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S-P-no-300

On the Properties of Market Equilibrium Functions

by

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I. Introduction

Partial equilibrium models of firm or consumer behavior are well established in the literature (e.g. Fuss and McFadden; Deaton and Muellbauer). These models, which treat prices as exogenous explanatory variables, have provided a basis for much empirical analysis of firm/-consumer and market level supply/demand functions. At the market level, however, it is often of interest to investigate supply/demand adjustments when some of the prices are allowed to respond to changing supply/demand shifters given market equilibrium.

Heiner showed that the short run, market equilibrium factor demands which result from a single output industry facing perfectly elastic input supplies will conform to the traditional law of demand (downward sloping factor demand schedules) _if_ the industry's output demand schedule is "normal" (that is, falling with respect to output price). Also, Heiner shows that short run factor behavior of such an industry is bounded by two well known, polar cases: first, the case where industry output demand is totally inelastic (i.e., industry output is held constant); and second, the case where industry output demand is infinitely elastic (i.e., output price is held constant).

Heiner's proof of the commonly held assertions concerning the law of demand for market level, short run factor behavior is seminal in that firms are not analyzed in partial equilibrium isolation. In contrast to the earlier methodology associated with Samuelson (1947), Heiner shows that if firms adjust in the market equilibrium context of an industry comprised of firms whose collective output response can affect output price, then the traditional partial equilibrium results concerning the law of demand for such individual firms may no longer hold. In this context, Heiner's results indicate that, given a "normal" output demand function, the traditional law of demand would apply to market level (industry) short run factor behavior but not necessarily to individual firm behavior.

Braulke (1984, 1987) generalizes Heiner's results to the short run multiproduct industry context where an arbitrary number and combination of input and output markets confronting the industry have less than infinitely elastic demand or supply schedules. In particular, the Braulke results indicate that a full analogy to the traditional short run, partial equilibrium theory of the multi-output, multi-input firm (as in Silberberg or Varian) characterizes short-run industry behavior under market equilibrium if one assumes the industry faces "normal conditions" in all it's markets. Importantly, as in the single output case above, the traditional partial equilibrium results at the individual firm level may no longer hold in this market equilibrium context.

However, the "normal conditions" stated by Heiner and Braulke may not be satisfied for netputs that are final products. In particular, consumer theory does not imply that the Marshallian demand functions will satisfy the "normal conditions", (i.e., the symmetric negative semi-definiteness of Marshallian price effects). This suggests that the Heiner-Braulke results would not be valid in general when concerned with markets for consumer goods. Also, their results are restricted to the behavior of a single industry under market equilibrium. Again, under certain conditions (discussed below), their industry results may not hold for multi-industry market equilibrium (even if the "normal conditions" are satisfied). This indicates a need to generalize the properties of industry behavior under market equilibrium.

The objective of this paper is to develop further the implications of market equilibrium, where some prices are endogenously determined, for the analysis and modeling of production/consumption behavior. For example, in a small open economy, while prices for non-tradeables are endogenous, the prices for internationally traded goods are exogenous. In this context, it is of interest to consider the effects of changing exogenous prices on resource allocation in a market equilibrium framework. Following Braulke, we focus on multi-input/multi-output industries and analyze several issues that have apparently not been addressed previously in the literature. In particular, we add a household sector to the analysis (i.e., as an "industry" aggregated over the individual level constrained optimization of utility) and explicitly consider the allocation problem when the output (input) of an industry is purchased (sold) by more than one other industry. Thus, we model the household sector as an "industry" which consumes the outputs of several producing industries and provides labor for the production activities. The analysis also includes the case of products with multiple uses (examples: household labor which can be sold to any of the producing industries; oil which can be used for residential, manufacturing and other commercial uses; etc.). We focus our attention on a short-run analysis where the number of firms in each industry or the number of consumers is given. While deriving a number of new results, we show that several of the Heiner/Braulke results do not hold under these generalizations.

The plan of the paper is as follows. Section II developes the notation and characterizes the sector level market equilibrium comprised of separate multi-output/multi-input industries including the household sector. Sections III and IV derive and summarize the properties of the associated compensated and uncompensated market equilibrium functions, respectively. Some implications of the framework, in particular for multi-market welfare analysis and the empirical specification of market equilibrium supply/demand functions, are presented in Section V. Last, concluding remarks are found in Section VI.

II. The Characterization of Market Equilibrium

Consider an economy constituted of (n-1) industries marketing a vector of commodities purchased and sold in competitive markets. Associate with each commodity an index $m = 1,2,\ldots$, and denote by M the set of these indexes, $M = \{1,2,\ldots\}$. Define the associated price vector $p = \{p_m: m \in M\}$ where p_m is the market price of the $m^{\underline{th}}$ commodity.

