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NON-PARAMETRIC DEMAND ANALYSIS:

A DUALITY APPROACH

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I - Introduction:

Non-parametric analysis of consumer behavior allows the investigation of a finite number of data points with no ad hoc specification of functional forms for utility or demand functions (Afriat; Diewert, 1973; Varian, 1982, 1983; Diewert and Parkan). An interesting result from the literature is that a finite number of observations allows consumers choices to be rationalized on the basis of a concave utility function. This is somewhat surprising given that consumer theory is often presented assuming a quasi-concave utility function (e.g. Lau; Diewert, 1974, 1982) and that all quasi-concave functions cannot be transformed into a concave function by a monotonic increasing transformation (e.g. see Arrow and Enthoven, p. 781). While this result has been used to justify the use of concave utility functions in demand analysis (e.g., Varian, 1982, 1983; Diewert and Parkan), it suggests the existence of a gap between the theory of consumer choice and the development of testable hypotheses based on a finite number of demand observations: in this case, it is not possible to distinguish empirically between a concave and a quasi-concave utility function.

The objective of this paper is to further explore the testable hypotheses of consumer theory in the context of a non-parametric approach. In particular, the implications of duality theory (as developed by Samuelson; Lau; Diewert, 1974, 1982) for non-parametric demand analysis are investigated. We show that, when duality relationships hold, the non-parametric implications of the primal problem (the maximization of a direct utility function subject to a budget constraint) differ from those of the dual problem (minimization of an indirect utility function subject to a budget constraint). In other words, even when the two problems are theoretically equivalent, they can be empirically different based on a finite number of observations. For example, it is possible for consumption data to be consistent with the primal problem while being inconsistent with the dual problem. This suggests some important limitations of duality theory for empirical demand analysis. Furthermore, it is shown that imposing the implications of duality in the non-parametric approach implies additional restrictions that can be used in testing consumer behavior. The results further illustrate the existence of a gap between consumer theory and the empirical analysis of consumption based on a finite number of observations.

II - Utility Theory and Duality:

In this section, we briefly review consumer theory and set up the notation for the paper. Consider the consumer problem: maximize the utility function $U(x)$ with respect to the n-dimensional consumption vector x subject to the budget constraint $p'x \leq I$, where p is a n-vector of commodity prices and $I > 0$ is consumer income. Defining normalized prices as $v = p/I$, this can be written as the primal problem[1/]

$$g(v) = \operatorname*{Sup}_{x} \{U(x) \colon v'x \leq 1,\ x \in R_n^+\} \tag{1}$$

where $g(v)$ denotes the indirect utility function. If $U(x)$ is a continuous function for $x \in R_n^+$, then it achieves a maximum over the feasible set and $g(v)$ in (1) is a continuous, non-increasing,[2/] quasi-convex[3/] function for $v \in R_n^{++}$ (Diewert, 1974).

Now, consider the corresponding dual problem:

$$\overline{U}(x) = \operatorname*{Inf}_{v} \{g(v): v'x \leq 1,\ v \in R_n^+\}. \tag{2}$$

If $g(v)$ is continuous for $v \in R_n^+$, then it achieves a minimum over the feasible set and $\overline{U}(x)$ in (3) is a continuous, non-decreasing,[4/] quasi-concave[5/] function for $x \in R_n^{++}$.

The conditions under which $\overline{U}(x)$ in (2) is equal to $U(x)$ establish the existence of duality between the direct ($U(x)$) and indirect ($g(v)$) utility functions. Variations in the statement of these conditions can be found in the literature. Following Diewert (1974, 1982), the following condition will be assumed:

Condition A: $U(x)$ is a continuous, non-decreasing and quasi-concave function for $x \in R_n^+$.[6/]

Under condition A, $g(v)$ is continuous, non-increasing and quasi-convex for $v \in R_n^{++}$. In order for $g(v)$ to be continuous as well for $v \in R_n^+$, the function $\{g(v): v \in R_n^{++}\}$ can be extended to the non-negative orthant ($v \in R_n^+$) by continuity from below.[7/] After extending $g(v)$ to the non-negative orthant, then under condition A, problem (2) generates $\overline{U}(x)$ such that $\overline{U}(x) = U(x)$ for $x \in R_n^{++}$. Moreover, given $\overline{U}(x) = U(x)$ for

$x \in R_n^{++}$, then their extension (by continuity from above) to the non-negative orthant ($x \in R_n^+$) would also coincide (see Diewert, 1974).

