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GENERALIZED OPTIMAL HEDGE RATIO ESTIMATION

A major problem faced by commodity traders is to select the proportion of spot positions that should be covered by opposite positions on futures markets. This is the problem of choosing an optimal hedge ratio (Johnson; Stein; Heifner). A frequently recommended solution is to set the hedge ratio equal to the ratio of the covariance between spot and futures prices to the variance of the futures price (Benninga, Eldor and Zilcha; Kahl, 1983). But in order to implement this seemingly simple rule, the relevant covariance and variance must be estimated from available data. This paper considers alternative methods for empirical estimation of this optimal hedging rule.

The conventional approach to estimating optimal hedge ratios is to use the slope coefficient from a simple regression of spot price levels on futures price levels (Ederington), or spot price changes on futures price changes (Carter and Loyns). Some researchers have also regressed spot market returns on futures market returns, where returns are defined as the proportional price change from period to period (Brown, 1985). The question of whether levels, changes or returns should be used in the simple regression approach to optimal hedge ratio estimation has become somewhat controversial (Brown, 1986; Kahl, 1986; Hill and Schneeweis; Bond, Thompson and Lee; Witt, Schroeder and Hayenga). This paper shows that none of these simple regression approaches are appropriate except under very special and restrictive circumstances. The reason is that the slope parameter from the simple regression model only gives a ratio of the unconditional covariance between the dependent and explanatory variable to the unconditional variance of the explanatory variable. Yet the covariance and variance in the optimal hedging rule are clearly conditional moments that depend on information available at the time the hedging decision is made. A generalized approach based on time series econometrics is provided as an alternative to simple regression. The generalized approach takes proper account of relevant conditioning information and therefore should provide improved estimates of optimal hedge ratios.

In the next section, the optimal hedging rule is derived from the usual mean-variance model. Two important points about the role of conditioning information and the way in which optimal hedge ratios should be estimated are then discussed. Next a generalized approach to optimal hedge ratio estimation is outlined. The advantages of the procedure are demonstrated by showing how the conventional simple regression methods are special cases under a set of restrictions on equilibrium prices. The following sections then provide two single equation methods for implementing the generalized approach and discuss hypothesis testing and model selection. In an example of optimal hedge ratio estimation for corn, soybean and wheat storage in Michigan, it is shown that simple regression may lead to errors in the estimation of optimal hedge ratios. Finally, implications for empirical estimation of optimal hedge ratios are summarized in the conclusion.

Derivation of the Optimal Hedging Rule

Consider the behavior of an agent that takes out spot and futures positions for some commodity at time t-1. The agent's profit in period t, the date at which the positions are liquidated, is denoted

$$\pi_t = p_t q_{t-1} - c(q_{t-1}) - (f_t - f_{t-1})b_{t-1}$$

where π_t is profit, p_t is the spot price in period t, q_{t-1} is the spot position chosen at t-1, c is an increasing and convex cost function, f_t is the futures price quoted at period t for delivery at some future date, and b_{t-1} is sales of futures contracts in t-1 (purchases if negative).

The model can be interpreted in a number of different ways. The agent might be a producer so that $c(q_{t-1})$ represents production costs. Or the agent may be a commodity trader so that $c(q_{t-1})$ represents the cost of purchasing, storing and transporting the commodity. It is also possible for the agent to sell short in the spot market ($q_{t-1} < 0$) in order to obtain a deterministic revenue equal to minus $c(q_{t-1})$.[1] However, the model does not accomodate producers with a stochastic production technology. An alternative framework is required for this case (Rolfo; Grant).

The agent chooses q_{t-1} and b_{t-1} to maximize a linear function of the mean and variance of income, conditional on available information:

$$\max_{q_{t-1},b_{t-1}} E(\pi_t|X_{t-1}) - \frac{\lambda}{2} Var(\pi_t|X_{t-1})$$

where X_{t-1} is a set of information available at t-1 and λ is a measure of the agent's risk aversion. First-order conditions for this problem are[2]

$$(1) \qquad E(p_t|X_{t-1}) - c'(q_{t-1}) - \lambda(\sigma_p^2 q_{t-1} - \sigma_{pf} b_{t-1}) = 0$$

$$(2) \qquad E(f_t|X_{t-1}) - f_{t-1} - \lambda(\sigma_f^2 b_{t-1} - \sigma_{pf} q_{t-1}) = 0$$

where: $\sigma_p^2 = Var(p_t|X_{t-1})$ is the conditional variance of the spot price;

$\sigma_f^2 = Var(f_t|X_{t-1})$ is the conditional variance of the futures price;

$\sigma_{pf} = Cov(p_t,f_t|X_{t-1})$ is the conditional covariance between spot and futures prices.

Notice that all of the relevant moments are conditional on information available at time t-1.

A crucial assumption in deriving the optimal hedge ratio is that the futures market is unbiased,

$$(3) \qquad E(f_t | X_{t-1}) = f_{t-1}.$$

Under this assumption, (2) implies the simple hedging rule

$$(4) \qquad r = \frac{\sigma_{pf}}{\sigma_f^2}$$

where $r = b_{t-1}/q_{t-1}$. This rule occurs frequently in the literature (Anderson and Danthine; Kahl, 1983; Bond, Thompson and Lee). If the futures market is biased, so that (3) does not hold, then (4) is still the minimum variance rule but it is no longer mean-variance efficient (Heifner).

