model the comparison was between investing after-tax equity at a rate of (r+i+ir)(1-T) versus the opportunity of investing in land. The present value of the two investments were then set equal to each other.

The minimum sell price model makes an alternative comparison. It compares the present value of after-tax proceeds from the sale of land with the alternative of keeping the land for an additional n periods. The proceeds from the land sold are assumed to be invested at the after-tax rate of (r+i+ir)(1-T). Equating the present value of the two investments, the minimum sell price model solves for the selling price which equates the present value of the two alternatives. We now proceed to construct the minimum sell price model.

The after-tax proceeds from the sale of land we write as: S_1 equal to:

$$(35) \quad S_1 = (V_s(1-s)-V_0(1+c))(1-.4T) + V_0(1+c)$$

It equals the after-tax capital gains adjusted for sales commissions and original closing fees, plus the original purchase price and adjustments for closing fees $V_0(1+c)$.

The alternative investment, to hold the asset another n periods, is much the same as it was in the maximum bid price model. It includes B_3, the

---

[3]The limit of V_n, comparable to (32), is found be replacing $R(1+g)^n$ with R_n--the cash income in the first period and letting n approach its limit elsewhere in the model.

inflating cash returns less the present value of property taxes equal to B_2.[4] It also includes after-tax capital gains adjusted for closing costs and realtor's fees plus the original investment of V_0 adjusted for closing fees which we write as S_4 equal to:

$$(36)\quad S_4 = \frac{(V_n(1-s)-V_0(1+c))(1-.4T) + V_0(1+c)}{(1+(r+i+ir)(1-T))^n}$$

Equating the two investments we write:

$$(37)\quad S_1 = B_3 - B_2 + S_4$$

Then we substitute for S_1, B_2, B_3, S_4 equations (35), (21), (22), and (36) and solve for V_s. The solution is:

$$(38)\quad V_s = \frac{R(1-T)(..)}{(.)(1-s)(1-.4T)} - \frac{VT_p(1-T)(..)}{(.)(1-s)(1-.4T)} - \frac{V_0.4T(1+c)(1-(...)^{-1})}{(1-s)(1-.4T)} + \frac{V_n}{(...)}$$

After substituting for V and V_n we obtain the result:

$$(39)\quad V_s = \frac{R(1-T)(..)S_5}{(.)k_1(1-k_2)(1-s)(1-.4T)} - \frac{V_0.4T(1+c)(1-(...)^{-1})}{(1-s)(1-.4T)}$$

where

$$(40)\quad S_5 = k_1(1-k_2) - \frac{T_p(1-T)(..)}{(.)} + \frac{(1+g)^n(1-s)(1-.4T)}{(...)}$$

A simplified approach for analyzing the minimum sell model is to assume the alternative to selling land now is to hold it an infinitely long period. Letting n become large, as the assumption implies, V_s becomes:[5]

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[4] In the property tax equation, we use V, the maximum bid price not V_s the minimum sell. This reflects our belief that the tax basis must be the most a buyer would pay.

[5] In determining the model in the limit it is helpful to remember that:

$$\lim_{n\to\infty} S_5 = (1+c)$$

$$(41)\quad \text{limit } V_s = \frac{R(1-T)(1+c)}{((.)(1+c)+T_p(1-T))(1-s)(1-.4T)} - \frac{V_0 .4T(1+c)}{(1-s)(1-.4T)}$$

The derivatives of V_s which can be unambiguously signed are:

$dV_s/dR > 0$,

$dV_s/dV_0 < 0$,

$dV_s/dT_p < 0$,

$dV_s/dr < 0$,

$dV_s/di < 0$,

$dV_s/dg > 0$, and

$dV_s/dc < 0$.

Building Financial Considerations Into Our Maximum Bid and Minimum Sell Models

The total returns and cost principle requires that all benefits and costs associated with an asset be included in the PV models. One important cost or benefit associated with the transfer of asset control is the financial arrangement.

Suppose, as a term of the trade, a buyer is able to finance his asset acquisition at a concessionary interest rate r* which is less than the current market rate (r+i+ir). Such a favorable rate may result from the transfer of a previous loan contracted by the seller in periods of lower rates or may be a land contract offered by the seller as an inducement to complete the trade.