Consider a particular industry, say the $j^{\underline{th}}$ industry, facing a production technology represented by the implicit concave production function $f(y^j,\alpha^j) = 0$, where $y^j = \{y^j_m: y^j_m \neq 0$ in at least one situation, $m \in M\}$ is the netput vector of the $j^{\underline{th}}$ industry, y^j_m being the quantity of the $m^{\underline{th}}$ netput and α^j is a technology parameter. By definition, $y^j_m =$ 0 in all situations would correspond to commodities not used nor produced by the $j^{\underline{th}}$ industry while $y^j_m \neq 0$ in at least one situation implies that the corresponding $m^{\underline{th}}$ commodity is a netput of the $j^{\underline{th}}$ industry. We use the convention that positive elements of y^j denote outputs while negative elements denote inputs.

Let $p^j = (p_m: y_m^j \neq 0$ in at least one situation, $m \in M)$ be the vector of market prices for the vector y^j. Assuming that economic decisions in the $j^{\underline{th}}$ industry are made to maximize profit,1/ we have

$$V^j(p^j,\alpha^j) = p^{j\prime}\bar{y}^j(p^j,\alpha^j) = \underset{y^j}{\text{Max}}\ (p^{j\prime}y^j: f(y^j,\alpha^j) = 0) \qquad (1)$$

where $\bar{y}^j(p^j,\alpha^j)$ is the profit maximizing netput decision vector and $V^j(p^j,\alpha^j)$ is the indirect profit function or quasi-rent for the $j^{\underline{th}}$ industry, $j = 1,\ldots,n-1$.

Expression (1) defines a partial equilibrium model of production where decisions depend on relevant market prices which are treated as exogenous. The economic implications of model (1) are well known (e.g., Lau; Fuss and McFadden):

> The indirect profit function $V^j(p^j,\alpha^j)$ is homogeneous of degree one and convex in prices. Moreover, under differentiability, it satisfies Hotelling's lemma (2a)
>
> $$\frac{\partial V^j(p^j,\alpha^j)}{\partial p^j} = \bar{y}^j(p^j,\alpha^j).$$

This implies the following:

> The choice functions $\bar{y}^j(p^j,\alpha^j)$ are homogenous of degree zero in prices p^j and, under differentiability, the matrix $\partial\bar{y}^j/\partial p^j$ is symmetric, positive semi-definite. (2b)

While some of the m commodities produced by the (n-1) industries are intermediate products used in the production of other goods, others may be final products purchased by households as input/factors used in household production of utility. Also, households sell labor to the producing industries.2/ Denote by y^n the quantity vector of final products consumed and labor supplied by households and by p^n the vector of corresponding market prices. Since final products are purchased by households, by convention we define the elements of the vector y^n to be negative for consumer goods and positive for labor supply.

Assume that consumption-labor decisions are made in a way consistent with a representative household maximizing utility subject to a budget constraint.3/ Let $U(y^n,\alpha^n)$ be the (direct) utility function of the representative household, α^n being a preference shifter, and denote exogenous non-labor household income by x. Then, the household decisions can be represented by

$$W(p^n,x,\alpha^n) = U(\bar{y}^n(p^n,x,\alpha^n),\alpha^n) = \underset{y^n}{\text{Max}}\ \{U(y^n,\alpha^n):\ x + p^{n\prime}y^n = 0\} \qquad (3)$$

where $\bar{y}^n(p^n,x,\alpha^n)$ are Marshallian choice functions and $W(p^n,x,\alpha^n)$ is the indirect utility function. Expression (3) defines a partial equilibrium model where consumption-labor decisions depend on exogenous non-labor income x and prices p^n treated as exogenous variables. The economic implications of model (3) are well known (e.g. Deaton and Muellbauer). Given that $U(y^n,\alpha^n)$ is decreasing and quasi-concave in y^n, the indirect utility function $W(p^n,x,\alpha^n)$ is homogenous of degree zero and quasi-convex in (p^n,x). Also, the choice functions $\bar{y}^n(p^n,x,\alpha^n)$ are homogeneous of degree zero in (p^n,x).

Additional properties of $\bar{y}^n(p^n,x,\alpha^n)$ can be obtained by considering the function

$$V^n(p^n,\alpha^n,\bar{U}) = p^{n\prime}\bar{y}^n_u(p^n,\alpha^n,\bar{U}) = \max_{y^n} \{p^{n\prime}y^n : \bar{U} \leq U(y^n,\alpha^n)\} \quad (4)$$

where $\bar{y}^n_u(p^n,\alpha^n,U)$ are compensated Hicksian choice functions holding utility constant and $V^n(p^n,\alpha^n,U)$ is the negative of the expenditure function. The functions $W(p^n,x,\alpha^n)$ and $V^n(p^n,\alpha^n,U)$ are dual: they are inverse functions of each other as $W(p^n,-V^n(p^n,\alpha^n,U),\alpha^n) = U$ or $-V^n(p^n,\alpha^n,W(p^n,x,\alpha^n)) = x$.

