



**AgEcon** SEARCH  
RESEARCH IN AGRICULTURAL & APPLIED ECONOMICS

*The World's Largest Open Access Agricultural & Applied Economics Digital Library*

**This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.**

**Help ensure our sustainability.**

Give to AgEcon Search

AgEcon Search  
<http://ageconsearch.umn.edu>  
[aesearch@umn.edu](mailto:aesearch@umn.edu)

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

378.77427  
D34  
S73  
97-62

# **Staff Paper**

## **A DYNAMIC DUAL MODEL OF ASYMMETRIC INVESTMENT UNDER UNCERTAINTY**

**Kyösti S. Pietola and Robert J. Myers**

**No. 97-62**

Waite Library  
Dept. of Applied Economics  
University of Minnesota  
1994 Buford Ave - 232 ClaOff  
St. Paul MN 55108-6040 USA



Department of Agricultural Economics  
**MICHIGAN STATE UNIVERSITY**  
East Lansing , Michigan

MSU is an Affirmative Action/Equal Opportunity Institution

378.77427  
D34  
S73  
97-62

**A DYNAMIC DUAL MODEL OF ASYMMETRIC  
INVESTMENT UNDER UNCERTAINTY**

by

Kyösti S. Pietola and Robert J. Myers

Contact author:

Robert J. Myers  
Dept. of Agricultural Economics  
Michigan State University  
East Lansing, MI 48824

Waite Library  
Dept. of Applied Economics  
University of Minnesota  
1994 Buford Ave - 232 Cla Off  
St. Paul MN 55108-6040 USA

Email: myersr@pilot.msu.edu

*JEL Code:* C51

*Key Words:* asymmetric investment, dynamic duality, investment under uncertainty

*Abstract:* Dynamic dual models of firm investment are extended by incorporating asymmetric investment behavior, and by allowing for stochastic transition equations and firm investment response to uncertainty. The resulting model is amenable to econometric estimation.

---

Kyösti Pietola is senior researcher at the Agricultural Economics Research Institute (AERI) in Helsinki and Robert J. Myers, is associate professor in the Department of Agricultural Economics, Michigan State University. This research has been financed by AERI, The Academy of Finland, and the Finnish Cultural Foundation. We also acknowledge the Michigan Agricultural Experiment Station for its support of this research. We would like to thank Jeff Wooldridge for helpful comments and discussion on this research, although any errors are the authors' responsibility.

## A DYNAMIC DUAL MODEL OF ASYMMETRIC INVESTMENT UNDER UNCERTAINTY

### 1. Introduction

The dual approach to modeling dynamic investment under uncertainty has proven extremely valuable in applied research (e.g. Epstein, 1981; Epstein and Denny, 1983; Vasavada and Chambers, 1986; Stefanou, 1989). Advantages of the dual approach are that it allows for a flexible representation of the underlying production technology and generates closed-form dynamic decision rules that are amenable to empirical estimation.

A weakness of existing dynamic dual models is that they generally assume the future paths of exogenous state variables are deterministic and known with certainty to the investing firm. Often the simple assumption of static expectations is made (current prices will continue forever with certainty). Nonstatic expectations, where firms expect certain kinds of growth or depreciation patterns in the exogenous state variables, have been incorporated into some studies (e.g. Epstein and Denny, 1983; Luh and Stefanou, 1996). However, state variable paths are still assumed deterministic and known. This creates an inconsistency because the future paths of exogenous state variables are usually stochastic and not known with certainty to the investing firm.

Another weakness of existing dynamic dual models is that they typically assume investment decisions are symmetric, in the sense that positive and negative investments are assumed to follow the same decision rule (for an exception see Chang and Stefanou, 1988 and Oude Lansink, 1997). Recent research on the theory of investment under uncertainty has shown that irreversibility and asymmetric adjustment costs lead to asymmetry in investment decision rules (Pindyck, 1991; Abel and Eberly, 1994; Chavas, 1994). In these models, investment responds differently during capital

expansion and contraction phases. However, none of these models has been applied empirically using a dual approach.