Consider the effects of financial arrangements on the maximum bid price model. First assume that at an interest rate r* the buyer acquires a loan of (1-D)V*. Let DV* be the amount of the loan paid as a downpayment where D is the percentage of the loan paid as a downpayment. In our earlier model V was the maximum bid price. With the financial arrangement, V* is the maximum bid price because the cost of acquiring control of the land has changed. Without tax considerations, it is an easy manner to compare the cost of controlling the asset at a concessionary rate r^*. To do so, first solve for the payment P required to retire the debt at interest rate r^* in q periods. It is:

$$(42) \quad (1-D)V^* = \frac{P}{r^*}\left(1 - \frac{1}{(1+r^*)^q}\right)$$

and solving for the payment P we obtain:

$$(43) \quad P = (1-D)V^*r^*/\left(1 - \frac{1}{(1+r^*)^q}\right)$$

Next we discount these constant payments of P for q periods at the current rate of (r+i+ir). If the discounted sum of the payments plus the downpayment of DV^* is equal to an amount V less the downpayment on the contract price V^*, we can form the equality:

$$(44) \quad V - DV^* = \frac{(1-D)V^*r^*}{\left(1 - \frac{1}{(1+r^*)^q}\right)(1+r+ir)} + \ldots + \frac{(1-D)V^*r^*}{\left(1 - \frac{1}{(1+r^*)^q}\right)(1+r+ir)^q}, \text{ and}$$

After summing geometrically we obtain:

$$(45) \quad V - DV^* = \frac{(1-D)V^*r^*}{(r+i+ir)} \frac{\left(1 - \frac{1}{(1+(r+i+ir))^q}\right)}{\left(1 - \frac{1}{(1+r^*)^q}\right)}$$

For r^* equal to r+i+ir, V^* equals V. For r^* less than r+i+ir V is less than V^*. Thus concessionary interest rates allow the buyer to increase his maximum bid.[6]

Since our model is an after-tax model, we now introduce the tax effect into our results. The tax effects enter in two parts. First the discount

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[6]That $V^* > V$ for $r^* < (r+i+ir)$ can be shown with some effort using equation (41). For convenience, let D=0 and at first let q=1. Under these assumptions the rate of V to V^* is:

$\frac{V}{V^*} = \frac{1+r^*}{(1+r+i+ir)} < 1$. Next let n approach ∞. Then the ratio of V to V^* can be written as: $\frac{V}{V^*} = \frac{r^*}{(r+i+ir)} < 1$. In both cases the implication is that V^* is greater than V. Since the ratio $\frac{V}{V^*}$ decreases monotonically, V^* is greater than V for all q.

rate which adjusts the payments on the concessionary loan must be an after-tax rate of $(r+i+ir)(1-T)$. Second, the interest, but not the principal creates a tax shield which must be adjusted for taxes. We can adjust the discount rate for taxes easily. It is somewhat of a larger task to include the effects of the tax shield.

Consider first of all interest costs. In the first period $r^*(1-D)V^*$ is paid as interest which solving from equation (43) equals:

$$(46) \quad P - \frac{P}{(1+r^*)^q} = r^*(1-D)V^*$$

Since the payment P exceeds the interest charge by $\frac{P}{(1+r^*)^q}$, this difference must equal the principal payment. Now consider the second period. The outstanding principal is:

$$(47) \quad (1-D)V^* - \frac{P}{(1+r^*)^q} = \frac{P}{(1+r^*)} + \ldots + \frac{P}{(1+r^*)^{q-1}} = \frac{P}{r^*}\left(1 - \frac{P}{(1+r^*)^{q-1}}\right)$$

The familiar form of the equation above allows us to calculate interest costs and principal payments as before. Interest costs in the second period are:

$$(48) \quad r^*\left((1-D)V^* - \frac{P}{(1+r^*)^q}\right) = \left(P - \frac{P}{(1-r^*)^{q-1}}\right)$$