There are two important points that need to be made regarding estimation of the hedging rule (4). First, the model used to derive (4) represents the decision process of an individual hedger. The decision rule (4) therefore tells us what information a hedger will seek when implementing the rule (i.e. information about σ_{pf} and σ_f^2). But by itself, (4) does not tell us anything about how to <u>estimate</u> σ_{pf} and σ_f^2. The parameters σ_{pf} and σ_f^2 are conditional moments of market prices. Therefore, to determine how σ_{pf} and σ_f^2 should be estimated, we need a market model of the equilibrium (or disequilibrium) relationship between spot and futures prices; a model that takes the behavior of all market participants into account. This essential point was missed by Witt, Schroeder and Hayenga when they argued that the way in which hedge ratios should be estimated depends on the type of decision problem the agent is facing. It is true that different decision problems will lead to different hedging rules but the type of problem does not prescribe how the parameters on the right hand side of those hedging rules should be estimated.

The second point is that the usual simple regression approaches to optimal hedge ratio estimation will be inadequate except under special and restrictive circumstances. This is because the simple regression approaches estimate a ratio of the unconditional covariance between spot and futures prices (price changes, returns) to the unconditional variance of futures prices (price changes, returns). But from the decision problem outlined above, a ratio of the conditional covariance between spot and futures price levels to the conditional variance of futures price levels is actually required. Unless the slope parameters from the simple regression models happen to equal the required ratio of conditional moments, each of the three simple regression approaches will be misleading. In the next section we show that the conditions under which the simple regression approaches will be appropriate are very restrictive and seem unlikely to be satisfied for seasonally produced storable commodities.

A Generalized Approach to Estimation

This section begins by outlining a generalized approach to optimal hedge ratio estimation. The approach is based on a model of the data generating process for equilibrium spot and futures market prices. After outlining the generalized approach, the simple regression approaches to optimal hedge ratio estimation are shown to be special cases that are only valid under strong restrictions on the data generating process for equilibrium prices.

Suppose spot and futures market price data are generated by the following linear equilibrium model:

(5) $$p_t = X_{t-1}\alpha + u_t$$

(6) $$f_t = X_{t-1}\beta + v_t$$

where X_{t-1} is a vector of variables known at t-1 that help predict p_t and f_t; α and β are vectors of unknown parameters; and u_t and v_t are error terms having zero mean and no serial correlation (conditional on X_{t-1}). The errors may be contemporaneously correlated and have a time independent (contemporaneous) covariance matrix Ω. Examples of variables that might appear in X_{t-1} are a constant plus lagged values of spot and futures prices, production, storage, exports, consumer income etc., all dated t-1 and earlier. Equations (5) and (6) can be considered reduced forms from a fully specified structural model of equilibrium spot and futures price determination for the commodity of interest.[3] For now it is assumed that the specification of (5) and (6) is known and only the unknown parameters α, β and Ω need to be estimated. The problem of choosing a specification for the data generating process is considered later.

Before discussing estimation of optimal hedge ratios using this model, it should be emphasized that the individual agent decision model presented in the previous section places no restrictions on the parameter vectors α and β in the equilibrium pricing model. Yet it is clearly the conditional moments of the equilibrium prices modeled by (5) and (6) that must be estimated to implement the optimal hedging rule. Thus, the appropriate estimation strategy depends on the data generating process (5) and (6), not on the individual hedger's decision rule.

To implement the optimal hedging rule, we need to estimate the conditional covariance matrix of spot and futures prices. But by applying the conditional (on X_{t-1}) covariance operator to (5) and (6), it is found that

$$\sigma_p^2 = \mathrm{Var}(u_t|X_{t-1})$$
$$\sigma_f^2 = \mathrm{Var}(v_t|X_{t-1})$$
$$\sigma_{pf} = \mathrm{Cov}(u_t,v_t|X_{t-1}).$$

Thus, the conditional covariance matrix of p_t and f_t is just equal to Ω, the contemponaneous covariance matrix of the error terms u_t and v_t. The problem has been reduced to one of obtaining an estimate of Ω.

To estimate Ω, the first step is to estimate the unknown parameters α and β using ordinary least squares (OLS) on each equation. The second step is to take the vectors of OLS residuals, $\hat{u}$ and $\hat{v}$, and use their sample variances and covariance as an estimator of Ω:

$$(7) \qquad \hat{\Omega} = \frac{1}{T}\begin{bmatrix} \hat{u}'\hat{u} & \hat{u}\hat{v} \\ \hat{v}'\hat{u} & \hat{v}\hat{v} \end{bmatrix} .$$

Since there are identical exogenous regressors on the right hand side of both equations, (7) is the maximum likelihood estimator of Ω, under the assumption of normally distributed errors (Harvey, p. 99). The third and final step is to set the hedge ratio equal to the ratio of the estimated conditional covariance between spot and futures prices to the estimated conditional variance of the futures price:

$$(8) \qquad \hat{r} = \frac{\hat{v}'\hat{u}}{\hat{v}'\hat{v}}.$$

Equation (8) has three desirable properties as an optimal hedge ratio estimator. First it is based on a generally specified market model for spot and futures price determination. Second, it takes proper account of relevant conditioning information by using deviations from the (time varying) conditional means of spot and futures prices to estimate the required covariance matrix. Third, under the assumption of normally distributed errors it is the maximum likelihood estimator of the optimal hedge ratio, given the data generating process defined by (5) and (6). These desirable properties persist even if p_t and f_t are nonstationary.[4]

The Conventional Approaches

We now evaluate the conventional simple regression approaches to optimal hedge ratio estimation within the context of the previously outlined generalized approach. First, consider using the slope coefficient from a simple regression of spot price levels on futures price levels as an optimal hedge ratio estimator:

$$p_t = \gamma + \delta f_t + \varepsilon_t.$$

Define the vectors of T observations on p_t, f_t and the constant as

$$p = [p_1\ p_2\ \dots\ p_T]'$$
$$f = [f_1\ f_2\ \dots\ f_T]'$$
$$i = [1\ 1\ \dots\ 1]'$$

and let $\bar{p}$ and $\bar{f}$ denote the sample means of p and f respectively. Then the OLS estimate of δ can be expressed as

$$(9) \qquad \hat{\delta} = \frac{(f - \bar{f}i)'(p - \bar{p}i)}{(f - \bar{f}i)'(f - \bar{f}i)}.$$

Notice that $\hat{\delta}$ is just a ratio of the unconditional sample covariance between spot and futures prices to the unconditional sample variance of futures prices.

