Linkages Among Estimated Technological Parameters, Production, Supply and Input Demand Elasticities

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Abstract

Unbiased estimates of technological parameters of an underlying production function can be obtained by first estimating a cost-share equation. From the parameter estimates of the cost share equation, it is possible to obtain all relevant elasticities, including production elasticities, own and cross input demand elasticities, the elasticity of input demand with respect to product price, and the output supply elasticities. The approach has important implications for, and represents a new way to estimate aggregate supply elasticities.

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Linkages among Estimated Technological Parameters, Production, Supply and Input Demand Elasticities for Agricultural Production Functions

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For six decades, agricultural economists have made attempts to estimate production functions describing technical relationships between inputs and outputs (Spillman, 1923, 1924; Furtan and Gray, 1981).[1] This paper derives some of the important relationships that exist between the estimated parameters of a production function and various production and demand elasticities.

The output from the agricultural production process is assumed to be a random variable, with the mean given by the parameters of the production function. A constant variance across all economic units that produce that same output is assumed. The Gauss-Markov theorem results in unbiased estimates of production function parameters estimated via ordinary least squares if the independent variables (inputs) are uncorrelated with the error term. The error term can represent the effect of either unobservable variables (which are uncorrelated with the amount of inputs used) or observable variables (as observed by the manager), which are correlated with the inputs used.

The conditional demand and supply functions (factor demands) for the firm are functions of prices determined by the marketplace. Therefore, these prices are uncorrelated with the error term. From the symmetry conditions, the partial derivatives representing the change in the quantity demanded of each input with respect to a change in the price of all other inputs should all be equal. In order to obtain efficient estimates of the parameters, these restrictions should be accounted for explicitly within the estimation process. This will reduce the number of parameters to be estimated, thus conserving degrees of freedom and possibly eliminating multicollinearity problems (Fuss, McFadden and Mundlak, p. 229).

Suppose a production function:

$$f(\mathbf{X}, \mathbf{a}),$$

where

$\mathbf{X} = [x_1,...,x_n]$ is a vector of inputs

$\mathbf{a} = [a_1,...,a_n]$ is a vector of parameters to be estimated.

The corresponding profit function is

$g(p, \mathbf{P}, \mathbf{a})$

where

p is the price of the output

$\mathbf{P} = [p_1,..., p_n]$ is a vector of input prices.

The difficulty with the estimation of profit functions is that data on economic profits are usually not readily available. Moreover, pure competition in the output markets may constitute an untenable assumption. Information on factor costs (input prices) is normally more readily available. Therefore, cost functions are frequently used to estimate technological parameters of the underlying production function. However, estimation problems may arise as a result of the lack of independence between the output level and the error term.[2]

Revenue functions and factor-share equations have been suggested as alternatives. Revenue functions have been used by Diewert, McFadden, and Askari and Cummings, with emphasis on multiproduct production processes. However, the usefulness of the revenue-function approach is limited because of the need for an aggregate input measure.

Under appropriate conditions, factor-share equations will provide efficient estimates of the parameters of the production function. Hence, input-demand functions as well as cost and supply functions can be derived. Cost-share functions have the advantage of providing an estimate for every technological parameter of the underlying production function. Therefore, it is possible to obtain either the input-demand functions or the output-supply function. Elasticities with respect to prices for each of the above functions are expressed in terms of the technological parameters. In the remainder of this paper, the linkages between the production elasticities and the input demand and output supply-price elasticities are derived.

Production, Price and Derived Demand

Mosak's pioneering 1938 work dealing with the interrelationships between the production function parameters, prices and derived demand for inputs leads to the development of expressions for input demand and supply price elasticities. The production function for the firm is defined as $y = f(x_1,...,x_n)$ where y represents the output of the firm, x_i represent quantities of inputs or factors of production, p is the product price and p_i is the price of the ith input.