Now the principal payment is $P/(1+r^*)^{q-1}$. Similarly we could calculate interest and principal payments for the remainder of the periods. So, we adjust the interest payments by multiplying them by $(1-T)$ and find the geometric sum of both interest and principal payments, discounted at the current after-tax rate to obtain:

$$(49) \quad V-DV^* = \frac{P}{(1+r^*)^q(1+(r+i+ir)(1-T))} + \ldots + \frac{P}{(1+r^*)(1+(r+i+ir)(1-T))^q}$$

$$+ \frac{P(1-(1+r^*)^q)(1-T)}{(1+(r+i+ir)(1-T))} + \ldots + \frac{P(1-(1+r^*)^{-1})(1-T)}{(1+(r+i+ir)(1-T)^q}$$

Simplifying we obtain:

$$(50)\qquad V-DV^* = \frac{P(1-T)}{(1+(r+i+ir)(1-T))} +\ldots+ \frac{P(1-T)}{(1+(r+i+ir)(1-T))^q}$$

$$+ \frac{TP}{(1+r^*)^q(1+(r+i+ir)(1-T))} +\ldots+ \frac{TP}{(1+r^*)(1+(r+i+ir)(1-T))^q}$$

The present value sum of V-DV* conveniently divides itself into the calculation of two separate sums:

$$(51)\qquad F_1 = \frac{P(1-T)}{(1+(r+i+ir)(1-T))} +\ldots+ \frac{P(1-T)}{(1+r+i+ir)(1-T))^q}$$

$$= \frac{P}{r+i+ir}\left(1 - \frac{1}{(1+(r+i+ir)(1-T))^q}\right)$$

and

$$(52)\qquad F_2 = \frac{TP}{(1+r^*)^q(1+(r+i+ir)(1-T))} +\ldots+ \frac{TP}{(1+r^*)(1+(r+i+ir)(1-T))^q}$$

$$= \frac{TP}{((r+i+ir)(1-T)-r^*)}\left(\frac{1}{(1+r^*)^q} - \frac{1}{(1+(r+i+ir)(1-T))^q}\right)$$

Finally we write:

$$(53)\qquad V = DV^* + F_1 + F_2$$

which after substituting for P can be written as:

$$(54)\qquad V = V^*f^7,$$

where

$$(55)\qquad f = D + \frac{(1-D)r^*}{(r+i+ir)}\ \frac{(1-(\ldots)q^{-1})}{(1-(\ldots)q^{*-1})} + \frac{T(1-D)r^*(\ldots.)}{(.)^*}$$

---

[7] In the special case where r^* equals $(r+i+ir)(1-T)$, f can be expressed as:

$$f = 1-T + DT + \frac{qTr^*(1-D)}{(1+r^*)^{q+1}(1+r^*)}$$

where

$$(\ldots)q = (1+(r+i+ir)(1-T))^q,$$

$$(\ldots)q^* = (1+r^*)^q$$

$$(.)^* = ((r+i+ir)(1-T)-r^*), \text{ and}$$

$$(\ldots.) = \frac{((\ldots)q^{*^{-1}}-(\ldots)q^{-1})}{(1-(\ldots)q^{*^{-1}})}$$

A Maximum Bid Price Model With Financial Considerations (V^*)

We now compare an investment of ($V^*f + cV^*$) which could be invested at the after-tax rate of return of $(r+i+ir)(1-T)$ versus the present value of an after-tax stream of earnings from land. The maximum bid price under such a comparison can be obtained from the equality below.[8]

$$(56) \quad B_1^* = B_2 - B_3 + B_4^*$$

where

$$B_1^* = fV^* + cV^*, \text{ and}$$

$$B_4^* = \frac{(V_n(1-s)-V^*(1+c))(1-.4T)}{(\ldots)} + \frac{V^*(1+c)}{(\ldots)}$$