In such a case, the indirect utility function g(v) completely characterizes the direct utility function (and vice-versa). Such duality relationships have been investigated in details in the literature (e.g. Houthakker; Samuelson; Lau; Diewert, 1974, 1982; Blackorby et al.). By suggesting alternative formulations of consumer theory, duality relationships have stimulated much research on the characterization of consumer behavior (e.g. Christensen et al.; Anderson; Lau, 1977; Christensen and Manser; Weymark).

For our purpose, it will be useful to discuss the implications of duality in the context of saddle point criteria. First, consider the saddle point problem:

Find $x^* \in R_n^+$ and $\lambda_p^* \in R^+$ such that

$$L(x,\lambda_p^*,v) \leq L(x^*,\lambda_p^*,v), \leq L(x^*,\lambda_p,v) \text{ for all } x \in R_n^+,\ \lambda_p \in R^+ \tag{3}$$

where $L(x,\lambda_p,v) = U(x) + \lambda_p[1-v'x]$.

It is well known that if a saddle point exists, then x^* in (3) is a (not necessarily unique) solution to problem (1), i.e. $x^* = \text{argmax}$ $\{U(x)\colon v'x \leq 1,\ x \in R_n^+\}$. However, for the converse to be true, we need the following condition:

Condition B:

- there exists an $\bar{x}$ such that $v'\bar{x} < 1$ (Slater's condition).
- the set $K = \{(z_o,z_1)\colon z_o \geq -U(x),\ z_1 \leq 1-v'x, \text{ for some } x \in R_n^+\}$ is convex.

If condition B is satisfied, then x^* being a solution to problem (1) implies that (x^*, λ_p^*) is a saddle point solution to problem (3) (see Karlin; Sposito). Thus, under condition B, the saddle point criterion (3) is both necessary and sufficient for x^* to be a solution of the maximimization problem (1). Note that condition B is satisfied if the utility function U(x) is quasi-concave[8/] and prices are finite.[9/] In such a situation, finding a saddle-point solution to (3) is therefore equivalent to finding a solution to problem (1).

Similarly, on the dual side, consider the saddle point problem:

Find $v^* \varepsilon R_n^+$ and $\lambda_d^* \varepsilon R$ such that

$$V(v,\lambda_d^*,x) \geq V(v^*,\lambda_d^*,x), \geq V(v^*,\lambda_d,x) \text{ for all } v \varepsilon R_n^+,\ \lambda_d \varepsilon R^+ \qquad (4)$$

where $V(v,\lambda,x) = g(v) + \lambda[v'x-1]$

Using the same arguments as in the primal case, if the indirect utility function g(v) is quasi-convex and quantities are finite, then the saddle point criterion (4) is both necessary and sufficient for v^* to be a (not necessarily unique) solution of the minimization problem (2), i.e. $v^* = \text{argmin } \{g(v): v'x \leq 1,\ v \varepsilon R_n^+\}$. Under such conditions, finding a saddle point solution to (4) is therefore equivalent to finding a solution to problem (2). These formulations are now used in the context of a non-parametric analysis of consumer demand.

III - The Non-Parametric Approach

The non-parametric approach to demand analysis consists in analyzing a finite body of data with no ad hoc specification of functional forms for demand equations (see Afriat; Diewert, 1973; Varian, 1982, 1983). We assume that we have T observations on (normalized) prices and quantities consumed: (v_t, x_t), $t=1,\ldots,T$. We also assume that these prices and quantities are positive and finite. The non-parametric approach investigates the implications of utility theory for these observations.