When will the simple regression estimate (9) equal the desired optimal hedge ratio estimator (8)? Suppose the conditional means of spot and futures prices are constant so that (5) and (6) take the special form

(10) $p_t = \alpha_0 + u_t$, and

(11) $f_t = \beta_0 + v_t$.

Then the OLS estimates of α_0 and β_0 are just the sample means $\bar{p}$ and $\bar{f}$. In this special case

$\hat{u} = p - \bar{p}i$, and

$\hat{v} = f - \bar{f}i$.

Equations (8) and (9) are clearly equivalent under this definition of $\hat{u}$ and $\hat{v}$. Thus, for the case where equilibrium spot and futures price levels equal a constant plus a serially uncorrelated error, the simple regression approach using price levels is appropriate for optimal hedge ratio estimation. Otherwise simple regression using price levels will generally lead to errors.

The equilibrium model (10) and (11) under which simple regression using price levels is an appropriate optimal hedge ratio estimator is very restrictive. If equilibrium prices equalled a constant plus a serially uncorrelated error, then there would be substantial arbitrage opportunities since relatively low price realizations would generally be followed by higher ones (and vice versa). Both common sense and empirical evidence reject this model but it is precisely the model that lies implicitly behind the use of simple regression with price levels to estimate optimal hedge ratios.

Second, consider using the slope coefficient from a simple regression of spot price changes on futures price changes as an optimal hedge ratio estimator:

$$\Delta p_t = \gamma + \delta \Delta f_t + \varepsilon_t$$

where $\Delta p_t = p_t - p_{t-1}$ and $\Delta f_t = f_t - f_{t-1}$. Define the vectors of T observations on Δp_t and Δf_t as

$$\Delta p = [\Delta p_1 \; \Delta p_2 \; \ldots \; \Delta p_T]'$$

$$\Delta f = [\Delta f_1 \; \Delta f_2 \; \ldots \; \Delta f_T]'$$

and let $\overline{\Delta p}$ and $\overline{\Delta f}$ denote the sample means of Δp and Δf respectively. Then the OLS estimate of δ can be expressed as

$$(12) \qquad \hat{\delta} = \frac{(\Delta f - \overline{\Delta f} i)'(\Delta p - \overline{\Delta p} i)}{(\Delta f - \overline{\Delta f} i)'(\Delta f - \overline{\Delta f} i)}.$$

The estimate $\hat{\delta}$ is the ratio of the unconditional sample covariance between spot and futures price changes to the unconditional sample variance of futures price changes.

When will the simple regression estimate (12) equal the desired optimal hedge ratio estimator (8)? Suppose spot and futures prices follow a random walk, possibly with drift, so that (5) and (6) take the special form

$$p_t = \alpha_0 + p_{t-1} + u_t$$

$$f_t = \beta_0 + f_{t-1} + v_t.$$

These equations can be written equivalently as

$$\text{(13)} \quad \Delta p_t = \alpha_0 + u_t$$

$$\text{(14)} \quad \Delta f_t = \beta_0 + v_t.$$

In this special case, the OLS estimates of α_0 and β_0 are just the sample means $\overline{\Delta p}$ and $\overline{\Delta f}$ and so

$$\hat{u} = \Delta p - \overline{\Delta p}i, \text{ and}$$

$$\hat{v} = \Delta f - \overline{\Delta f}i.$$

Equations (8) and (12) are clearly equivalent under this definition of $\hat{u}$ and $\hat{v}$. Thus for the case where equilibrium spot and futures prices follow a random walk, possibly with drift, the simple regression approach using price changes is appropriate for optimal hedge ratio estimation. Otherwise, simple regression using price changes will generally lead to errors.

Although the model defined by (13) and (14) is fairly simple, the random walk hypothesis for equilibrium commodity prices does have empirical support, particularly in the case of futures markets (Kamara). Thus, equation (14), perhaps with the additional restriction that $\beta_0 = 0$, seems like a reasonable market model for the evolution of futures prices. On the other hand, the spot price equation is too restrictive since spot prices cannot always be expected to follow a random walk. Spot prices of storable seasonally produced commodities exhibit systematic movements over the course of the crop year to reflect carrying changes and changes in stock levels. As an example, consider the model of equilibrium spot prices for storable commodities developed by

Samuelson. If we make the simplifying assumption that the cost of physically storing the commodity is negligible, Samuelson's model indicates that the expected proportional rise in spot prices over the storage season should equal the interest rate, R:

$$\frac{E(p_t | X_{t-1}) - p_{t-1}}{p_{t-1}} = R.$$

This implies

$$\Delta p_t = Rp_{t-1} + u_t$$

which means (13) is misspecified through exclusion of the Rp_{t-1} term. Furthermore, the Samuelson model is a very simple one. Taking convenience yields, heterogeneous expectations, physical storage costs etc. into account, there is little reason to believe (13) is a good model of equilibrium spot price determinination (Goldman and Sosin). Thus, there is also little reason to believe a priori that simple regression using price changes will give the desired generalized estimate of the optimal hedge ratio.