In this paper we develop a truly stochastic dual model of investment under uncertainty. We allow exogenous state variables to follow geometric Brownian motion with drift and use stochastic dynamic programming to characterize duality relations. Firms are assumed to have rational expectations but perceive future values of exogenous state variables as stochastic, so that uncertainty can directly influence the investment decision. We also allow investment decisions to respond asymmetrically during capital expansion and contraction phases. While stochastic transition equations and asymmetric investment behavior have been incorporated previously into primal approaches to modeling investment under uncertainty (e.g. Abel, 1983; Abel and Eberly, 1994), our major contribution is that we incorporate these features into a dual model which is more amenable to empirical estimation.

## 2. The Model

Consider a vector of exogenous state variables  $Z = (\ln Y, \ln W, \ln Q)'$  consisting of the logarithms of output  $Y$ , variable input prices  $W$ , and rental rates on capital goods  $Q$ . The state variables are all functions of time but time subscripts have been dropped to simplify the notation. We assume that the state vector follows geometric Brownian motion with drift:

$$(1) \quad \Delta Z = \mu(Z)\Delta t + P v$$

where  $\Delta$  indicates a small change,  $\mu(\cdot)$  is a non-random function or drift parameter,  $t$  is time,  $P$  is a matrix satisfying  $PP' = \Sigma$  and  $v$  is an i.i.d. error vector satisfying  $E(v) = 0$ ,  $E(v_i v_j) = 0$  for  $i \neq j$ , and  $E(v_i^2) = \Delta t$ .

The production process is characterized by a transformation function  $F(X, Y, K, I) = 0$  where  $X$  is a vector of variable inputs,  $K$  is a vector of capital stocks and  $I$  is a vector of gross investments.<sup>1</sup> The transformation function is augmented with gross investment to account for adjustment costs because scarce resources have to be withdrawn from production to install new capital stock (Lucas, 1967). The capital stock evolves over time according to:

$$(2) \quad \Delta K = (I - \delta K) \Delta t$$

where  $\delta$  is a diagonal matrix of constant depreciation rates.

Firms are assumed to be risk-neutral and minimize the expected value of discounted production costs subject to transition equations for capital and for the exogenous state variables:

$$(3) \quad J(Z_0, K_0) = \min_I E_0 \left\{ \int_0^{\infty} e^{-rt} C(Z, K, I) dt \right\}$$

subject to (1) and (2). Here  $r$  is a constant discount rate and  $C(\cdot)$  is an instantaneous cost function defined by:

$$(4) \quad C(Z, K, I) = \min_X \{ W'X : F(X, Y, K, I) = 0 \} + Q'(U + \gamma)K$$

where  $U$  is the identity matrix and  $\gamma$  is a diagonal matrix with diagonal elements equal to zero when  $I \leq 0$  and some non-zero value when  $I > 0$ . Notice that the way rental costs have been defined allows for a proportional expansion (or contraction) in rental cost when gross investment is positive, as compared to the base case of zero or negative gross investment. This asymmetry in rental costs may be due to a difference between acquisition price and salvage values for capital goods, or to an asymmetry in the costs of adjusting the capital stock. The parameters in  $\gamma$  therefore capture the degree of asymmetry in investment response.

Using standard stochastic dynamic programming, the Hamilton-Jacobi-Bellman (HJB) equation corresponding to (3) is:

$$(5) \quad rJ = \min_I \{ C + \nabla_Z J \mu(Z) + \nabla_K J (I - \delta K) + 0.5 [\text{vec}(\nabla_Z^2 J)]' [\text{vec}(\Sigma)] \}$$

where  $\nabla_i J$  is the gradient vector of  $J$  with respect to  $i$  evaluated at  $(t_0, Z_0, K_0)$ ,  $\nabla_Z^2 J$  is the hessian matrix of  $J$  with respect to  $Z$  evaluated at  $(t_0, Z_0, K_0)$ , and  $\text{vec}$  is the column-stacking operator.