The elasticity of input demand with respect to the jth input price (η_{ipj}) is defined as

(1) $\eta_{ipj} = \lambda M_{11,j+1,i+1} M_{11}^{-1}$,

where λ is some undetermined Lagrangean multiplier, which under profit maximization conditions is equal to $1/p$. M is defined as negative of the familiar bordered Hessian

$$
\begin{array}{lll}
0 & p_1 & \dots\ p_n \\
p_2 & f_{11} & f_{1n} \\
. & . & . \\
. & . & . \\
. & . & . \\
p_n & f_{1n} & \dots\ f_{nn}
\end{array}
$$

Now define (Bocher Corollary 3 p. 33):

$M_{ijkl} = (M_{ij}M_{kl}\text{-}M_{il}M_{kj})M^{-1}$

$\forall\ i,j,k,l = 1, \ldots n$

For the particular case, equation (1) can be restated as:

(2) $\eta_{1p1} = \lambda M_{1122}M_{11}^{-1}$

$\eta_{1p2} = \lambda M_{1132}M_{11}^{-1}$

$\eta_{2p1} = \lambda M_{1123}M_{11}^{-1}$

The elasticity of supply with respect to the *i*th input price is given by:

(3) $\varepsilon_{pi} = \lambda(\partial c/\partial p_i - x_i)$. Mosak establishes the following relationship:

(4) $\varepsilon_{pi} = -\eta_{ip}$, where η_{ip} is the demand elasticity for the *i*th input with respect to the product or output price. The elasticity of product supply with respect to the output price is given by:

(5) $\varepsilon_p = \lambda\partial c/\partial p$.

Then:

(6) $\Sigma\varepsilon_{pi} = -\varepsilon_p$.

In the following section, we will propose a more direct linkage between the input demand elasticities and the underlying production function parameters, by making use of a somewhat different approach.

Relationships among Elasticities

Assuming a smooth, continuous, monotonically-increasing convex production function (Fuss, McFadden and Mundlak):

(7) $y = f(x_1,...,x_n)$,

where y is output, and the x_i are inputs with given relevant prices. Then, the production elasticities are given by

(8) $\alpha_i = \partial y/\partial x_i\, x_i y^{-1} = p_i p^{-1} x_i y^{-1}$

where p is the output price, and p_i is the ith input price, $i = 1,\ldots, n$

The second term of the equality substitutes for the relevant price ratio for each marginal product. The sum of the production elasticities is called the function coefficient (Carlson, Allen) and defines the degree of homogeneity or returns to scale (ζ):

(9) $\sum\alpha_i = \zeta \;\; i=1,\ldots n$

where each production elasticity is assumed to be non-negative over the relevant range of the production function. Furthermore, each production elasticity may change as input utilization changes. Define ρ_{ij} as

(10) $\rho_{ij} = (\partial\alpha_i/\partial x_j)x_j\alpha_i^{-1}$

$(i, j = 1,...,n)$

Griliches has shown that

(11) $dy/dp = (\partial y/\partial x_i)(dx_i/dp) + ... + (\partial y/\partial x_n)(dx_n/dp)$.

Define η_{ip} as the elasticity of demand for the ith factor with respect to the output price p. Then the elasticity of supply with respect to output price (e_p) is

$$\varepsilon_p=(dy/dp)(p/y) =(\partial y/\partial x_i)(x_i y^{-1})(dx_i/dp)(px_i^{-1})$$
$$+ ... + (\partial y/\partial x_n)(x_n y^{-1})(dx_n/dp)(px_n^{-1}),$$

or,

(12) $\varepsilon_p = \Sigma\alpha_i\eta_{ip}$.

Similarly, the elasticity of output supply with respect to the *i*th input price (ε_{pi}) is:

(13) $\varepsilon_{pi} = \alpha_i \Sigma \eta_{jpi}$.

Direct- and Cross-Variability among Production Elasticities

In the general case, the production elasticity of input *i* changes in response to adjustments in the use of input *j* as measured by ρ_{ij}. Thus,

(14) $\rho_{ij} < 0$, $\rho_{ij} = 0$ or $\rho_{ij} > 0$ depending on if the inputs are competitive, complementary or independent in the production process.

In general, the direct elasticities (ρ_{ii}) would be expected to have a negative sign and be smaller in absolute value than the positive cross elasticities ρ_{ij}. To the extent that the relevant second cross-partial derivatives of the production function are positive, the additional use of input x_i would *increase* the productivity of the remaining inputs.[3]

From equation (8),

(15) $x_i = \alpha_i p p_i^{-1} y$, or

(16) $y = \alpha_i^{-1} p_i p^{-1} x_i$.

Differentiating equation (15) with respect to output price *p*:

(17) $$\partial x_i/\partial p = \alpha_i(y p_i^{-1} + p p_i^{-1} \partial y/\partial p) + p y p_i^{-1}\, \partial\alpha_i/\partial x_i\;\; \partial x_i/\partial p + \sum_{j\neq i} p y p_i^{-1}\, \partial\alpha_i/\partial x_j\;\; \partial x_j/\partial p$$

The elasticity of factor demand for the ith input with respect to the output price *p* follows from equation (17)

(18) $$\eta_{ip} = (1+\varepsilon_p + \sum_{j\neq i}\rho_{ij}\varepsilon_{jp})(1-\rho_{ii})^{-1}$$

where

ε_{ip} is the supply elasticity with respect to output price and η_{jp} is the factor *j* demand elasticity with respect to output price.