First, consider the primal problem (1) where the utility function $U(x)$ is continuous. If $U(x)$ does not satisfy the monotonicity and curvature properties stated in condition A, then it is well known that it will never be possible to observe choices from the decreasing regions of the utility functions nor will it ever be possible to observe the non-convex regions of the indifference curves (e.g. Weymark). Thus, consumer data could not distinguish between a continuous utility function and some appropriate utility function satisfying condition A (i.e. non-decreasing and quasi-concave). In this sense, condition A implies no loss of generality as far as empirical implications of the theory are concerned. For this reason, we will treat condition A as a maintained hypothesis and assume that it is satisfied in the discussion presented below.

Considering the primal problem (1), then the following results are obtained (see the proof in the Appendix).

Lemma 1: The following statements are equivalent

a) $x_t = \text{argmax } \{U(x): v_t' x \leq 1,\ x \in R_n^+\}$ for a set of normalized prices $\{v_t,\ t = 1,\ldots,T\}$, where $U(x)$ satisfies condition A.

b) there exist V_t, λ_t, t=1,...,T, such that

$$\lambda_t \geq 0 \tag{5a}$$

$$V_s \leq V_t + \lambda_t \, [x_s' \, v_t - 1]. \tag{5b}$$

c) there exists a concave function f(x) that rationalizes the data in the sense that

$$x_t = \text{argmax } \{f(x) : v_t' \, x \leq 1, \ x \in R_n^+\}.$$

Theorem 1 is an extension of the non-parametric results obtained by Afriat, Diewert (1973) and Varian (1982, 1983). For example, the necessary and sufficient conditions for the data (x_t, v_t) to be consistent with utility maximization are given in equations (5). Note that these conditions differ from the ones presented by Varian (1982, 1983).10/ The reason is that we did not impose non-satiation on the utility function U(x). Without non-satiation, it is possible that the budget constraint will not be binding. From the complementary slackness condition, this implies that the multiplier λ_t can be zero. Thus, while previous results have been restricted to the case where $\lambda_t > 0$ (see Afriat; Diewert, 1973; Varian, 1982, 1983), we consider the more general case where $\lambda_t \geq 0$.11/

Under non-satiation, note that (5) becomes $\lambda_t > 0$, and $\lambda_s \, x_t' \, [v_t - v_s] \leq V_s - V_t \leq \lambda_t \, x_s' \, [v_t - v_s]$, as the marginal utility of (normalized) income λ_t is positive and the budget constraint is binding, i.e. $v_s' \, x_s = 1$. Then, under differentiability, this implies the well known Roy's identity $\partial g(v)/\partial v = -\lambda^* \, x^*$ (see Anderson and Takayama, p. 506).

Now, consider the dual problem (2). The following result holds:12/

Lemma 2: The following statements are equivalent:

a) $v_t = \text{argmin}\ \{g(v)\colon v'x_t \le 1,\ v \in R_n^+\}$ for a set of quantities $\{x_t,\ t=1,\ldots,T\}$, where $g(v)$ is a continuous, non-increasing and quasi-convex function.

b) there exist $\bar{V}_t$, $\bar{\lambda}_t$, $t=1,\ldots,T$, such that

$$\bar{\lambda}_t \ge 0 \tag{6a}$$

$$\bar{V}_t \le \bar{V}_s + \bar{\lambda}_t\ [v_s'\ x_t - 1]. \tag{6b}$$

c) there exists a convex function $h(v)$ that rationalizes the data in the sense that

$$v_t = \text{argmin}\ \{h(v)\colon v'x_t \le 1,\ v \in R_n^+\}.$$

Equations (6) present necessary and sufficient conditions for the data (x_t, v_t) to be consistent with the dual (indirect) utility minimization problem (2). Note that, under non-satiation of $U(x)$, (6) becomes $\bar{\lambda}_t > 0$, $\bar{\lambda}_s\ v_t'\ [x_t - x_s] \le \bar{V}_t - \bar{V}_s \le \bar{\lambda}_t\ v_s'\ [x_t - x_s]$, as the marginal utility of (normalized) income $\bar{\lambda}_t$ is positive and the budget constraint is binding, i.e. $v_s'\ x_s = 1$. Then, under differentiability, this implies the well known Wold's identity, $\partial U(x)/\partial x = \lambda^* v^*$.