The following properties will be of interest in this paper (e.g., Deaton and Muellbauer):

> The function $V^n(p^n,\alpha^n,U)$ is linear homogeneous, decreasing and convex in p^n. Under differentiability it satisfies Shephard's lemma (5a)
>
> $$\frac{\partial V^n(p^n,\alpha^n,U)}{\partial p^n} = \bar{y}^n_u(p^n,\alpha^n,U).$$

Noting the identity $\bar{y}^n(p^n,-V^n(p^n,\alpha^n,U),\alpha^n) = \bar{y}^n_u(p^n,\alpha^n,U)$, this implies the following:

> The Hicksian choice functions $\bar{y}^n_u(p^n,\alpha^n,U)$ are homogeneous of degree zero in prices p^n and, under differentiability, satisfy the Slutsky equation:
>
> $$\frac{\partial \bar{y}^n_u}{\partial p_n} = \frac{\partial \bar{y}^n}{\partial p_n} - \frac{\partial \bar{y}^n}{\partial_x}.\bar{y}^{n\prime} = \text{a symmetric, positive} \quad (5b)$$

semi-definite matrix.

Now, consider a small open economy where the prices for internationally traded goods are exogenous, while the prices of nontradeables are endogenous. In order to introduce a subset of prices which are endogenously determined by the market equilibrium context, partition the set of commodities M into two subsets: the subset K associated with the endogenous prices, and the subset $R = M\backslash K$, the complement of K relative to M. Denote by $J = \{1,\ldots,n\}$ the set of industries and consumer sector in the economy either purchasing or selling the subset of commodities K, i.e. $J = \{j: y_m^j \neq 0$ in at least one situation, for all $j \in J$ and for any $m \in K\}$. Let $p^j = (p_K^j, p_R^j)$ where $p_R^j = \{p_m: y_m^j \neq 0$ in at least one situation, $m \in R$, $R = M\backslash K\}$ is the price vector for all netputs other than those in K facing the $j^{\underline{th}}$ industry. Thus, $p_K = \{p_K^j, j\epsilon J\}$ represents the vector of endogenous prices associated with (6) below, while $p_R = \{p_R^j, j\epsilon J\}$ represents the vector of "exogenous" prices facing the industries in J.[4/]

Given this notation and allowing each commodity to be possibly sold or purchased by more than one industry, market equilibrium for the endogenous commodity set K is characterized by

$$\sum_{j \in J} y_m^j = 0, \quad m \in K \tag{6}$$

which simply states that excess demand is zero for all y_m, $m \in K$. Note that equation (6) allows for products with multiple uses in different industries (including the household sector), each industry competing for the allocation of these products. For example, this is typically the case for household labor which is allocated among the various producing industries. In addition, this formulation allows for conglomerates as industries producing outputs also produced by more specialized firms. Substituting $\bar{y}^j(p^j,\alpha^j)$, $j \neq n$, and $\bar{y}^n(p^n, x, \alpha^n)$ into (6) and solving for p_K, the vector of all endogenous prices facing the industries in J, (and, assuming that a solution exists via the implicit function theorem) yields the uncompensated market equilibrium price functions

$$p_K(p_R, x, \alpha), \tag{7}$$

where $p_R = \{p_R^j: j \in J\}$ is the vector of all exogenous prices facing the consumer and the industries in the set J, and $\alpha = \{\alpha^j: j \in J\}$.5/ Alternatively, substituting $\bar{y}^j(p^j,\alpha^j)$, and $\bar{y}_u^n(p^n,\alpha^n,U)$ into (6) and solving for p_K (and, assuming that a solution exists via the implicit function theorem) yields the compensated market equilibrium price functions

$$p_{K,u}(p_R, \alpha, U). \tag{8}$$

Expressions (7) and (8) allow the following definitions of market equilibrium (to be contrasted with traditional partial equilibrium) choice functions for all commodities (netputs) in M by the industries in J:

$$\tilde{y}^j(p_R, x, \alpha) = \bar{y}^j(p_K^j(p_R, x, \alpha), p_R^j, x, \alpha^j), \quad j \in J \tag{9}$$

and

$$\tilde{y}_u^j(p_R, \alpha, U) = \bar{y}_u^j(p_{K,u}^j(p_R, \alpha, U), p_R^j, \alpha^j, U), \quad j \in J, \tag{10}$$

where the lower case superscript, j, denote netputs or prices faced by the j^{th} industry and $\bar{y}_u^j = \bar{y}^j$, $j = 1, \ldots, n-1$. Expression (9) gives uncompensated market equilibrium choice functions while (10) gives compensated equilibrium choice functions holding household utility constant. In either case, the functions do not depend on p_K as the prices p_K now endogenously adjust to changing market conditions given market equilibrium (6). The properties of such functions will be analyzed in detail in the following sections. We will assume throughout the paper that these functions are differentiable.[6/]

Also, we investigate the following aggregate welfare measure

$$V^J(p_R, \alpha, U) = \sum_{j \in J} V^j(p^j_{K,u}(p_R, \alpha, U), p^j_R, \alpha^j, U), \tag{11}$$

which sums the quasi-rents of the industries in the set J, minus household expenditure (holding utility constant), letting the prices p_K adjust to changing market conditions. Such a measure will be of interest in multi-market welfare analysis (see section V below). The properties of these functions are discussed next.