After substituting for B_1^*, B_2, B_3, and B_4^* equation (57) is obtained:

$$(57) \quad V^*f + cV^* = \frac{R(1-T)(..)}{(.)} - \frac{V(1-T)T_p(..)}{(.)}$$

$$+ \frac{(V_n(1-s)-V^*(1+c)(1-.4T)}{(\ldots)} + \frac{V^*(1+c)}{(\ldots)}$$

Replacing V with the right-hand side of equation (31), V_n with the right-hand side of equation (34), and the right-hand side of equation (40) to simplify the result the expression for V^* below is obtained:

$$(58) \quad V^* = \frac{R(1-T)(..)S_5}{k_3k_1(1-k_2)(.)}$$

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[8]The opportunity cost is $V^*f + cV^*$ because the closing cost is calculated as a percentage of the contract price V^* while the present value of the contract price repaid over q periods at interest rate r^* is fV^*.

where

$$k_3 = f+c - \frac{.4T(1+c)}{(\ldots)}$$

In the limit of n, a more simplified model is obtained. It is:

$$(59) \quad \lim_{n\to\infty} V^* = \frac{R(1-T)(1+c)}{(f+c)((1+c)(.)+T_p(1-T))}$$

The derivatives which can be unambiguously signed are:

$\frac{dV^*}{dR} > 0$,

$\frac{dV^*}{dT_p} < 0$,

$\frac{dV^*}{dr^*} < 0$,[9]

$\frac{dV^*}{dc} < 0$,

$\frac{dV^*}{dr} < 0$, and

$\frac{dV^*}{di} < 0$.

Minimum Sell Price Models with Financing Considerations

The effects of financial arrangements on the maximum bid price model have now been calculated. Now consider how the minimum sell price of land may be affected by alternative financial arrangements. Four minimum sell price models with financing will be considered. The first model assumes the seller has a loan with a concessionary interest rate along with a "due on sale" clause in the financial instrument. At the time of the sale the buyer does not receive the concessionary rate loan and the seller does not finance the loan. In the second model we assume the seller is free from debt, but must offer to finance

[9]We can learn more about the properties of V* in the limit if we also let q become large. In that instance $df/dr^* > 0$. The implication of these results are that $dV^*/dr^* < 0$.

the buyer with a land contract to consummate the transaction. In the third model the seller must not only provide financing to the seller but he also loses his concessionary rate loan due to a due on sale clause. The fourth and final model involves a seller financing the asset for the buyer, while the seller continues to hold a financial instrument <u>not</u> subject to a due on sale clause. This model we call the land contract sale with an outstanding financial instrument.

Minimum Sell Model With A "Due on Sale" Clause With No Seller Financing (V_s^c)

Suppose a seller holds a loan with a balance of L to be repaid at interest rate r_s* over the next q* periods. This loan unfortunately, must be paid in full at the time of sale of the asset. When the loan balance is repaid at the time of the sale the seller pays L to the financial intermediary which originally provided the loan. On the other hand, if the seller repays the loan over q* periods, the proceeds from keeping the land another n periods is reduced by the amount Lf* where f* equals f when d=0 and with r* replaced by r_s*. The equality between these two options is expressed as:

(60) $$S_1 - L = B_3 - B_2 + S_4 - Lf^*$$

After substituting for S_1, B_2, B_3, and S_4 and replacing V_s with V_s^c to indicate a minimum sell price with a due on sale clause, the equality below is obtained:

(61) $$V_s^c = \frac{R(1-T)(..)S_5}{(.)k_1(1-k_2)} - \frac{.4T(1+c)V_0(1-(...)^{-1})}{(1-s)(1-.4T)} + \frac{(1-f^*)L}{(1-s)(1-.4T)}$$

$$= V_s + \frac{(1-f^*)L}{(1-s)(1-.4T)}$$

It should be obvious that when r_s^* equals (r+i+ir), f* equals one and V_s^c equals V_s.[10] Moreover, for r_s^* less than (r+i+ir), f* is less than 1 so that V_s^c is greater than V_s. Similarly, V_s^c is less than V_s when r* is greater than (r+i+ir).