Comparing (5) with (6), note that the inequalities (6b) differ from those obtained in the context of the primal problem (5b). This suggests that, given a finite number of observations, the dual approach to utility theory has in general different implications for consumer behavior compared to the primal approach. For example, consider the case where condition A is satisfied and duality between the primal problem (1) and the dual problem (2) holds. Then, the two problems give theoretically equivalent representations of consumer preferences. They can also provide equivalent representations of consumer behavior (see Samuelson; Lau; Weymark). However, from (5) and (6), the two problems can be empirically different in the context of a finite number of observations. In particular, it is possible for consumption data to be consistent with the primal problem (1) while being inconsistent with the dual problem (2) (or vice-versa). This indicates an important limitation of duality theory in empirical demand analysis: by exploring only selected points of the consumption set, the primal problem applied to a set of consumer data is not empirically equivalent to the dual problem. In other words, there is a gap between consumer theory (assumed to hold at all points) and the empirical analysis of consumer behavior based on a finite number of observations.

In this context, should some of the duality restrictions be imposed in non-parametric demand analysis? We have argued above that, given the continuity of preferences, condition A implies no loss of generality in empirical consumption analysis. In this case, duality relationships between U(x) and g(v) would be expected to hold for x_t and v_t in the positive orthant (and by extension to the non-negative orthant), implying that $U(x_t) = g(v_t)$ (see Diewert, 1974, 1982). This implies that $V_t = \bar{V}_t$ in Lemma 1 and Lemma 2. Furthermore, consider the case where the function U(x) is continuous, increasing and quasi-concave for $x \in R_n^+$,[13] and define the closed convex set $M(U) = \{v: g(v) \leq U, v \in R_n^+\}$ where $U = g(v^*)$, $v^* \in R_n^{++}$ being a boundary point of M(U). Then, the set of normalized supporting hyperplanes to M(U) through the point v^* constitutes the solution set to the primal problem (1) (see Diewert, 1974). In this case, the primal and dual problems provide equivalent representations of consumer behavior (e.g. Weymark). From Lemma 1 and Lemma 2, the following result is obtained:

Proposition 1: The following statements are equivalent:

a) $x_t = \text{argmax}\ \{U(x)\colon v_t' x \le 1,\ x \in R_n^+\}$ where $U(x)$ is a continuous, increasing and quasi-concave function,

$v_t = \text{argmin}\ \{g(v)\colon v' x_t \le 1,\ v \in R_n^+\}$ where $g(v)$ is a continuous, decreasing and quasi-convex function,

and $U(x_t) = g(v_t)$, $t=1,\ldots,T$.

b) there exist V_t, λ_t, $\bar{\lambda}_t$, $t=1,\ldots,T$, such that

$$\lambda_t > 0,\ \bar{\lambda}_t > 0 \tag{7a}$$

$$V_s \le V_t + \lambda_t\ [x_s' v_t - 1] \tag{7b}$$

$$V_t \le V_s + \bar{\lambda}_t\ [v_s' x_t - 1]. \tag{7c}$$

Equations (7) provide directly testable conditions that the data must satisfy if it is to be consistent with the dual formulation of consumer theory as stated above. Testing these conditions consists in checking whether there exist solutions to a set of linear inequalities. The existence of such solutions can be checked easily by solving a linear programming problem (see Diewert, 1973).

Note that equations (7) are more restrictive than equations (5) in the primal problem or equations (6) in the dual problem. For example, while the existence of a solution to (7) implies the existence of a solution to (5) and (6), the converse does not necessarily hold. To the extent that it is reasonable to assume convexity of preferences and non-satiation of $U(x)$ as a maintained hypothesis, this suggests that the empirical testing of consumer behavior should be done using (7) rather than (5) or (6). Previous non-parametric demand analysis (which is based on (5); see Diewert, 1973; Varian, 1982, 1983) explores only the implications of the primal problem (1). In contrast, equations (7) reflects the empirical implications of both (1), the primal problem, and (2), the dual problem. Since the non-parametric implications of the two problems are not equivalent, more precise results would be obtained using Proposition 1 (instead of Lemma 1 or Lemma 2) in the non-parametric analysis of consumer behavior.