III. Compensated Market Equilibrium Functions

In this section, we analyze the properties of the market equilibrium functions (8), (10) and (11). The properties of the compensated market equilibrium prices $p_{K,u}$ in (8) are presented in the following lemma (see the proof in Appendix A).

Lemma 1: The equilibrium price functions $p_{K,u}(p_R, \alpha, U)$ in (8) are linear homogeneous in p_R and satisfy

$$\frac{\partial p_{K,u}}{\partial p_R} = -\left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K}\right)^{-1} \sum_{j \epsilon J}\left(\frac{\partial \bar{y}^j_{K,u}}{\partial p_R}\right), \tag{12}$$

where the matrix $\left[\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K}\right]$ is assumed non-singular, p_R = $\{p_r: r \epsilon R\}$, and R = M\K, the complement of K in M.

The properties of the aggregate willingness-to-pay function V^J are presented next (see the proof in Appendix A).

<u>Proposition 1</u>:

$V^J(p_R, \alpha, U)$ is homogeneous of degree one and convex in prices p_R. It satisfies

$$\frac{\partial V^J}{\partial p_R} = \sum_{j \epsilon J} \bar{y}^j_{R,u}(p^J_R, \cdot) \tag{13a}$$

where $y_R = \{y_r: r \epsilon R, R = M\backslash K\}$ and $\frac{\partial^2 V^J}{\partial {p_R}^2} = \sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_R}$

is a symmetric, positive semi-definite matrix. Moreover,

$$\frac{\partial V^J}{\partial \alpha} = \sum_{j \epsilon J} p^{j\prime} \left. \frac{\partial \bar{y}^j_u(p^j, \alpha^j, U)}{\partial \alpha} \right|_{p_k = p_{k,u}(p_R, \alpha, U)}. \tag{13b}$$

Proposition 1 states that the market equilibrium function V^J has properties similar to its partial equilibrium counterpart V^j (see (2) and (5)). Moreover, expression (13a) is similar to Hotelling's or Shephard's lemma obtained in a partial equilibrium context (see (2a) and (5a)). More specifically, from (13a), the derivative of V^J with respect to p_R generates the sum of equilibrium choice functions for $\bar{y}_{R,u}$ across the industries in J. If the commodity set R is produced (or purchased) by a single industry, then (13a) can provide a convenient derivation of the specification of the choice function $\tilde{y}_{R,u}$ in this industry. In this case, (13a) and the convexity of V^J in p_R implies that $\left[\frac{\partial \tilde{y}_{R,u}}{\partial p_R}\right]$ is a symmetric, positive semi-definite matrix. This is the result obtained by Heiner. However, in the case where y_r is allocated among several industries, then (13a) implies that $\left[\sum_{j\epsilon J} \frac{\partial \tilde{y}^j_{R,u}}{\partial p_R}\right]$ is a symmetric, positive semi-definite matrix. In this context, it is no longer necessarily true that the individual industry equilibrium supply (demand) functions are upward (downward) sloping or symmetric. From (13a), specifying the function V^J would allow recovery of the aggregate function $(\sum_{j\epsilon J} \tilde{y}^j_{R,u})$, but not the individual industry equilibrium functions $\tilde{y}^j_{R,u}$.

The relationships between the partial equilibrium quasi-rents, V^j, and the aggregate, market equilibrium willingness-to-pay, V^J, are the topic of the following proposition (see the proof in Appendix A).

Proposition 2:

The function $\sum_{j\epsilon J} V^j(p_K, p^j_R, .)$ is more convex than $V^J(p_R, .)$ in the sense that

$$\left[\frac{\partial^2 V^J}{\partial p_R^2} - \frac{\partial^2 \sum_{j\epsilon J} V^j}{\partial p_R^2}\right]$$

is a (symmetric), negative semi-definite matrix.

Given (2), (5) and (13), proposition 2 states that price adjustments through market equilibrium tend to reduce the need for compensated quantity adjustments in the economy. In other words, letting p_K adjust tends to reduce the supply elasticities (or the absolute value of demand elasticities) in the sense that $0 \leq \sum_{j \epsilon J} \frac{\partial \tilde{y}^j_{R,u}}{\partial p_R} \leq \sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_R} \geq 0$. This result extends the Heiner/Braulke analysis to the allocation case where the commodity set R is produced (or purchased) by more than one industry in the set J.

Finally, the properties of the compensated market equilibrium functions $\tilde{y}^j_u$ are presented in the following proposition (the proof is omitted; it follows directly from (10) and (12) and from proposition 2).