In the limit, V_s^c equals the limit of V_s plus the value of L(1-f*)/(1-s) (1-.4T) since n does not occur in f*. Thus the limit model for V_s^c can be written as:

$$\text{(62)} \quad \text{limit } V_s^c = \frac{R(1-T)(1+c)}{((.)(1+c)+T_p(1-T))(1-s)(1-.4T)} - \frac{V_0 .4T(1+c)}{(1-s)(1-.4T)} + \frac{L(1-f^*)}{(1-s)(1-.4T)}$$

In (62) above, the ambiguous derivatives are:

$dV_s^*/dR > 0$,

$dV_s^*/dV_0 < 0$,

$dV_s^*/dT_p < 0$,

$dV_s^*/dr < 0$,

$dV_s^*/di < 0$,

$dV_s^*/dg > 0$,

$dV_s^*/dc < 0$,

$dV_s^*/dL > 0$, for $r_s^* < (r+i+ir)$, and

$dV_s^*/dr^* < 0$.

Debt Free Seller Providing Financing (V_s^*)

Consider now the case where the seller offers an inducement to the buyer, a concessionary interest rate loan of $(1-D)V_s^*$, where V_s^* is the sale price.

---

[10] This can be shown by replacing (r+i+ir) with r* when D equals zero in equation (56). This results in the expression:

$$f = \frac{1-(1+r^*)1-T))^{-q}}{1-(1+r^*)^{-q}} - \frac{((1+r^*)^{-q}-(1+r^*(1-T))^{-q})}{(1(1+r^*)^{-q})} = \frac{1-((1+r^*)^{-q})}{1-((1+r^*)^{-q})}$$

The seller is quite willing to offer the loan for q periods provided V_s* is increased to offset the loss in interest income. Of course, the higher the seller's tax bracket, the more important will be the tax savings which occurs from having the firm's returns taxed at the capital gains tax rate (.4T) rather than at the income tax rate T.

Let the contract sale price after sales commission be $V_s^*(1-s)$ and let the percentage of the sale price V_s* subject to capital gains tax be:

$$(63) \quad w = (1-(V_0(1+c)/V_s^*))$$

Current tax laws provide that if the purchase price is to be repaid over a number of years then v percent of the principal payments should be taxed at the capital gains tax rate along with w percent of the downpayment. Any interest received by the seller is, of course, taxed at the seller's marginal income tax rate. Therefore, to calculate the present value of a seller financed sale, the after-tax benefits of the loan which is scheduled to be financed for q periods at a concessionary interest rate r* must also be calculated.

Define S_1* as the net present value of after-tax payments P received by the seller for the land sold plus the downpayment less the sales commission. It can be expressed as:

$$(64) \quad S_1^* = wD(1-.4T)V_s^* + (1-w)DV_s^* - sV_s^*(1-.4T)$$

$$+ \frac{(wP(1-.4T)+(1-w)P}{(1+r^*)^q(1+(r+i+ir)(1-T))} +\ldots+ \frac{(wP(1-.4T)+(1-w)P}{(1+r^*)(1+(r+i+ir)(1-T))^q}$$

$$+ \frac{P(1-(1+r^*)^{-q}(1-T)}{(1+(r+i+ir)(1-T))} +\ldots+ \frac{P(1-(1+r^*)^{-1})(1-T)}{(1+(r+i+ir)(1-T))^q}$$

Since the downpayment and sales commission were paid initially, the loan payment P must be sufficient to retire the amount $V_s^*(1-D)$ at interest rate r* in q periods. We express this relationship below and solve for P

$$(65a)\quad (1\text{-}D)V_s{}^* = \frac{P}{r^*}(1 - \frac{1}{(1+r^*)^q})$$

and

$$(65b)\quad P = \frac{r^*(1\text{-}D)V_s{}^*}{(1-(1+r^*)^{-q})}$$

We can substitute into $S_1{}^*$ for P to obtain an expression for $V_s{}^*$. Simplifying (64) after summing geometrically produces the result:

$$(66)\quad S_1{}^* = V_s{}^*f^{**} + V_0(1+c).4T(D + \frac{r^*(1\text{-}D)(....)}{(.)^*})$$

where

$$f^{**} = (1-.4T)(D\text{-}s) + \frac{r^*(1\text{-}D)}{(r+i+ir)}\ \frac{(1-(...)q^{-1})}{(1-(...)q^{*-1})} + \frac{.6Tr^*(1\text{-}D)(....)}{(.)^*}$$