### 3. Dynamic Duality

Dynamic duality requires derivation of the characteristic properties of the value function  $J$  from those of the cost function  $C$ . Then any  $J$  which satisfies these characteristic properties is the value function for some cost function  $C$  defined by:

$$(6) \quad C = \max_{w, Q} \{ rJ - \nabla_Z J \mu(Z) - \nabla_K J (I - \delta K) - 0.5 [\text{vec}(\nabla_Z^2 J)]' [\text{vec}(\Sigma)] \}$$

Furthermore, optimal decision rules can be derived from  $J$  using a version of Shephard's lemma (Epstein, 1981; McLaren and Cooper, 1980). In particular, for our model it can be shown that:

$$(7.1) \quad \begin{aligned} \dot{K}^* &= (\nabla_{K, \ln Q} J)^{-1} \{ r(\nabla_{\ln Q} J)' - (U + \gamma)K \circ Q - (\nabla_{Z, \ln Q} J) \mu(Z) \\ &\quad - (\nabla_{\ln Q} \mu(Z))' (\nabla_Z J)' - 0.5 \nabla_{\ln Q} [\text{vec}(\nabla_Z^2 J)' \text{vec}(\Sigma)] \end{aligned}$$

$$(7.2) \quad \begin{aligned} X^* \circ W &= r(\nabla_{\ln W} J) - (\nabla_{Z, \ln W} J) \mu(Z) - [\nabla_{\ln W} \mu(Z)]' (\nabla_Z J)' \\ &\quad - (\nabla_{K, \ln W} J) \dot{K} - 0.5 \nabla_{\ln W} [\text{vec}(\nabla_Z^2 J)' \text{vec}(\Sigma)] \end{aligned}$$

where  $\circ$  indicates element by element multiplication,  $\dot{K}^*$  is the time derivative of the optimal capital path, and  $X^*$  is the optimal variable input allocation. These decision rules are derived from differentiating (5) with respect to  $\ln Q$  and  $\ln W$  and using the envelope theorem. All that remains is to derive the characteristic properties of  $J$  from those of  $C$ .

Suppose that the primal problem (3) satisfies:

- (A.1)  $C \geq 0$ .
- (A.2)  $C$  is increasing in  $Y$ , decreasing in  $K$ , and increasing (decreasing) in  $I$  if  $I > 0$  ( $I < 0$ ).
- (A.3)  $C$  is convex in  $I$  and concave in  $(W, Q)$ .
- (A) (A.4) Well-defined optimal decision rules exist (the integral in (3) converges).
- (A.5) There is a unique steady state capital stock  $\bar{K}(Y, W, Q)$  which is globally stable.
- (A.6)  $C$  is positively linearly homogeneous in  $(W, Q)$ .



These properties are standard and follow directly from applying the usual regularity conditions to the product transformation function  $F$  (see footnote 1 and Epstein and Denny, 1983). Because we allow for asymmetric investment response we might expect the optimal investment path to be discontinuous at zero investment, which generates kinks in the optimal path of  $K$ . Nevertheless, necessary conditions for optimality imply that  $J$  and  $\nabla_K J$  (the shadow price of installed capital) are continuous along the optimal path of  $K$  (Dixit and Pindyck, 1994; Abel and Eberly, 1994).

Given the conditions (A) on the primal problem it can be shown that  $J$  satisfies:<sup>2</sup>

(B.1)  $J$  is real valued and non-negative.

(B.2) (i)  $(r + \delta)(\nabla_K J)' - (U + \gamma)Q - (\nabla_{Z,K} J)\mu(Z) - (\nabla_K^2 J)\dot{K}^*$   
 $- 0.5 \nabla_K [\text{vec}(\nabla_Z^2 J)' \text{vec}(\Sigma)]' < 0$

(ii)  $(\nabla_K J)' < (>) 0$  if  $I^* > (<) 0$  where  $I^*$  is optimal investment from (7.1).

(iii)  $r(\nabla_Y J)' - (\nabla_{Z,Y} J)\mu(Z) - (\nabla_{K,Y} J)\dot{K}^*$   
 $- 0.5 \nabla_Y [\text{vec}(\nabla_Z^2 J)' \text{vec}(\Sigma)]' > 0$

(B)

where  $\dot{K}^*$  is the time derivative of the optimal capital path from (7.1).

(B.3)  $\{rJ - \nabla_Z J\mu(Z) - \nabla_K J(I - \delta K) - 0.5 [\text{vec}(\nabla_Z^2 J)]' [\text{vec}(\Sigma)]\}$  is convex in  $I$  and concave in  $(W, Q)$ .