Finally, note that the multipliers λ_t and $\overline{\lambda}_t$ are allowed to be different in Proposition 1. More restrictive assumptions on the nature of the objective function in (1) or (2) can imply that the primal and dual multipliers are the same. This is the case when U(x) and g(v) are connected by Legendre transformation, where the utility function U(x) is differentiable, strictly increasing and strictly quasi-concave,[14/] and at equilibrium $\lambda_p^* = \lambda_d^* = \frac{\partial U(x)}{\partial x}.\ x = -\frac{\partial g(v)}{\partial v}.\ v$ (see Samuelson; Lau). Under such conditions, the optimal solutions to problems (1) and (2) are unique and $\lambda_t = \overline{\lambda}_t$ in (7). The following result is then a special case of Proposition 1.

<u>Corollary 1</u>: The following statements are equivalent

a) $x_t = \text{argmax}\ \{U(x)\colon v_t' x \leq 1,\ x \in R_n^+\}$ where U(x) is a differentiable, strictly increasing and strictly quasi-concave function,

v_t = argmin $\{g(v): v'x_t \leq 1,\ v \in R_n^+\}$ where $g(v)$ is a differentiable, strictly decreasing and strictly quasi-convex function,

$U(x_t) = g(v_t)$ and $\lambda_p^*(v_t) = \lambda_d^*(x_t)$, $t=1,\ldots,T$.

b) there exist V_t, λ_t, $t=1,\ldots,T$, such that

$$\lambda_t > 0 \tag{8a}$$

$$V_s \leq V_t + \lambda_t\ [x_s'\ v_t - 1] \tag{8b}$$

$$V_t \leq V_s + \lambda_t\ [v_s'\ x_t - 1]. \tag{8c}$$

Again, equations (8), which are more restrictive than (7), could be used in the empirical investigation of consumer behavior. To the extent that the duality representations considered in Proposition 1 and Corollary 1 are commonly assumed as maintained hypotheses in demand analysis, then equations (7) or (8) should provide a useful basis for a non-parametric analysis of consumption data.

IV - Conclusion

While duality theory suggests that, under some regularity conditions, the primal and dual approach to consumer behavior are equivalent, a non-parametric demand analysis based on a finite number of observations indicates that the primal and dual problems are in general not empirically equivalent. This presents additional evidence of the existence of a gap between consumer theory and the analysis of a finite number of observations on consumption.

By exploiting duality relationships between the primal and the dual problems, we derived a series of non-parametric tests of consumer behavior. Such tests should be useful to evaluate the consistency of a finite number of consumption data with various hypotheses. For example, equations (8) in Corollary 1 could be used to test whether consumption behavior can be equivalently modeled from some parametrically specified direct or indirect (differentiable) utility function. Thus, the non-parametric analysis proposed extends the empirically testable content of consumer theory. We hope that our results will help stimulate the testing of the theory and its use in demand analysis.

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Appendix

Proof of Lemma 1:

From section II, we know that, given finite prices and a continuous quasi-concave function $U(x)$, the solution x_t of the primal problem

$$g(v_t) = \underset{x}{\text{Max}} \; \{U(x):\; v_t' \, x \le 1,\; x \in R_n^+\}$$

is equivalent to the solution $(x_t \in R_n^+, \lambda_t \in R^+)$ of the saddle point problem

$$L(x,\lambda_t,v_t) \le L(x_t,\lambda_t,v_t) \le L(x_t,\lambda,v_t), \text{ for all } x \in R_n^+,\; \lambda \in R^+ \qquad \text{(A1)}$$

where $L(x,\lambda,v) = U(x) + \lambda[1-v'x]$.

The saddle point problem (A1) implies $\lambda_t[1-v_t' \, x_t] = 0$ and

$$L(x_s,\lambda_t,v_t) \le L(x_t,\lambda_t,v_t) = U(x_t) = g(v_t)$$

or

$$U(x_s) \le U(x_t) + \lambda_t \; [x_s' \, v_t - 1];\; s,\; t=1,\ldots,T;\; \lambda_t \in R^+,$$

which is expression (5).