For reasons already given (see footnote 6):

$$(67)\quad \frac{df^{**}}{dr^*} > 0$$

The opportunity cost is again associated with the alternative of holding the land another n periods equal to $(B_3 - B_2 + S_4)$. This allows us to write the equality as:

$$(68)\quad S_1{}^* = B_3 - B_2 + S_4$$

And after making the appropriate substitutions the expression above simplifies to:

$$(69)\quad V_s{}^* = \frac{R(1\text{-}T)(..)S_5}{f^{**}(.)k_1(1\text{-}k_2)} + \frac{.4TV_0(1+c)}{(...)f^{**}}$$

$$- (\frac{V_0(1+c).4T}{f^{**}})(D + \frac{r^*(1\text{-}D)(....)}{(.)^*})$$

A simplified version of the above model is obtained when only one holding period of infinite length is considered. Let n become large and the result is:

$$(70) \quad \text{limit } V_s{}^* = \frac{R(1-T)(1+c)}{f^{**}((.)(1+c)+T_p(1-T))} - \frac{V_0(1+c).4T}{f^{**}} \left(D + \frac{r^*(1-D)(....)}{(.)^*}\right)$$

The unambiguous derivatives in the above model are fewer than before. They are expressed as:

$dV_s{}^* / dR > 0$,

$dV_s{}^* / dV_0 < 0$,

$dV_s{}^* / dT_p < 0$,

$dV_s{}^* / dr < 0$,

$dV_s{}^* / di < 0$,

$dV_s{}^* / dg > 0$, and

$dV_s{}^* / dr^* < 0$.

The derivatives are the same as before and do not require further comment.

Seller Financing Models With and Without A Due on Sale Clause ($V_s{}^{c*}$ and $V_s{}^{1*}$)

Suppose the seller must not only provide financing but forfeit his concessionary loan: What would be the effect on the sale price? Let L be the seller's loan balance financed at rate $r_s{}^*$ for q^* periods. The present value of that loan is Lf^*. If the seller sells now, his net returns equal:

$$(71) \quad S_1{}^{**} = S_1{}^* - L$$

The opportunity cost meanwhile is reduced by Lf^*. This allows us to write:

$$(72) \quad S_1{}^* = B_3 - B_2 + S_4 + L(1-f^*)$$

And after appropriate substitutions permits us to write:

$$(73) \quad V_s{}^{c*} = V_s{}^* + \frac{L(1-f^*)}{f^{**}}$$

The fourth model ($V_s{}^{1*}$) involves selling the asset on a land contract with the seller being able to keep his loan. This model would just reverse the sign on the last term, thus allowing us to write the model as:

(74) $\quad V_s^{1*} = V_s^* - \frac{L(1-f^*)}{f^{**}}$

In both cases, the limit is the same as for the V_s^* model with the addition of the second term. The applicable derivatives in addition to those derived for V_s^* are:

$dV_s^{c*}/dL \quad > 0,$

$dV_s^{c*}/dr_s^* < 0,$

and

$dV_s^{1*}/dL \quad < 0,$

$dV_s^{1*}/dr_s^* > 0.$

Examination of the model results reveals interesting relationships. For example, the maximum bid model without financing value of $2,132 is $228 less than the minimum sell model without financing. This difference in value is attributed to the relaxation of the perfect capital market assumption, and reflects the incurrence of costs by one of the market participants while not borne by the other.