(B.4) Optimal decision rules are given by (7.1) and (7.2).

(B.5) The optimal decision rule (7.1) defines a unique, globally stable steady state  $K(Y, W, Q)$ .

(B.6)  $J$  is positively linearly homogeneous in  $(W, Q)$ .

Property (B.1) is self evident and immediate from the primal problem. Properties (B.2) are obtained by differentiating the HJB equation (5) with respect to  $K$ ,  $I$ , and  $Y$ , rearranging, and using the properties of  $C$ . Notice that we have generalized Epstein and Denny (1983) here by allowing gross investment to take negative as well as positive values. Property (B.3) follows directly from the convexity properties of the cost function (see (A.3) and the dual cost function (6)). Finally, properties (B.4), (B.5), and (B.6) follow immediately from properties (A.4), (A.5) and (A.6) of the primal problem. These conditions can also be used to show that  $J$  is non-decreasing in  $(W, Q)$ .

The key value function property for empirical implementation is the concavity restriction (B.3). Even in the standard case of nonstochastic transition equations where  $vec(\Sigma) = 0$ , this concavity restriction is somewhat difficult to impose on  $J$ . To see this, note that the first derivative of  $J$  appears on the right-hand-side of (6) so that, even in the nonstochastic case, convexity restrictions on  $C$  impose third derivative restrictions on  $J$ . Thus, a complete characterization of the necessary conditions for  $J$  requires third derivative properties. The conventional solution to this problem is to assume static expectations,  $\mu(Z) = 0$ , and that the shadow price of installed capital  $\nabla_K J$  is linear in prices  $(W, Q)$ . Then concavity of  $J$  in  $(W, Q)$  is sufficient to ensure that  $C$  will be convex in  $I$  and concave in  $(W, Q)$ .

The assumption of static expectations can be relaxed slightly. For example, Luh and Stefanou (1996) have shown that if *all* first derivatives of the value function ( $\nabla_Z J$  as well as  $\nabla_K J$ ) are linear in prices  $(W, Q)$ , and  $\mu$  is convex, then concavity of  $J$  in  $(W, Q)$  remains sufficient to ensure that  $C$  is convex in  $I$  and concave in  $(W, Q)$  (see equation (6)). But while this allows certain kinds of expected growth or depreciation patterns in the exogenous state variables, firms are still implicitly assumed to know the future path of all state variables with complete certainty, so that uncertainty does not alter the decision to invest.

In our case of stochastic transition equations,  $vec(\Sigma) \neq 0$ , the right-hand-side of (6) contains second derivatives of  $J$  as well as first derivatives. This clearly exacerbates the problem of analyzing convexity relations between  $J$  and  $C$  because convexity properties on  $C$  now impose fourth-order curvature restrictions on  $J$ . Thus, a complete characterization of the necessary conditions on  $J$  in the stochastic case requires fourth derivative properties. Nevertheless, the following proposition which is proven in the appendix can be used to obtain conditions on  $J$  which are sufficient to ensure that the convexity restrictions (B.3) are satisfied.

**Proposition:** If (a)  $J$  is concave in  $(W, Q)$

(b)  $\nabla_K J$  is linear in  $(W, Q)$

(c)  $\nabla_Z^2 J$  is linear in  $(W, Q)$

(d)  $\mu(Z)$  is non-increasing and convex

then the convexity restriction (B.3) is satisfied.

Notice that the Luh and Stefanou sufficient conditions under convex  $\mu$  and  $vec(\Sigma) = 0$  require  $\nabla_Z J$  to be linear in  $(W, Q)$ , while we only require  $\nabla_Z^2 J$  to be linear in  $(W, Q)$  in our stochastic model. In other words, we allow  $\nabla_Z J$  to be quadratic rather than requiring linearity, as in Luh and Stefanou. The usefulness of our generalization is that it allows a shift in uncertainty to alter investment decisions while still generating fairly tractable decision rules for econometric estimation. The restriction (d) on  $\mu$  can be made with little loss of generality. For example, a stationary VAR process or differenced VAR process with unit roots estimated in the logarithms of prices satisfy the required restrictions on  $\mu$ .