Proposition 3:

The market equilibrium functions $\tilde{y}^j_u(p_R, .)$ are homogeneous of degree zero in p_R. Moreover, given $R \epsilon M \backslash K$, they satisfy

$$\frac{\partial \tilde{y}^j_{R,u}}{\partial p_R} = \frac{\partial \bar{y}^j_{R,u}}{\partial p_R} - \frac{\partial \bar{y}^j_{R,u}}{\partial p_K} \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K} \right)^{-1} \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_R} \right) \quad (14)$$

where

$$\left[\sum_{j \epsilon J} \frac{\partial \tilde{y}^j_{R,u}}{\partial p_R} - \sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_R} \right]$$

is a symmetric, negative semi-definite matrix.

If the set of commodities R is produced (or purchased) by a single industry, expression (14) reduces to Heiner's results. However, (14) indicates that having a commodity allocated among several industries (e.g. oil might be a typical example) modifies Heiner's results in some significant ways. In particular, it shows that the matrix $\left(\frac{\partial \tilde{y}^j_{R,u}}{\partial p_R}\right)$ need not be symmetric nor positive semi-definite. In other words, industry equilibrium choice functions $\tilde{y}^j_u$ would not have the properties discussed by Heiner. In this context, the positive semi-definiteness would apply only to the matrix $\left[\sum_{j \varepsilon J} \frac{\partial \tilde{y}^j_{R,u}}{\partial p_R}\right]$ (from proposition 1).

IV. Uncompensated Market Equilibrium Functions

This section focuses on the properties of uncompensated market equilibrium functions (7) and (9), and on the role of income effects. The properties of the uncompensated equilibrium market prices p_K in (7) are presented in the following lemma (the proof is similar to Lemma 1 and is omitted).

Lemma 2: The equilibrium price functions $p_K(p_R, x, \alpha)$ in (7) are linear homogeneous in (p_R, x) and satisfy

$$p_{K,u}(p_R, U, \alpha) = p_K(p_R, -V_n(p^n_R, p^n_{K,u}(p_R, U, \alpha), U, \alpha), \alpha) \tag{15a}$$

and

$$\frac{\partial p_K}{\partial \beta} = -\left(\sum_{j \varepsilon J} \frac{\partial \bar{y}^j_K}{\partial p_K}\right)^{-1} \left(\sum_{j \varepsilon J} \frac{\partial \bar{y}^j_K}{\partial \beta}\right), \; \beta = (p_R, x, \alpha), \tag{15b}$$

where the matrix $\left[\sum_{j \varepsilon J} \frac{\partial \bar{y}^j_K}{\partial p_K}\right]$ is assumed non-singular.

The properties of the uncompensated equilibrium functions (9) are presented next (see the proof in Appendix A).

Proposition 4:

The market equilibrium functions $\tilde{y}^j(p_R, x, .)$ are homogeneous of degree zero in (p_R, x). Moreover, for $j \epsilon J$, they satisfy

$$\tilde{y}^j_u(P_R, U, \alpha) = \tilde{y}^j(P_R, -V^n(p^n_R, p^n_{K,u}(P_R, \alpha, U), U, \alpha), \alpha), \quad (16)$$

implying the Slutsky-like equation

$$\frac{\partial \tilde{y}^j_u}{\partial p_R} = \frac{\partial \tilde{y}^j}{\partial p_R} - \frac{\partial \tilde{y}^j}{\partial x}\left(\tilde{y}^n_R + \tilde{y}^n_K \frac{\partial p^n_{K,u}}{\partial p_R}\right) \quad (17)$$

where $\frac{\partial \tilde{y}^j}{\partial x} = \frac{\partial \bar{y}^j}{\partial x} + \frac{\partial \bar{y}^j}{\partial p_K} . \frac{\partial p_K}{\partial x}$ from (9), and $\frac{\partial p_K}{\partial x}$ and $\frac{\partial p_K}{\partial p_R}$ are given in (15b).

Note that, in the absence of income effects where $\frac{\partial \tilde{y}^j}{\partial x} = 0$,[7/] then (17) implies that $\frac{\partial \tilde{y}^j}{\partial p_R} = \frac{\partial \tilde{y}^j_u}{\partial p_R}$, where the properties of $\frac{\partial \tilde{y}^j_u}{\partial p_R}$ are given in propositions 1 and 3. However, the existence of income effects alters the properties of the equilibrium functions $\tilde{y}^j$ in some significant ways. In particular, equation (17) indicates that uncompensated functions $\tilde{y}^j$ differ from compensated functions $\tilde{y}^j_u$ both quantitatively and qualitatively. This is a market equilibrium analogy to traditional partial equilibrium results of consumer theory on the differences between Marshallian (uncompensated) and Hicksian (compensated) price response.

With the explicit incorporation of the household sector as an "industry" within the market equilibrium context, equations (17) indicate that Heiner's results are clearly for "compensated" market equilibrium. Thus, even in the absence of product allocation among more than one industry, the presence of income effects in the household sector is sufficient in general to invalidate Heiner's results concerning the

symmetry and positive semi-definiteness of the matrix $\left(\frac{\partial \tilde{y}_R^j}{\partial p_R}\right)$. The Slutsky-like equation (17) illustrates how the income effects $\frac{\partial \tilde{y}^n}{\partial x}$ influence the properties of the equilibrium functions $\tilde{y}^j$. In particular, non-vanishing income effects would imply that compensated and uncompensated price responses can be empirically different in a market equilibrium context. This difference is an empirical issue which can be investigated using the above results.