To the two well-known fundamental relationships [equations (4) and (6)] derived by Mosak, we add a third relationship that also follows from equations (4) and (6):

(19) $\Sigma\eta_{ip} = \varepsilon_p$.

The elasticity of supply with respect to the output price is equal to the sum of the respective input demand elasticities with respect to the output price p.

Differentiating equation (16) with respect to input price results in

(20) $\partial y/\partial p_i = \alpha_i^{-1}(x_i p^{-1} + p_i p^{-1}\, \partial \mathrm{x_i}/\partial \mathrm{p_i}) - \alpha_i^{-2} p_i x_i p^{-1}\, \partial\alpha_i/\partial x_i\ \ \partial x_i/\partial p_i$ -

$\alpha_i^{-2}\ \Sigma p_i x_i p^{-1}\, \partial\alpha_i/\partial x_j\ \ \partial x_j/\partial p$ for $j \neq i$

From equation (20), the elasticity of supply with respect to each of the factor prices is:

(21) $\varepsilon_{pi} = 1 + \eta_{ipi} - \eta_{\mathrm{ipi}}\rho_{ii} - \lambda\eta_{jpi}\rho_{ij}$ for $j \neq i$

where η_{ipi} and η_{ipj} are the own- and cross-price input-demand elasticities.

It is important to note that the fundamental proposition of neoclassical profit maximizing behavior implies symmetric cross-price effects are symmetric (Varian p. 33; Fuss, Mcfadden and Mundlak). Therefore,

(22) $\eta_{ipj}\ x_i p_j^{-1} = \eta_{jpi}\ x_j p_i^{-1}$.

Differentiating equation (15) with respect to input price p_i first and then with respect to input price p_j, the factor demand own price elasticity η_{ipi} and cross price elasticity η_{ipj} is obtained as

(23) $\eta_{ipi} = (\varepsilon_{pi} - 1 + \Sigma\ \rho_{ij}\eta_{jpi})(1 - \rho_{ii})^{-1}$ for $j \neq i$

(24) $\eta_{ipj} = (\varepsilon_{pj} + \Sigma\rho_{\mathrm{ij}}\eta_{jpj})(1 - \rho_{ii})^{-1}$ for $j \neq i$.

Following Uzawa, the linkage between the cross- elasticity of input demand for factor i with respect to the price of factor j, and the elasticity of substitution between inputs i and j (G_{ij}) is

(25) $G_{ij} = \eta_{ipj}/S_i = \Sigma x_i f_i(x_i x_j)\lambda\partial x_i/\partial p_j$, where S_i is the cost share for factor i.

Differentiating equation (16) with respect to output price yields

(26) $\partial y/\partial p = \alpha_i^{-1}(-p_i x_i p^{-2}+p_i p^{-1}\partial x_i/\partial p) - \alpha_i^{-2} p_i x_i p^{-1}\ \partial\alpha_i/\partial x_i\ \partial x_i/\partial p - \alpha_i^{-2}$

$\Sigma p_i x_i p^{-1}\partial\alpha_i/\partial x_j\ \partial x_j/\partial p$ for $j \neq i$.

Thus, the elasticity of supply with respect to output price (ε_p) can be expressed in terms of the input demand elasticities with respect to the output price as

(27) $\varepsilon_p = \eta_{ip} - 1 - \eta_{ip}\rho_{ii} - \Sigma\eta_{jp}\rho_{ij}$ for $j \neq i$.

Successive substitutions would allow each of the relevant elasticities to be expressed in terms of the other elasticities.[4]

Directly-Variable Production Elasticities

In this case, assume that $\rho_{ii} \leq 0$ and $\rho_{ij} = 0$. This implies that changes in input use affect only the own production elasticities. Moreover, the greater the level of input use, the lower the production elasticity for that input. These assumptions are consistent with the neoclassical model of the three-stage production process. Equations (17) and (20) can be rewritten as

(28) $\partial x_i/\partial p = \alpha_i(yp_i^{-1} + pp_i^{-1}\ \partial y/\partial p) + (pyp_i^{-1}\ \partial\alpha_i/\partial x_i\ \partial x_i/\partial p)$

and

(29) $\partial y/\partial p_i = \alpha_i^{-1}(x_i p^{-1}+p_i p^{-1}\ \partial x_i/\partial p_i) - \alpha_i^{-2}(p_i x_i p^{-1}\ \partial\alpha_i/\partial x_i\ \partial x_i/\partial p_i)$.