Third, consider using the slope coefficient from a simple regression of spot market returns on futures market returns as an optimal hedge ratio estimator:

$$(15) \quad \frac{\Delta p_t}{p_{t-1}} = \gamma + \delta\frac{\Delta f_t}{f_{t-1}} + \varepsilon_t.$$

From the two previous cases using price levels and price changes, it might be thought that the specialization of (5) and (6) which is consistent with the simple regression approach using returns is:

$$(16) \quad \frac{\Delta p_t}{p_{t-1}} = \alpha_0 + u_t$$

$$(17) \quad \frac{\Delta f_t}{f_{t-1}} = \beta_0 + v_t.$$

Indeed it is true that the OLS estimate of δ in (15) gives a ratio of the sample covariance between the OLS residuals $\hat{u}$ and $\hat{v}$ from estimating α_0 and β_0 in (16) and (17), to the sample variance of $\hat{v}$.[5]

However, there is a problem. Rearranging (16) and (17) gives

$$p_t = (1 + \alpha_0)p_{t-1} + p_{t-1}u_t$$
$$f_t = (1 + \beta_0)f_{t-1} + f_{t-1}v_t.$$

Thus, in this case the conditional (on p_{t-1} and f_{t-1}) covariance matrix of p_t and f_t is not just equal to Ω, the covariance matrix of the error terms. In fact, the conditional covariance matrix of p_t and f_t is heteroscedastic with

$$\text{Cov}(p_t, f_t | X_{t-1}) = p_{t-1}f_{t-1} \text{ Cov}(u_t, v_t | X_{t-1}), \text{ and}$$
$$\text{Var}(f_t | X_{t-1}) = f_{t-1}^2 \text{ Var}(v_t | X_{t-1}).$$

Now it is the conditional moments of price <u>levels</u> (not returns) which enters into the optimal hedging rule (4). Therefore, given the price determination model (16) and (17),

$$(18) \quad r = \frac{p_{t-1}f_{t-1} \text{ Cov}(u_t, v_t | X_{t-1})}{f_{t-1}^2 \text{ Var}(v_t | X_{t-1})}.$$

As stated earlier, the OLS estimate of δ in (15) gives an estimate of the ratio of $\text{Cov}(u_t, v_t | X_{t-1})$ to $\text{Var}(v_t | X_{t-1})$. Therefore, this estimate of δ will be an appropriate optimal hedge ratio estimator if: (a) equations (16) and (17) adequately represent the determination of equilibrium spot and futures prices; and (b) $p_{t-1} = f_{t-1}$ so that the prices in (18) cancel and r just equals the ratio of covariance to variance of the error terms u_t and v_t. If either of these conditions does not hold then simple regression using returns will generally lead to errors in optimal hedge ratio estimation.

The equilibrium model (16) and (17), together with the condition $p_{t-1} = f_{t-1}$, impose strong restrictions on the evolution of equilibrium spot and futures prices. Even if (16) and (17) seem reasonable, it is illogical to expect current spot price to equal the current futures price for delivery at a future date. Thus, simple regression using returns data will generally be inappropriate for estimation of the hedging rule (4).

Single Equation Estimation Methods

The generalized approach to optimal hedge ratio estimation in the previous section is more complicated than the simple regression methods. Two equations must be estimated using multiple regression and then the vectors of OLS residuals have to be cross-multiplied to get the optimal hedge ratio estimate (8).[6] However, in this section we show that the generalized optimal hedge ratio estimate (8) can be obtained from OLS estimation of the parameters in a single regression equation. This simplifies the estimation approach considerably since only one equation has to be estimated and the optimal hedge ratio is equal to an estimated parameter so there is no need to cross-multiply vectors of residuals.

The single-equation approach to generalized optimal hedge ratio estimation is motivated by the following proposition:

Proposition 1 Given the data generating process

$$p_t = X_{t-1}\alpha + u_t$$
$$f_t = X_{t-1}\beta + v_t$$

where u_t and v_t are serially uncorrelated with contemporaneous covariance matrix Ω, then the generalized optimal hedge ratio estimator (8) is equal to the OLS estimate of δ in the regression equation

$$(19) \quad p_t = \delta f_t + X_{t-1}\alpha + \varepsilon_t.$$

Proof See the appendix.

Equation (19) is called an augmented reduced form since the reduced form equation for the spot price is "augmented" by using the current futures price, f_t, as an additional regressor. Estimation proceeds by simply running OLS on (19) and using the estimate of δ as the optimal hedge ratio.

It is important to note that estimation of (19) is just a mechanism for computing the generalized optimal hedge ratio estimate (8). The augmented reduced form has no structural interpretation and so simultaneous equations bias (and other estimation problems) from using f_t as an additional regressor are irrelevant. OLS estimation of (19) is just a direct single-equation method for computing (8).

A careful examination of (19) reveals how this optimal hedge ratio estimator generalizes the simple regression approach. In simple regression using price levels, it is assumed that all parameters in α other than the constant term are zero. In other words, using simple regression assumes implicitly that p_t is equal to a constant plus a serially uncorrelated error. By including $X_{t-1}\alpha$ in the regression we take account of relevant conditioning information that may be important in optimal hedge ratio estimation.

A further simplification is to assume X_{t-1} contains only a constant and one through q lags of spot and futures prices. Then (19) becomes

$$(20) \quad p_t = \gamma + \delta f_t + \sum_{i=1}^{q} \alpha_i p_{t-i} + \sum_{j=1}^{q} \alpha_{q+j} f_{t-j} + \varepsilon_t.$$

The advantage of this specification is that it uses only the same data that would be needed to implement a simple regression approach (i.e. data on p_t and f_t). Although estimating (20) does require multiple regression rather than simple regression, the marginal cost of using (20) over taking a simple regression approach is essentially zero. But estimation of (20) may lead to significant improvements in optimal hedge ratio estimation since it takes account of (at least some) relevant conditioning information.