V. Some Implications

In this section, we explore some of the implications of our results. In particular, we briefly discuss the implications of proposition 1 for welfare analysis. Also, we illustrate the usefulness of proposition 4 in empirical research.

1 - Welfare Analysis:

Recall from (11) that $V^J(p_R, \alpha, U)$ is an aggregate welfare measure across all industries (including the household sector) affected by induced adjustments in the price vector p_K through market supply-demand equilibrium. Hence, it provides a basis for conducting welfare compensation tests. To see this, let $\theta = (p_R,\alpha)$ and consider a change in the parameters θ from θ^o to θ^1. Using $\bar{U}$ as a reference a level of utility, the change in V^J associated with the change in θ is given by

$$\Delta V^J = V^J(\theta^1,\bar{U}) - V^J(\theta^o,\bar{U})$$

$$= \int_{\theta^o}^{\theta^1} \frac{\partial V^J(\theta,\bar{U})}{\partial \theta}\, d\theta \qquad (18)$$

where ΔV^J is the aggregate willingness-to-pay for the change across all industries (including the household sector), allowing for induced adjustments in p_K. If $\Delta V^J \geq 0$, it would follow that the change in θ passes the potential Pareto improvement test in the sense that aggregate welfare is increasing, i.e. that the gainers can compensate the losers so that no one is made worse off. Alternatively, if $\Delta V^J < 0$, it would follow that the change in θ fails the potential Pareto improvement test in the sense that the gainers cannot compensate the losers and at least one industry of the economy is made worse off by the change in θ.

Perhaps more importantly, note that the results presented in section 3 have relevant implications for the empirical measurement of ΔV^J in (18). In the case of an exogenous price change where $\theta = p_R$, equation (13a) implies that (18) takes the form

$$\Delta V^J = \int_{\theta^o}^{\theta^1} \sum_{j\epsilon J} \tilde{y}^j_{R,u}(p_R,\alpha,U)\,d\theta \tag{19}$$

where ΔV^J is measured by the sum over $j\epsilon J$ of the changes in the areas between the market equilibrium functions $\tilde{y}^j_{R,u}$ and the corresponding prices p_R. These areas are the traditional producer and consumer surplus measures, except that they are measured from market equilibrium (rather than partial equilibrium) functions. They measure economy-wide welfare impacts of changes in the price vector p_R. This generalizes some results obtained by Just and Hueth in the context of a vertically structured sector and provides a practical way of evaluating the welfare impact of exogenous price changes (e.g. due to government intervention) on all the industries (including the household sector) affected by the change.

Alternatively, in the case where $\theta = \alpha$, then equation (18) can provide a basis for investigating the welfare impact of technical change in some industry (or a change in consumer preferences). Given $\theta = \alpha$, equation (13b) implies that (18) takes the form

$$\Delta V^J = \int_{\theta^o}^{\theta^1} \sum_{j\epsilon J} p^{j\prime} \left.\frac{\partial \bar{y}^j_u(p^j,\alpha^j,U)}{\partial\alpha}\right|_{p_{K,u}(p_R,\alpha,U)} d\theta \tag{20}$$

Again, equation (20) measures the economy-wide welfare impact of technical change in the parameters α, allowing for induced adjustments in the price vector p_R. It provides a practical way of evaluating the welfare impact of a technical change in some industry on all the industries (including the household sector) affected by the change.

2 - Specification of Market Equilibrium Supply-Demand Functions:

Note that equations (7) and (9) constitute the reduced form of the structural model (1), (4) and (6) where equations (1) and (4) in the structural model correspond to partial equilibrium models of production and household decisions. Obviously, if the structural model is completely specified and known, then its associated reduced form representing market equilibrium functions will also be known. However, in many cases the structural model is not readily available. For example, because information on some of the relevant variables is lacking, it may not be possible to estimate all the partial equilibrium supply/demand functions in (1) and (4). Also, collinearity problems may make it difficult to estimate accurately the effects of all the explanatory variables included in (1) and (4). In such cases, it may be advantageous to consider a direct specification and estimation of the reduced form equations (7) and (9). This reduced form approach would of course be appropriate if the objective of the researcher is only to examine market equilibrium behavior.