Overall, the regularity conditions on  $J$  in the stochastic case are comparable to those in the deterministic case. The only significant additional restriction required is condition (c) of the proposition, a condition which is weaker than that used in Luh and Stefanou (1996) in their deterministic model.

#### 4. An Illustrative Example

Suppose there is a single variable input which is defined as the numeraire. Then  $W=1$  for all  $t$  and rental prices  $Q$  are expressed relative to the variable input price. Following Epstein (1981), consider a second-order approximation to the value function of the form:

$$(8) \quad J(\cdot) = a_0 + [A_1' \ A_2' \ A_3'] \begin{bmatrix} K \\ \ln Y \\ \ln Q \end{bmatrix} + \frac{1}{2} [K' \ \ln Y' \ \ln Q'] \begin{bmatrix} B_{11} & B_{21}' & 0 \\ B_{21} & B_{22} & B_{32} \\ 0 & B_{32}' & B_{33} \end{bmatrix} \begin{bmatrix} K \\ \ln Y \\ \ln Q \end{bmatrix} \\ + Q' M^{-1} K$$

where  $a_0$  is a parameter and the  $A$ ,  $B$ , and  $M$  matrices are made up of unknown parameters. It is easy to verify that (8) is consistent with the relevant value function properties derived above. In particular, the zero restrictions in the  $B$  matrix ensure that  $\nabla_K J$  is linear in  $Q$ ; and simple differentiation shows that  $\nabla_Z^2 J$  is linear in  $Q$ . Notice also that although we are allowing for a discontinuity between capital expansion and contraction phases in investment decisions, both  $J$  and  $\nabla_K J$  are continuously differentiable in  $K$ , as required along the optimal capital path (Abel and Eberly, 1994).

For purposes of illustration, suppose we make the simple assumption of static expectations,  $\mu(\cdot) = 0$ , and that  $J(\cdot)$  is given by (8). Then differentiating  $J$ , substituting into (7.1) and (7.2), and rearranging gives the decision rules:<sup>3</sup>

$$(9.1) \quad \dot{k}_i = \frac{r}{q_i} M_i [A_3 + B_{32} \ln Y + B_{33} \ln Q] - \sum_{j=1}^3 M_{ij} (1 + \gamma_j) k_j + (r - 0.5\sigma_i^2) k_i$$

$$(9.2) \quad x = \alpha + A_2' r \ln Y + A_3' r \ln Q + (A_1' + B_{21} \ln Y)(rK - \dot{K}) + K' B_{11} (0.5rK - \dot{K}) \\ + 0.5r [\ln Y' B_{22} \ln Y + 2 \ln Y' B_{32}' (\ln Q - 1) + \ln Q' B_{33} (\ln Q - 2)]$$

for  $i = 1, 2, \dots, n$  capital goods and  $x =$  the aggregate variable input. Here,  $\dot{k}_i$  is the  $i$ th element of  $\dot{K}$ ,  $q_i$  is the  $i$ th rental price,  $M_i$  is the  $i$ th row of  $M$ ,  $M_{ij}$  is the  $ij$ th element of  $M$ ,  $\sigma_i^2$  is the variance of proportional changes in the  $i$ th rental price,  $\gamma_j$  is the  $j$ th diagonal element of  $\gamma$ , and  $\alpha$  is a parameter determined by other parameters in the system. Notice from the decision rules that uncertainty about future state variable paths influences the decision to invest, and investment may respond asymmetrically during capital expansion and contraction phases. These equations are highly nonlinear but can be estimated using full information maximum likelihood methods.

## 5. Concluding Comments

Existing dynamic dual models of investment typically assume investing firms know future state variable paths with complete certainty, and that investment decision rules are symmetric during capital expansion and contraction phases. Yet most state variables are more appropriately modeled as

stochastic processes, and irreversibility and asymmetric adjustment costs may induce an asymmetric investment response. In this paper we derive a truly stochastic model of investment under uncertainty where firms perceive state variables as geometric Brownian motion with drift. Stochastic dynamic programming is used to characterize duality relations, and value function restrictions are comparable to those used in much existing empirical work assuming deterministic state variable paths. We also allow for a shift in rental rates during capital expansion and contraction phases which introduces an asymmetry into the investment decision rules generated by the model. The resulting model is amenable to econometric estimation.