So far we have been dealing with a very general equilibrium model defined by (5) and (6). Now suppose we want to impose the restriction that the futures market is unbiased, but we still want to specify a general model for spot prices. This leads to a data generating model of the form

$$(21) \quad p_t = X_{t-1}\alpha + u_t$$

$$(22) \quad \Delta f_t = v_t$$

where, as before, u_t and v_t have contemporaneous covariance matrix Ω. The generalized approach to optimal hedge ratio estimation is slightly more complicated in this case since the two equations have different regressors and OLS is no longer an appropriate estimator. A seemingly unrelated regression approach is now required. However, there is no real difficulty. Just estimate α by maximum likelihood under the assumption of normally distributed errors and then use the maximum likelihood residuals in (8) instead of the OLS residuals to get the generalized optimal hedge ratio estimator:

$$(23) \quad \hat{r}_u = \frac{\Delta f' \hat{u}_{ml}}{\Delta f' \Delta f}$$

where $\hat{r}_u$ is the hedge ratio estimated under the assumption of an unbiased futures market and $\hat{u}_{ml}$ is a vector of residuals from estimating (21) and (22) by maximum likelihood.

Is there a direct single-equation approach to estimation in this case? Consider the following proposition:

Proposition 2 Given the data generating process

$$p_t = X_{t-1}\alpha + u_t$$

$$\Delta f_t = v_t$$

where u_t and v_t are serially uncorrelated with contemporaneous covariance matrix Ω, then the generalized optimal hedge ratio estimator (23) is equal to the OLS estimate of δ in the regression equation

$$(24) \quad p_t = \delta \Delta f_t + X_{t-1}\alpha + \varepsilon_t.$$

Proof See the appendix.

Equation (24) is an augmented reduced form with Δf_t instead of f_t as the augmenting variable.

This proposition indicates that if you have prior information that the futures market is unbiased, but have a more general model specified for spot prices, then OLS estimation of δ in (24) will give a generalized optimal hedge ratio estimator with desirable properties. As before, a simplification would be to have only a constant and one through q lags of spot and futures prices in X_{t-1}. In this case, one would estimate

$$(25) \quad p_t = \gamma + \delta \Delta f_t + \sum_{i=1}^{q} \alpha_i p_{t-i} + \sum_{j=1}^{q} \alpha_{q+j} f_{t-j} + \varepsilon_t$$

and use the OLS estimate of δ as the optimal hedge ratio. Again, this procedure has zero marginal cost over using simple regression but may lead to an improved optimal hedge ratio estimate.

Hypothesis Testing and Model Selection

Up until now we have assumed that the form of the equilibrium model (5) and (6) is known. In practice this will rarely be the case. Thus, choosing a model specification is an important aspect of generalized optimal hedge ratio estimation. This involves specifying the variables and lag lengths to include in X_{t-1}. As is usually the case, the approach to model specification is somewhat ad hoc with economic theory, hypothesis testing, and common sense used as guides.

The first set of variables that should be included in X_{t-1} is lagged values of p_t and f_t. Both dynamic economic theory and common sense suggest that past values of these series may help predict future values. Further, we may be able to place restrictions on the way these lagged values are specified. In particular, if p_t and f_t are nonstationary then this implies a set of unit root restrictions on α and β (Dickey, Bell and Miller).

There is a growing number of sophisticated methods of testing for unit roots (Phillips). Here we outline a simple procedure due to Dickey and Fuller. Consider models of the form

$$\Delta p_t = \alpha_0 p_{t-1} + \sum_{i=1}^{q} \alpha_i \Delta p_{t-i} + u_t$$

$$\Delta f_t = \beta_o f_{t-1} + \sum_{i=1}^{q} \beta_i \Delta f_{t-i} + v_t.$$

Dickey and Fuller suggest estimating these models using OLS and then testing the null hypothesis that α_0 (and β_0) are zero against the alternative that they are negative. The usual t-statistic can be used but under the null of a unit root the usual t distribution is inappropriate. Dickey and Fuller provide the correct probability tables.

If evidence of a unit root is found in both the spot and futures price equations then equations (5) and (6) can be specified using first differences, Δp_t and Δf_t. If valid, imposing this restriction will improve estimation efficiency. Estimation then proceeds as before but using Δp_t and Δf_t instead of p_t and f_t.[7] For example, equations (20) and (25) now have the identical form:

$$(26) \quad \Delta p_t = \gamma + \delta \Delta f_t + \sum_{i=1}^{q} \alpha_i \Delta p_{t-i} + \sum_{j=1}^{q} \alpha_{q+j} \Delta f_{t-j} + \varepsilon_t.$$

Thus, when p_t and f_t have a unit root OLS estimation of δ in (26) would give the generalized optimal hedge ratio estimate irrespective of whether the futures market is biased or unbiased.[8]

Another problem is choosing the lag length q. Economic theory usually provides little guidance on this question but various statistical criteria have been suggested. One approach is to start with q deliberately large and test down using sequential likelihood ratio tests (Harvey, p. 279). Various procedures based on minimizing an objective function (e.g. Akaike's information criterion) have also been suggested (see Judge et al. Section 16.6).

The final choices are which variables to include other than lagged prices, whether these variables should be differenced, and specifying their appropriate lag lengths. Economic theory will guide which variables to consider and the way in which they enter can be determined in the same way as for lagged prices.

Model specification is perhaps the most difficult aspect of the generalized approach to optimal hedge ratio estimation. However, the simple regression approaches to optimal hedge ratio estimation do not overcome this difficulty. They too may be misspecified since they imply certain restrictive forms of equilibrium spot and futures price equations. In fact, since these restricted equilibrium models are just special cases of the more generally specified time-series models suggested in our generalized estimation method, the simple regression approaches are much more susceptible to specification error than our approach. This is because they will usually omit relevant variables.