The results presented in the previous sections can then help specify the properties of market equilibrium functions. To illustrate this, consider the differentials of equation (9):

$$d\tilde{y}^j = \frac{\partial \tilde{y}^j}{\partial p_R}.dp_R + \frac{\partial \tilde{y}^j}{\partial x}.dx + \frac{\partial \tilde{y}^j}{\partial \alpha}.d\alpha, \qquad j \epsilon J. \tag{21}$$

From (17) of proposition (4), expression (21) can be alternatively written, for $j \epsilon J$, $k \epsilon M$, as

$$d\tilde{y}^j_k = \sum_{r \epsilon R} \frac{\partial \tilde{y}^j_{k,u}}{\partial p_r} dp_r + \frac{\partial \tilde{y}^j_k}{\partial x} (dx + \sum_{r \epsilon R} (\tilde{y}^n_r + \sum_{k \epsilon K} \tilde{y}^n_k \frac{\partial p_k}{\partial p_r}) dp_r) + \frac{\partial \tilde{y}^j_k}{\partial \alpha} d\alpha .$$

Multiplying this expression by $(\frac{p^j_k}{x})$ yields the following Rotterdam-type model of market equilibrium supply-demand functions:

$$w^j_k \cdot \frac{d\tilde{y}^j_k}{\tilde{y}^j_k} = \sum_{r \epsilon R} \sigma^j_{kr} \, d\ln p_r + \beta^j_k \, (d\ln x + \sum_{r \epsilon R} (w^n_r + \sum_{k \epsilon K} w^n_k \, \gamma_{kr}) d\ln p_r) + w^j_k . \theta_k . d\ln\alpha \qquad (22)$$

where $w^j_k = \frac{p^j_k \, y^j_k}{x}$ is a budget share, $\sigma^j_{kr} = \frac{\partial \tilde{y}^j_{k,u}}{\partial p_r} . \frac{p^j_k . p^J_r}{x}$,

$\beta^j_k = p^j_k . \frac{\partial \tilde{y}^j_k}{\partial x}$, $\gamma_{kr} = \frac{\partial \ln p_k}{\partial \ln p_r}$ and $\theta_k = \frac{\partial \tilde{y}^j_k}{\partial \alpha} . \frac{\alpha}{\tilde{y}^j_k}$.

Theoretical restrictions on equation (22) follow from the results obtained in previous sections. In particular, from proposition 3, the homogeneity restriction takes the form

$$\sum_{r \epsilon R} \sigma^j_{kr} = 0 \qquad (23a)$$

Also, from proposition 1, the following symmetry restrictions hold

$$\sum_{j \epsilon J} \sigma^j_{kr} = \text{a symmetric, positive semi-definite matrix.} \qquad (23b)$$

Such restrictions appear in a simple form and are easy to impose or test in the context of specification (22). This illustrates how the theoretical properties derived in this paper could be used in the empirical investigation of market equilibrium supply-demand functions.

VI - Concluding Remarks:

This paper has developed the properties of market equilibrium supply-demand functions when some of the prices are allowed to adjust to an exogenous change through market equilibrium. It generalizes previous results found in the literature in several ways. In particular, we add a household sector to the Heiner/Braulke framework and derive the implications for multi-industry market equilibrium functions. Also, contrary to previous work, we consider the allocation problem where one product can be sold to (or purchased from) more than one industry. Our results should be of significant interest in the analysis of economic adjustments across sectors of a small open economy.

The addition of a household sector to the the Heiner/Braulke framework yields the implications of possible income effects associated with the changing prices of consumer goods. In this context, we derive a Slutsky-like equation which provides some useful insights into the role of income effects on the properties of market equilibrium functions. Similarly, adding multiple industries (versus assuming a single multi-input/multi-output industry) allows explicit analysis of the allocation issue. In particular, several of the Heiner/Braulke results are found not to hold under these generalizations.

Finally, we illustrate the usefulness of our results for multi-market welfare analysis and in the specification/estimation of market equilibrium adjustments in an economy when prices adjust to some exogenous changes. One interesting extension of our short run analysis would be to consider entry and exit and the determination of the number of firms in each producing sector. Additional work on the market equilibrium determination of the number of multiproduct firms in a multi-industry economy (including a consumer sector) appears needed. We hope that our approach will stimulate further research on these important topics.

Appendix A

Proof of Lemma 1:

Using the implicit function rule, the differentiation of (8) with respect to p_R, R = M\K, yields (12). Post multiplying (12) by p_R yields

$$\frac{\partial p_{K,u}}{\partial p_R} \cdot p_R = - \left(\sum_{j \varepsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K} \right)^{-1} \left(\sum_{j \varepsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_R} p_R \right)$$

Note that $\frac{\partial \bar{y}^j_{K,u}}{\partial p_R} p_R = - \frac{\partial \bar{y}^j_{K,u}}{\partial p_K} p_K$ from the homogeneity of degree zero of $\bar{y}^j_u$. It follows that $\frac{\partial p_{K,u}}{\partial p_R} . p_R = p_{K,u}$, which which proves the linear homogeneity of $p_{K,u}(p_R,.)$.

Q.E.D.

Proof of Proposition 1:

Given that $V^j(p_K, p^j_R)$ is linear homogeneous (see (2)) and that $p_{K,u}(p_R,.)$ is linear homogeneous from Lemma 1, it follows from (11) that $V^J(p_R,.)$ is homogeneous of degree one in p_R.