From equation (28)

(30) $\eta_{ip} = \partial x_i/\partial p\ px_i^{-1} = \alpha_i(ypp_i^{-1}x_i^{-1}+ ypp_i^{-1}x_i^{-1}\varepsilon_p) + (ypp_i^{-1}x_i x_i^{-1}\ \partial\alpha_i/\partial x_i\ \eta_{ip})$,

and

(31) $\eta_{ip} = 1+ \varepsilon_p + \rho_{ii}\eta_{ip} = (1 + \varepsilon_p)(1 - \rho_{ii})^{-1}$.

Similarly, from equation (29),

(32) $\varepsilon_{pi} = (\partial y/\partial p_i)p_i y^{-1}$

$$= \alpha_i^{-1}[x_i p_i p^{-1} y^{-1} + p_i p^{-1}\, \partial x_i/\partial p_i\, p_i y^{-1}] - \alpha_i^{-2} p_i^2 x_i^2 p^{-1} y^{-1} x_i^{-1}\, \partial \alpha_i/\partial x_i\, \partial x_i/\partial p_i,$$

and

(33) $\varepsilon_{pi} = 1 + \eta_{ipi}(1-\rho_{ii})$.

Equation (33) also implies that the elasticity of output supply with respect to an input price (ε_{pi}) is smaller than or equal to the elasticity of the demand for the input with respect to its own price (η_{ipi}). The relationship between the two elasticities depends specifically on the value of ρ_{ii}.

The elasticity of the demand for an input with respect to its own price (η_{ipi}) is obtained from the differentiation of equation (15) with respect to the own-input price.

(34) $\eta_{ipi} = \alpha_i(-pyp_i p_i^{-2} x_i^{-1} + \partial y/\partial p_i\, pyp_i p_i^{-1} y^{-1} x_i^{-1}) + \eta_{ipi} pyp_i p_i^{-1} x_i^{-1}\, \partial \alpha_i/\partial x_i,$

and

(35) $\eta_{ipi} = \varepsilon_{pi} - l + \eta_{ipi}\rho_{ii} = 2\rho_{ii}$.

The elasticity of supply with respect to product price can be obtained by combining equations (12) and (29) as

(36) $\varepsilon_p = \Sigma\alpha_i\eta_{ip} = \Sigma\alpha_i(1+\varepsilon_p)(1-\rho_{ii})^{-1}$.

Solving for ε_p

(37) $\varepsilon_p = [\Sigma\alpha_i(1-\rho_{ii})^{-1}][1- \Sigma\alpha_i(1-\rho_{ii})]^{-1}$.

Each production elasticity is divided by one minus its own ρ_{ii}. The ρ_{ii} takes into account the sensitivity to changes in the level of input use.

Using equation (37), the elasticity of input demand with respect to the product price (η_{ip}) can be restated as

(38) $\eta_{ip} = [1- \Sigma\alpha_i(1-\rho_{ii})^{-1}]^{-1}(1-\rho_{ii})^{-1}$.

Each input-demand elasticity with respect to the product price decreases at the rate at which the elasticity of production decreases with the expanded use of the input.

Cobb-Douglas Production Functions

The Cobb-Douglas function is a special case in which the elasticities of production are constant and therefore the ρ_{ij} and ρ_{ii} are equal to zero. Differentiating equation 15 with respect to the product price and equation 16 with respect to the input price yields

(39) $\partial x_i/\partial p = \alpha_i(yp_i^{-1} + \partial y/\partial p\ pp_i^{-1})$

(40) $\partial y/\partial p_i = \alpha_i^{-1}(x_i p^{-1} + \partial x_i/\partial p_i\ p_i p^{-1})$.

Then the elasticity of factor demand with respect to output price (η_{ip}) and the elasticity of supply with respect to the input price (ε_{pi}) can be obtained as

(41) $\eta_{ip} = \alpha_i(pyp_i^{-1}x_i^{-1} + pyp_i^{-1}x_i^{-1}\ \partial y/\partial p\ py^{-1})$, and

(42) $\varepsilon_{pi} = \alpha_i^{-1}(p_i x_i p^{-1}y^{-1} + p_i x_i p^{-1}y^{-1}\ \partial x_i/\partial p_i\ p_i x_i^{-1}\ pyp_i^{-1}x_i^{-1})$.