An Example

Optimal hedge ratios were estimated for corn, soybean and wheat storage in Michigan to illustrate the generalized approach. The hedger is assumed to be an agent that stores the commodity at harvest and intends to sell at the most advantageous time prior to the next harvest. To hedge, the agent sells futures in a contract maturing just before the next harvest (July for corn and soybeans and May for wheat). He or she then liquidates portions of the spot and futures positions at whatever time prior to the next harvest is deemed appropriate. It is assumed that the agent decides on a weekly basis what spot and futures positions to hold for the coming week. Thus, weekly data are appropriate.

Spot price data are for the Saginaw market in Michigan and were obtained from Mid-States Terminals, Toledo, Ohio. Futures price data are for the Chicago Board of Trade and were obtained from various issues of the Chicago Board of Trade Statistical Annual. These data are weekly observations taken at the mid-week (Wednesday) closing price on the relevant market. In addition to price data, weekly data on total commodity stocks available on the Friday of each week were also used in some of the optimal hedge ratio models. These data were obtained from various issues of the Chicago Board of Trade Statistical Annual. The estimation period runs from July 1977 to July 1985, a total of 417 observations.[9]

As shown above, there are different procedures for estimating optimal hedge ratios depending on the equilibrium model specified for spot and futures price determination. We start with some very simple models and build up to consider more general alternatives. Estimated optimal hedge ratios for all model specifications discussed here are shown in table 1 and were estimated using the single equation methods outlined above.

The simplest possible approach is to use (9), the slope coefficient from a simple regression of spot price levels on futures price levels, as a hedge ratio estimate. This gives 98% for corn, 87% for soybeans and 61% for wheat (table 1). But since the equilibrium model, (10) and (11), implied by this approach appears so contrary to the actual behavior of commodity prices, we would expect these to be poor estimates of optimal hedge ratios.

The first generalization, which is very easy to implement, involves regressing p_t on Δf_t as well as lagged values of p_t and f_t [see model (2) in table 1]. This implies an equilibrium model for spot price determination of the form

$$p_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i p_{t-i} + \sum_{j=1}^{q} \alpha_{q+j} f_{t-j} + u_t$$

$$f_t = f_{t-1} + v_t$$

(see Proposition 2). Thus, the futures market is assumed to be unbiased but a fairly general model is specified for spot price determination. Since evidence is presented below that p_t and f_t are nonstationary, identifying the lag length q by the usual hypothesis testing approach is fraught with difficulties. However, it was found that the estimated hedge ratio was very insensitive to additional lags once five lags had been included. Results reported in table 1 use a lag length of 15. The estimated hedge ratios from this generalized model change fairly dramatically compared to simple regression with price levels. The corn hedge ratio decreases by 11 percentage points, soybeans increases by 15 percentage points, and wheat increases by 33 percentage points. Since we can be more confident in results from the general model, we conclude that simple regression in price levels would lead to large errors in optimal hedge ratio estimation in this application.[10]

The second generalization adds lagged observations on the relevant commodity stock levels to the model just discussed [see model (3) in table 1]. This implies an equilbrium model of the form

$$p_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i p_{t-i} + \sum_{j=1}^{q} \alpha_{q+j} f_{t-j} + \sum_{k=1}^{q} \alpha_{2q+k} s_{t-k} + u_t$$

$$f_t = f_{t-1} + v_t$$

where s_t is commodity stocks in period t. The estimated hedge ratios change little compared to the previous case in which the stock variables were excluded (table 1). We conclude that in this application the stock variables can be excluded without introducing serious errors in estimated optimal hedge ratios. This is because the information contained in lagged commodity stocks does not change significantly the estimated conditional covariance matrix of spot and futures prices.

We experimented by including other variables in the generalized optimal hedge ratio estimation equation. For example, we specified multicommodity models in which lagged prices and stock levels of corn and wheat were included in the soybean equation etc. None of these specifications changed the estimated hedge ratios appreciably. Thus, for this application equation (2) in table 1 was found to be an adequate model for optimal hedge ratio estimation, given that the equilibrium spot price equation is specified in levels.

It is often suggested that commodity prices are nonstationary in the levels and that they have a unit root so their first differences are stationary. If this is true, more efficient optimal hedge ratio estimation may be achieved by imposing the unit root restriction and building models in first differences.[11] Table 2 contains the results from Dickey-Fuller tests for nonstationarity applied to spot prices, futures prices, and storage levels for each commodity. A lag length of 15 was used in the estimated equations, but experimenting with lag lengths between 5 and 20 revealed the inferences are not sensitive to this assumption. The first difference model in table 2 tests the null hypothesis that the levels are nonstationary while the second difference model tests the null hypothesis that the first differences are nonstationary. There is evidence of a unit root in all of the series tested.

The simplest optimal hedge ratio model using first differences is to regress Δp_t on Δf_t and use the slope parameter (12). This gave optimal hedge ratios of 89% for corn, 102% for soybeans and 95% for wheat. Perhaps surprisingly, these estimates are very close to those obtained with the generalized model using price levels. Furthermore, augmenting the simple regression of Δp_t on Δf_t with lagged price changes and lagged commodity stocks did not change the estimated hedge ratio appreciably (table 1). Further experimentation with model specification using a multicommodity approach also had little effect on estimated hedge ratios.

These results indicate that the simple regression in price changes is a reasonable approach to optimal hedge ratio estimation in this application. In turn, this implies that (13) and (14) represent a reasonable model of spot and futures price determination for the commodities studied, for the purpose of optimal hedge ratio estimation. Thus, although spot commodity prices probably do not follow a random walk with drift, the random walk model with drift appears to be a reasonable approximation in this case if the aim is to estimate optimal hedge ratios. However, a word of caution is in order. This is not a general result and other applications may require the use of the generalized approach to optimal hedge ratio estimation, even if modeled in first difference form. Furthermore, without a careful empirical investigation there is no way to anticipate whether simple regression in price changes is adequate or whether a generalized approach is necessary. The easiest way to answer this question is to apply the generalized approach and compare results.