Empirical Evaluation of the Maximum Bid and Minimum Sell Models

To demonstrate the numeric properties of the various models, solutions were derived for a baseline set of input assumptions. The base run assumptions were:

	Input Variable	Base Run Value
r :	Real rate of return	4.0%
T :	Constant proportional tax rate	15.0%
T_p :	Property tax rate	2.5%
i :	General rate of inflation in the discount rate	4.5%
g :	Inflation rate applied to asset returns	4.0%
R :	Constant real cash returns	\$150.00
n :	Number of years asset will be held by current owner	20
s :	Percentage of asset's value paid as realtor's fee	5.0%
c :	Percentage of purchase price paid in closing fees	2.5%
q :	Length of loan amortization period (in years)	20
D :	Downpayment percentage of purchase price	25.0%
V_0 :	Acquisition price	\$1,000.00
L :	Seller's outstanding loan balance	\$750.00
r^* :	Buyer's nominal interest rate paid on funds used to acquire the asset	5.0%
r_s^*:	Nominal interest rate on the loan held by seller at time of sale	7.5%

The base line results obtained with these parameter specifications were:

Maximum Bid Price Models

-model without financing: $V = \$2,132$
-model with financing: $V^* = \$2,570$

Minimum Sell Price Models

-model without financing: $V_s = \$2,424$

-model "due on sale" clause, no seller financing: $V_s^c = \$2,552$

-seller finances buyer, seller debt free at sale: $V_s^* = \$2,861$

-seller finances buyer, seller loses concessionary rate: $V_s^{c*} = \$2,938$

-seller finances buyer, seller maintains previous loan at concessionary rate: $V_s^{1*} = \$2,783$

The difference of \$438 between the maximum bid price models reflects the present value of the after-tax benefits to the purchaser who has access to concessionary interest rate loans.

The difference between V_s, the minimum sell without financial considerations and V_s^c, \$192, is the compensation the seller would require to give up his concessionary rate loan as a condition for the sale. The difference between V_s and V_s^*, \$501, is the after-tax present value of the seller's opportunity costs when the seller is required to finance the buyer's purchase. Of course, if the seller can return a concessionary interest rate loan of L, financed as r_s*, while financing the seller, his minimum sale price V_s^{1*} is reduced from V_s*. On the other hand, if in addition to financing the buyer, the seller also is faced to sacrifice a concessionary rate loan, then the largest minimum sale price of V_s^{c*} is required.

Table 1 highlights the sensitivity results on a per acre dollar basis for all seven of the models when each input parameter is increased and decreased by 25 percent. The percent change from the baseline analysis for each of the values displayed in Table 1 is presented in Table 2. All values responded in the direction predicted by derivatives taken on the models when n was very large; however, the magnitude of some of the changes were below those expected by the authors. In particular, note the small impact on asset value from the +/-25 percent change to the T, n and q parameters, respectively.

Summary and Conclusions

In this paper five principles have been suggested which if followed would lead to consistent results from present value models. The five principles were then used to construct maximum bid and minimum sell price models for land. Example solutions were obtained from the models. In all cases, results of the sensitivity analysis results were consistent with the derivatives of the limit models.

Hopefully this paper may serve as a starting point for future discussions which will help researchers obtain consistent results from their PV models. A natural analysis which should follow is the comparison of the sensitivity results with the direction and magnitude of actual changes in the land market.

Table 1. Sensitivity Results in Dollar Measures for Maximum Bid and Minimum Sell Models in Response to $\pm$25 Percent Changes in Parameter Values

Input Parameters	+25% / -25%	Maximum Bid V	Maximum Bid V^*	V_s	Minimum Sell V_s^c	Minimum Sell V_s^*	Minimum Sell V_s^{c*}	Minimum Sell V_s^{1*}
Base Line		2132	2570	2360	2424	2861	2938	2783
r	.05	1859	2346	2053	2169	2623	2769	2478
	.03	2498	2867	2769	2777	3176	3185	3167
T	.1815	2171	2607	2416	2480	2911	2987	2835
	.1125	2090	2530	2298	2363	2807	2885	2728
T_p	.03125	1958	2360	2163	2227	2625	2702	2548
	.01875	2340	2820	2595	2659	3142	3220	3065
i	.05625	1831	2322	2022	2143	2599	2752	2445
	.03375	2550	2909	2827	2828	3221	3222	3220
g	.05	2540	3061	2821	2885	3413	3490	3335
	.03	1837	2213	2025	2090	2460	2537	2383
R	187.50	2665	3212	2963	3027	3583	3660	3506
	112.50	1599	1927	1757	1821	2138	2215	2061
n	25	2165	2607	2403	2468	2912	2990	2835
	15	2082	2512	2295	2359	2783	2860	2706
s	.0625	2117	2551	2373	2439	2885	2963	2806
	.0375	2148	2589	2347	2411	2837	2914	2761
c	.03125	2117	2548	2356	2421	2857	2934	2779
	.01875	2148	2592	2363	2428	2865	2942	2787
q	25	2132	2652	2360	2434	2954	3046	2862
	15	2132	2479	2360	2412	2757	2818	2696
D	.3125	2132	2526	2360	2424	2811	2888	2735
	.1875	2132	2614	2360	2424	2911	2990	2833
V_0	1250	2132	2570	2347	2411	2853	2931	2776
	750	2132	2570	2373	2437	2868	2945	2791
L	937.50	2132	2570	2360	2440	2861	2957	2764
	562.50	2132	2570	2360	2408	2861	2919	2803
r^*	.0625	2132	2409	2360	2424	2660	2732	2588
	.0375	2132	2743	2360	2424	3081	3164	2997
r_s^*	.09375	2132	2570	2360	2321	2861	2814	2908
	.05625	2132	2570	2360	2522	2861	3054	2667