To show that (5) implies consumer utility maximization, define the function

$$f(x_s) = \underset{t}{\text{Min}} \; \{V_t + \lambda_t \; [v_t' \, x_s - 1],\; \lambda_t \in R^+\}. \qquad \text{(A2)}$$

It follows that the function $f(x_s)$ is continuous, non-decreasing and concave (and hence quasi-concave). Using (5b), (A2) implies that $f(x_s) = V_s$. Also, from (A2), we have

$$\underset{x}{\text{Max}} \; \{f(x):\; v_s' \, x - 1 \le 0,\; x \in R_n^+\} \le \underset{x}{\text{Max}} \; \{V_s + \lambda_s \; [v_s' \, x - 1]:$$

$$v_s' \, x - 1 \le 0,\; x \in R_n^+,\; \lambda_s \in R^+\} = V_s$$

It follows that $x_s = \text{argmax} \; \{f(x):\; v_s' \, x \le 1,\; x \in R_n^+\}$, i.e. that x_s is a solution to a utility maximization problem for $s=1,\ldots,T$, where the utility function $f(x)$ is defined in (A2).

Footnotes

1/ In our notation, R_n^+ denotes the non-negative orthant where $x = (x_1, \ldots x_n)' \varepsilon R_n^+$ means $x_i \geq 0$, for all i. Similarly, R_n^{++} denotes the positive orthant where $x_i \varepsilon R_n^{++}$ means $x_i > 0$, for all i.

2/ Given $v = (v_1, \ldots, v_n)'$, g(v) is non-increasing if $g(v) \leq g(\bar{v})$ for $\bar{v}_i \leq v_i$ for all i. It is decreasing if $\bar{v}_i < v_i$ for all i implies $g(v) < g(\bar{v})$.

3/ g(v) is quasi-convex if and only if $\{v: g(v) \leq k\}$ is a convex set for every scalar k.

4/ U(x) is non-decreasing (increasing) if -U(x) is non-increasing (decreasing).

5/ U(x) is quasi-concave if -U(x) is quasi-convex.

6/ Diewert (1974) assumes a weaker condition: rather than being continuous for $x \varepsilon R_n^+$, U(x) can be assumed to be only "continuous enough" for $x \varepsilon R_n^{++}$. However, the former condition implies the latter one (see Diewert, 1974).

7/ See Diewert (1974) for a discussion of the procedure for this extension.

8/ The quasi-concavity of U(x) implies that the set K is convex. To see this, choose two distinct points $\underline{x}$ and $\overline{\overline{x}}$ such that $U(\underline{x}) = U(\overline{\overline{x}})$. Also, choose $\underline{z} \varepsilon K$ given $\underline{x}$ and $\overline{\overline{z}} \varepsilon K$ given $\overline{\overline{x}}$. Define $\tilde{z} = \alpha \underline{z} + (1-\alpha)\, \overline{\overline{z}}$ and $\tilde{x} = \alpha\, \underline{x} + (1-\alpha)\, \overline{\overline{x}}$ for some $\alpha \varepsilon [0,1]$. The linearity of the budget constraint implies that $\tilde{z}_1 \leq 1 - v'\tilde{x}$. Also, the quasi-concavity of U(x) (where $U(\tilde{x}) \geq U(\underline{x}) = U(\overline{\overline{x}})$) implies that $\tilde{z}_o \geq -\, U(\tilde{x})$. Thus, the set K is convex.

9/ Positive income and finite prices guarantee that Slater's condition is satisfied.

10/ The conditions presented by Varian are:

$$\lambda_t > 0$$

$$V_s \leq V_t + \lambda_t\, [v_t'\, x_s - v_t'\, x_t)]$$

11/ As can be easily verified, our equations (5) reduce to the conditions presented by Afriat or Varian under non-satiation (where $\lambda_t > 0$, $v_t' x_t = 1$).

12/ With $\underset{v}{\text{Min}} \{g(v): v'x \le 1,\ v \in R_n^+\} = - \underset{v}{\text{Max}} \{-g(v): v'x \le 1,\ v \in R_n^+\}$ and the saddle point characterization (4), the proof of Lemma 2 follows from Lemma 1.

13/ Hence, $g(v)$ is continuous, decreasing and quasi-convex in R_n^{++}.

14/ Hence, $g(v)$ is differentiable, strictly decreasing and strictly quasi-convex.