The approaches discussed earlier for testing different lag lengths and model specifications could be applied to the (stationary) first difference model. However, in our example the estimated hedge ratios are not sensitive to lag length or model specification, once first differences have been taken. Thus these tests were not applied.

The simple regression in returns was not estimated because it is a logically inconsistent approach to optimal hedge ratio estimation in this application. The reason is that the simple regression in returns implies that the conditional distribution of spot and futures prices in heteroscedastic, as discussed above. But if this conditional distribution is really heteroscedastic, then the optimal hedge ratio (4) will change over time. Thus the constant slope parameter in the simple regression of spot market returns on futures market returns cannot be the optimal hedge ratio defined by (4).

Conclusion

The correct approach to optimal hedge ratio estimation depends on the data generating process that determines equilibrium spot and futures prices. Moreover, the usual simple regression approaches using price levels, price changes, and returns, implicitly assume particular forms for this data generating process. These forms appear to be quite restrictive *a priori*. A generalized approach to optimal hedge ratio estimation allows for a more flexible specification of equilibrium pricing models.

This paper derived a generalized approach to optimal hedge ratio estimation and provided direct single-equation OLS techniques for implementing it. The approach is very easy to apply and will usually lead to improved optimal hedge ratio estimates. The generalized approach was illustrated with an example of storage hedging in corn, soybeans, and wheat. The simple regression model in price levels was found to be inadequate but, perhaps surprisingly, the simple regression model in price changes provided estimates that were very close to those obtained with the generalized approach. Of course, this is not a general result and simple regression in price changes may be inadequate in other applications.

Two main extensions of the paper suggest themselves. The first is to expand the generalized approach to allow for conditionally heteroscedastic errors. That is, allow the conditional covariance matrix, Ω, to change systematically over time. This would lead to optimal hedge ratios that also may change over time. Therefore, optimal hedge ratio estimates would need to allow for this time variation. The second extension is to use out of sample data to compare the performance of simple regression hedging rules and generalized hedging rules for particular applications. The comparisons would be in terms of impacts on the mean and variance of total returns. This would provide information on the size of economic benefits from moving to the generalized approach.

FOOTNOTES

1 This implies $c(0) = 0$ and $c(q_{t-1}) < 0$ for $q_{t-1} < 0$.

2 Since $c(q_{t-1})$ is convex these conditions are necessary and sufficient for a maximum.

3 For optimal hedge ratio estimation we do not need to be concerned directly with this structural model. The reduced form is sufficient since the only information we are seeking is the conditional covariance matrix of the prediction errors u_t and v_t.

4 Hypothesis testing and inference in models with nonstationary variables is problematical because the properties and distributions of standard test statistics are generally unknown. However, the usual estimators remain appropriate. In fact, they are generally "superefficient" since they converge more rapidly to actual parameter values as sample size increases. We will return to the issue of nonstationarity when discussing hypothesis testing and model selection.

5 To see this just follow the same steps used in the previous cases of simple regression with price levels and simple regression with price changes.

6 There is also the problem of specifying which variables to include in X_{t-1}. This issue will be taken up in the next section.

7 In this case we are actually computing the conditional covariance matrix of Δp_t and Δf_t instead of p_t and f_t. However, since p_{t-1} and f_{t-1} are known at time t-1, these two conditional covariance matrices are identical.

8 Of course, if the futures market is biased then the estimated hedge ratio will be minimum variance but not mean-variance efficient.

9 Although there are a total of 417 observations, not all of these could be used to estimate models that contained lagged futures price variables. The problem is that the futures price series has annual breaks when there is a switch from one futures contract to the next (e.g. switching from the July 1980 contract to the July 1981 contract). It would clearly be unwise to allow observations on, say, the maturing July 1980 contract to be treated as lagged observations on the July 1981 contract. Thus, if there are 15 lagged futures prices in the model, then the first 15 observations on each contract were not used, except in the computation of lagged variables.

10 There is a question concerning the sense in which these errors, ranging from 11 to 33 percentage points, are "large." These seem like relatively important amounts but it is unclear exactly how much additional risk exposure is implied by errors of this magnitude. As a rule of thumb, we define deviations of greater than five percentage points in optimal hedge ratios as "large" and those less than five percentage points as "small."

11 This would not imply that the models specified in levels which have just been discussed are wrong. It merely suggests that if the unit root restriction holds it will be more efficient to impose it.

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APPENDIX

Proof of Proposition 1

The generalized optimal hedge ratio estimator (8) is

$$\hat{r} = \frac{\hat{v}'\hat{u}}{\hat{v}'\hat{v}}.$$

Letting $X = [X_1, X_2, \ldots, X_T]'$, this can be written

$$\text{(A.1)} \quad \hat{r} = \frac{(f-X\hat{\beta})'(p-X\hat{\alpha})}{(f-X\hat{\beta})'(f-X\hat{\beta})}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the OLS estimates of α and β. Using the definition of the OLS estimator, then (A.1) becomes

$$\hat{r} = \frac{f'M'Mp}{f'M'Mf}$$

where $M = I - X(X'X)^{-1}X'$. Since M is an indempotent matrix, this can be written

$$\text{(A.2)} \quad \hat{r} = \frac{f'Mp}{f'Mf}.$$

We now show that the OLS estimate of δ in (19) is equivalent to the right hand side of (A.2).

Let $Z = [f\ X]$, $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T]$, and $\gamma = [\delta\ \alpha']'$. Then (19) can be expressed equivalently as

$$\text{(A.3)} \quad p = Z\gamma + \varepsilon.$$

The OLS estimator of γ in (A.3) is

$$\hat{\gamma} = (Z'Z)^{-1}Z'p.$$

Using the definition of Z and computing $(Z'Z)^{-1}$ but partitioned inverse (see Theil p. 17), the first row of $\hat{\gamma}$ is

$$\hat{\delta} = \frac{[f' - f'X(X'X)^{-1}X']p}{f'Mf}.$$

Using the definition of M, this can be written

$$\hat{\delta} = \frac{f'Mp}{f'Mf}.$$

This shows the OLS estimate of δ in (19) gives the desired optimal hedge ratio estimator (8).