Differentiating (11) with respect to p_R, R = M\R, yields

$$\frac{\partial V^J}{\partial p_R} = \sum_{j \varepsilon J} \frac{\partial V^j}{\partial p_R} + \sum_{j \varepsilon J} \frac{\partial V^j}{\partial p_K} \frac{\partial p_{K,u}}{\partial p_R}.$$

But, from (2a), (5a) and (6), $\sum_{j \varepsilon J} \frac{\partial V^j}{\partial p_K} = \sum_{j \varepsilon J} \bar{y}^j_{K,u} = 0$. Noting from (2a) and (5a) that $\frac{\partial V^j}{\partial p_R} = \bar{y}^j_{R,u}$, this proves (13a).

Differentiating (13a) with respect to p_R yields

$$\frac{\partial^2 V^J}{\partial p_R^2} = \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_R} \right) + \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_K} \right) \frac{\partial p_{K,u}}{\partial p_R}$$

Using (12), this implies

$$\frac{\partial^2 V^J}{\partial p_R^2} = \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_R} \right) - \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_K} \right) \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K} \right)^{-1} \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_R} \right). \qquad \text{(A1)}$$

From (2b) and (5b)), the symmetry and positive semi-definiteness of $\frac{\partial \bar{y}^j_u}{\partial p^j}$ implies that $[u_1' \; u_2'] \begin{bmatrix} \sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_R} & \sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_K} \\ \sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_R} & \sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

is a symmetric, positive semi-definite matrix. Choosing $u_2' = -u_1' \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_K} \right) \left(\sum_{j \epsilon J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K} \right)^{-1}$ proves the positive semi-definiteness of the right hand side in (A1), which implies the convexity of $V^J(P_R, \cdot)$.

To prove (13b), differentiating (11) with respect to α yields

$$\frac{\partial V^J}{\partial \alpha} = \sum_{j \epsilon J} \left[\frac{\partial V^j}{\partial \alpha} + \frac{\partial V^j}{\partial p_K} \cdot \frac{\partial p_{K,u}}{\partial \alpha} \right]$$

Using (2a), (5a) and (6), it follows that

$$\frac{\partial V^J}{\partial \alpha} = \sum_{j \epsilon J} \frac{\partial V^j}{\partial \alpha} \Big|_{p_{K,u}(p_R, \alpha, U)}.$$

But, from (1) or (4), $\frac{\partial V^j}{\partial \alpha} = p^{j\prime} \frac{\partial \bar{y}^j_u(p^j, \alpha^j, U)}{\partial \alpha}$. This proves (13b).

Q.E.D.

Proof of Proposition 2:

Note from (A1) that

$$\frac{\partial^2 V^J}{\partial p_R^2} = \sum_{j \in J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_R} = -\left(\sum_{j \in J} \frac{\partial \bar{y}^j_{R,u}}{\partial p_K}\right)\left(\sum_{j \in J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K}\right)^{-1}\left(\sum_{j \in J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_R}\right)$$

which is a negative semi-definite matrix from (2b) and (5b) since $\sum_{j \in J} \frac{\partial \bar{y}^j_{K,u}}{\partial p_K}$ is a positive definite matrix from (3). Noting from (2) that $\bar{y}^j_u = \frac{\partial V^j}{\partial p^j}$, this concludes the proof.

Q.E.D.

Proof of Proposition 4:

The homogeneity proof is the same as in proposition 1 and is omitted. Expression (16) follows from the duality relationship between the expenditure function $-V^n$ and the indirect utility function W (see section II). Expression (17) is obtained by differentiating (16) and using Shephard's lemma (equation (5a)).

Q.E.D.

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ENDNOTES

1/ Note that the results presented below hold when each industry is composed of a fixed number of competitive firms facing the same prices, where each firm possibly faces a different technology (see Heiner). Since Braulke's results established that a full analogy to the traditional, short-run, partial equilibrium theory of the competitive firm characterizes the associated industry level behavior under these assumptions (and, the "normality" conditions discussed in Section I), we start our analysis from this point.

2/ If the households are also involved directly in the production of outputs that are marketed, then the arguments presented below can be easily modified in the context of household production theory (e.g. see Deaton and Muellbauer, Ch. 10).

3/ Note that the results presented here would hold when the household sector is composed of a fixed number of consumers facing the same prices where consumers can have different incomes but exhibit quasi-homothetic Gorman preferences (e.g. Gorman; Deaton and Muellbauer). That is, similar to the "representative firm" arguments of footnote 1, we are assuming the "representative consumer" in aggregating the individual level constrained optimization of utility up to the market level.

4/ Note that we drop the superscript J on the p_K and p_R vectors of endogenous and exogenous prices facing all of the industries in J for notational convenience.

5/ Again, we drop the J superscript on α for notational simplicity.

6/ Although the differentiability assumption is convenient for deriving our results, it could be relaxed (e.g. see Braulke, 1987).

7/ Note from (9) and (15) that $\frac{\partial \bar{y}^n}{\partial x} = 0$ is a sufficient condition for

$$\frac{\partial \bar{y}}{\partial x} = 0.$$