## ENDNOTES

1. Standard regularity conditions on  $F(\cdot)$  are: (a)  $F$  is continuous and twice differentiable; (b)  $F$  is strictly increasing in  $Y$  and strictly decreasing in  $X$ ,  $K$ , and the absolute value of  $I$ ; and (c)  $F$  is convex in  $I$  and  $X$ .
2. Analogous conditions for the deterministic case with  $\mu(Z) = 0$  are given in Epstein and Denny (1983). Here we derive the conditions allowing for stochastic transition equations and  $\mu(Z) \neq 0$ .
3. Because we have assumed a single variable input which is defined as the numeraire then it's decision rule is actually calculated by solving the HJB equation (5) for  $X$ , simplifying, and collecting terms.

## APPENDIX

**Proof of the Proposition**

We need to show that each term in (B.3) is convex in  $I$  and concave in  $(W, Q)$ . Convexity in  $I$  is immediate from condition (b) of the proposition. Furthermore, the first term,  $rJ$ , is concave in  $(W, Q)$  by condition (a) of the proposition; and the third and fourth terms,  $-\nabla_K J(I - \delta K)$  and  $-0.5[\text{vec}(\nabla_Z^2 J)]'[\text{vec}(\Sigma)]$ , are both linear and concave in  $(W, Q)$  by conditions (b) and (c) of the proposition. Turning to the second term,  $-\nabla_Z J \mu(Z)$ , we note from conditions (a) and (c) of the proposition that  $\nabla_Z J$  is non-negative and quadratic while  $\nabla_Z^2 J$  is negative semidefinite; and from condition (d) that  $\mu$  is non-increasing and convex. These conditions are sufficient to ensure that  $-\nabla_Z J \mu(Z)$  is concave in  $(W, Q)$ .



## REFERENCES

- Abel, A.B., 1983, Optimal investment under uncertainty, *American Economic Review* 73, 228-233.
- Abel, A.B. and J.C. Eberly, 1994, A unified model of investment under uncertainty, *The American Economic Review* 84, 1369-1384.
- Chang, C.-C. and S.E. Stefanou, 1988, Specification and estimation of asymmetric adjustment rates for quasi-fixed factors of production, *Journal of Economic Dynamics and Control* 12, 145-151.
- Chavas, J.-P., 1994, Production and investment decision under sunk cost and temporal uncertainty, *American Journal of Agricultural Economics* 76, 114-127.
- Dixit, A.K. and R.S. Pindyck, 1994, *Investment under uncertainty*, Princeton, New Jersey.
- Epstein, L.G., 1981, Duality theory and functional forms for dynamic factor demands, *Review of Economic Studies* 81, 81-95.
- Epstein, L.G. and M. Denny, 1983, The multivariate flexible accelerator model: Its empirical restrictions and an application to U.S. manufacturing, *Econometrica* 51, 647-673.
- Lucas, R.E., Jr., 1967, Optimal investment policy and the flexible accelerator, *International Economic Review* 8, 78-85.
- Luh, Y.-H. and S.E. Stefanou, 1996, Estimating dynamic dual models under non-static expectations, *American Journal of Agricultural Economics* 78, 991-1003.
- McLaren, K.R. and R.J. Cooper, 1980, Intertemporal duality: Application to the theory of the firm, *Econometrica* 48, 1755-62.
- Oude Lansink, A., 1997, Asymmetric adjustment of dynamic factors at the firm level, in *Micro Economic Models for Analyzing Policy Changes in Dutch Arable Farming*, Wageningen.

Pindyck, R.S., 1991, Irreversibility, uncertainty and investment, *Journal of Economic Literature* 29, 1110-1148.

Stefanou, S.E., 1989, Returns to scale in the long run: The dynamic theory of cost, *Southern Economic Journal* 55, 570-579.

Vasavada, U. and R.C. Chambers, 1986, Investment in U.S. agriculture, *American Journal of Agricultural Economics* 68, 950-960.