Proof of Proposition 2

The generalized optimal hedge ratio estimator (23) under the assumption of an unbiased futures market is

$$\text{(A.4)} \quad \hat{r}_u = \frac{\Delta f'\hat{u}_{ml}}{\Delta f'\Delta f}.$$

Now

$$\text{(A.5)} \quad \hat{u}_{ml} = p - X\hat{\alpha}_{ml}$$

where $\hat{\alpha}_{ml}$ is the maximum likelihood estimate of α from (21) and (22). Differentiating the likelihood function for this model with respect to $\hat{\alpha}_{ml}$ gives the first order condition

$$(A.6)\quad \hat{\alpha}_{ml} = (X'X)^{-1}X'p - \hat{r}_u(X'X)^{-1}X'\Delta f$$

where, by definition, $\hat{r}_u$ is the estimated covariance between Δf_t and u_t divided by the estimated variance of Δf_t. Substituting (A.6) into (A.5) gives

$$(A.7)\quad \hat{u}_{ml} = p - X(X'X)^{-1}X'p + \hat{r}_u X(X'X')^{-1}X'\Delta f.$$

Next, substitute (A.7) into (A.4) to get

$$\hat{r}_u = \frac{\Delta f'[p - X(X'X)^{-1}X'p + \hat{r}_u X(X'X)^{-1}X'\Delta f]}{\Delta f'\Delta f}$$

or

$$\hat{r}_u \Delta f'\Delta f = \Delta f'Mp + \hat{r}_u \Delta f'X(X'X)^{-1}X'\Delta f$$

Collecting terms in $\hat{r}_u$,

$$\hat{r}_u[\Delta f'\Delta f - \Delta f'X(X'X)^{-1}X'\Delta f] = \Delta f'Mp$$

or

$$r_u \Delta f'M\Delta f = \Delta f'Mp.$$

Dividing through by the scalar $\Delta f'M\Delta f$ gives

$$(A.8) \quad \hat{r}_u = \frac{\Delta f'Mp}{\Delta f'M\Delta f}.$$

We now show that the OLS estimate of δ in (24) is equivalent to the right hand side of (A.8).

Redefine Z to be $Z = [\Delta f \; X]$ and, as in the proof of Proposition 1, let $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T]'$ and $\gamma = [\delta \; \alpha']'$. Then (24) can be expressed equivalently as

$$(A.9) \quad p = Z\gamma + \varepsilon.$$

Using partioned inverse, as in the proof of Proposition 1, the first row of the OLS estimate of γ is

$$\hat{\delta} = \frac{[\Delta f' - \Delta f'X(X'X)^{-1}X']p}{\Delta f'M\Delta f}.$$

Using the definition of M then

$$\hat{\delta} = \frac{\Delta f'Mp}{\Delta f'M\Delta f}.$$

This shows that the OLS estimate of δ in (24) gives the desired optimal hedge ratio estimator (23), under the restriction of an unbiased futures market.

Table 1

Estimated Optimal Hedge Ratios Under Alternative Model Specifications

Model	Corn	Hedge Ratio $\hat{\delta}$ Soybeans	Wheat
(1) $p_t = \gamma + \delta f_t + \varepsilon_t$	0.98	0.87	0.61
(2) $p_t = \gamma + \delta \Delta f_t + a(L)p_{t-1} + b(L)f_{t-1} + \varepsilon_t$	0.87	1.02	0.94
(3) $p_t = \gamma + \delta \Delta f_t + a(L)p_{t-1} + b(L)f_{t-1} + c(L)s_{t-1} + \varepsilon_t$	0.85	1.04	0.94
(4) $\Delta p_t = \gamma + \delta \Delta f_t + \varepsilon_t$	0.89	1.02	0.95
(5) $\Delta p_t = \gamma + \delta \Delta f_t + a(L)\Delta p_{t-1} + b(L)\Delta f_{t-1} + \varepsilon_t$	0.85	1.02	0.94
(6) $\Delta p_t = \gamma + \delta \Delta f_t + a(L)\Delta p_{t-1} + b(L)\Delta f_{t-1} + c(L)\Delta s_{t-1} + \varepsilon_t$	0.84	1.03	0.93

<u>Notes</u> a(L), b(L) and c(L) are polynomials in the lag operator L, defined by $Ly_t = y_{t-1}$, p_t is the spot price, f_t is the futures price and s_t is commodity stocks at time t.

Table 2

Dickey-Fuller Tests

	First Difference Model Parameter	t	p-value	Second Difference Model Parmeter	t	p-value
Corn						
Spot Price	-0.000	-0.01	0.96	-0.784	-4.63	< 0.01
Futures Price	-0.001	-0.82	0.70	-0.729	-3.41	0.01
Stocks	-0.002	-1.11	0.60	-0.428	-5.41	< 0.01
Soybeans						
Spot Price	-0.000	-0.11	0.92	-1.074	-5.35	< 0.01
Futures Price	-0.004	-2.57	0.10	-1.364	-5.42	< 0.01
Stocks	-0.004	-1.31	0.50	-0.491	-5.13	< 0.01
Wheat						
Spot Price	-0.001	0.30	0.98	-1.225	-4.80	< 0.01
Futures Price	-0.004	-2.16	0.20	-0.931	-3.58	< 0.01
Stocks	0.001	-1.31	0.50	-0.263	-5.22	< 0.01

Notes <0.01 indicated less than 0.01.