Table 2. Sensitivity Results in Percentage Change for Maximum Bid and Minimum Sell Models in Response to ±25 Percent Changes in Parameter Values

Input Parameters	+25% -25% Value	Maximum Bid		Minimum Sell				
		V	V^*	V_s	V_s^c	V_s^*	V_s^{c*}	V_s^{1*}
r	.05	-12.8	- 8.7	-13.0	-10.5	- 8.3	- 5.8	-11.0
	.03	17.2	11.6	17.3	14.6	11.0	8.4	13.8
T	.1815	2.2	1.8	2.37	2.3	1.75	1.67	1.87
	.1125	- 2.0	- 1.6	- 2.63	- 2.5	- 1.89	- 1.80	- 1.98
T_p	.03125	- 8.2	- 8.2	- 8.35	- 8.1	- 8.3	- 8.0	- 8.44
	.01875	9.8	9.7	9.96	9.7	9.8	9.6	10.13
i	.05625	-14.1	- 9.7	-14.3	-11.6	- 9.2	- 6.3	-12.2
	.03375	19.6	13.2	19.8	16.7	12.6	9.7	15.70
g	.05	19.1	19.1	19.5	19.0	19.3	18.8	19.8
	.03	-13.8	-13.9	-14.2	-13.8	-14.0	-13.7	-14.4
R	187.50	25	25	25.6	24.9	25.2	24.6	26.0
	112.50	-25	-25	-25.6	-24.9	-25.3	-24.6	-25.9
n	25	1.6	1.4	1.82	1.8	1.78	1.77	1.87
	15	- 2.4	- 2.3	- 2.75	- 2.7	- 2.73	- 2.65	- 2.77
s	.0625	- .7	- .7	.55	.62	.84	.85	.83
	.0375	.8	.7	- .55	- .54	- .84	- .82	- .79
c	.03125	- .7	- .9	- .17	- .12	- .14	- .14	- .14
	.01875	.8	.9	.13	.17	.14	.14	.14
q	25	NA	3.2	NA	.41	3.25	3.68	2.84
	15	NA	- 3.5	NA	- .50	- 3.64	- 4.08	- 3.13
D	.3125	NA	- 1.71	NA	NA	- 1.75	- 1.70	- 1.72
	.1875	NA	1.71	NA	NA	1.75	1.77	1.80
V_0	1250	NA	NA	- .55	- .54	- .28	- .24	- .25
	750	NA	NA	.55	.54	.24	.24	.29
L	937.50	NA	NA	NA	.66	NA	.65	- .68
	562.50	NA	NA	NA	- .66	NA	- .65	.72
r^*	.0625	NA	- 6.3	NA	NA	- 7.03	- 7.01	- 7.01
	.0375	NA	6.3	NA	NA	7.69	7.69	7.69
r_s^*	.09375	NA	NA	NA	- 4.3	NA	- 4.22	4.49
	.05625	NA	NA	NA	4.0	NA	3.95	- 4.17

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