An alternative way for obtaining e_{pi} in terms of direct- and cross-price elasticities of input demand will allow for an explicit linkage between ε_{pi} and the cross elasticity η_{ipj}. Thus,

(43) $\varepsilon_{pi} = \partial y/\partial p_i\ p_i y^{-1} = (\partial y/\partial x_i\ \partial x_i/\partial p_i - \sum_{j\neq i} \partial y/\partial x_j\ \partial x_j/\partial p_i)p_i y^{-1}$.

Rewriting (43),

(44) $\varepsilon_{pi} = \alpha_i\eta_{ipi} + \sum_{j\neq i}\alpha_j\eta_{jpi} = \sum_{j\neq i,\, j=i}\alpha_j\eta_{jpi}$.

The expression for the cross-price elasticity of input demand is obtained from

(45) $\eta_{jpi} = \partial x_j/\partial p_i\ p_i x_j^{-1} = \partial(x_j = \alpha_j pp_j^{-1}y)/\partial p_i\ p_i x_j^{-1}$

$= \alpha_i\eta_{ipi} + \Sigma\alpha_j\eta_{jpi} = \Sigma\alpha_j\eta_{jpi}$.

Thus, $\varepsilon_{pi} = \eta_{jpi}$, and it follows that the elasticity of product supply with respect to the *i*th input price equals the input demand cross elasticities with respect to the *i*th input price.

Differentiating equation (15) with respect to the input price (p_i) yields

$$(46) \quad \partial x_i/\partial p_i = \alpha_i(-pyp_i^{-2} + \partial y/\partial x_i\, \partial x_i/\partial p_i\, pp_i^{-1} + \sum_{j\neq i} \partial y/\partial x_j \partial x_j/\partial p_i\, pp_i^{-1}).$$

The elasticity of factor demand with respect to the *i*th input price (η_{ipi}) is:

$$(47) \quad \eta_{ipi} = \sum_{i\neq j\ i=j} \alpha_i \eta_{jpi}^{-1} = \varepsilon_{pi}^{-1}.$$

Therefore,

$$(48) \quad \varepsilon_p = \Sigma\alpha_i(1+\varepsilon_p) = \Sigma\alpha_i\, \eta_{ip}.$$

For the Cobb-Douglas case, starting with the production function, $y = A\Pi x_i^{\delta i}$, the demand for an input and the corresponding price elasticities can be derived from the profit-maximizing equation. The elasticities are exactly the same as those derived here.

The *i*th input demand elasticity with respect to the output price is given by

$$(49) \quad \eta_{ip} = (\delta-1)^{-1}.$$

Substituting equation 49 into equation 47 yields

$$(50) \quad \varepsilon_p = \delta(\delta-1)^{-1}.$$

From the Cobb-Douglas production technology, the derived *i*th input demand has an elasticity with respect to its own price equal to

$$(51)\ \eta_{ipi} = [1-(\delta-\alpha_i][\delta-1]^{-1}.$$

And from equation 47:

$$(52)\ \eta_{ipi} = \varepsilon_{pi}-1 = [1-(\delta-\alpha_i)][\delta-1]^{-1}.$$

Concluding Comments

Unbiased estimates of technological parameters of an underlying production function can be obtained by first estimating the cost-share equations. From the parameter estimates of the cost share equations, it is possible to obtain all relevant elasticities, including production elasticities, own and cross input-demand elasticities, the elasticity of input demand with respect to the product price, and the output supply elasticities. Agricultural input demand elasticities with respect to the output price equals the elasticity of supply with respect to the output price, based on cost-share data.

This paper has derived some simple and previously unrecognized linkages among the various elasticities of production, supply and factor demand for three subclasses of the Generalized Power Production Function. The work by Mosak, although pioneering, could not have taken into account the new forms of production functions developed over the past forty years. Mosak did not provide explicit linkages, but rather a series of cumbersome expressions. The equations contained in this paper provide explicit linkages that can be used by agricultural economists for obtaining elasticities from a diverse array of specific functional forms.

Footnotes

[1]The production function is a technical relationship between inputs and output representing the minimum amount of inputs necessary to produce a unit of output, or the maximum output that can be produced by a given level of inputs. A serious problem arises in the measurement of the inputs which represent aggregate quantities (Varian, p. 119).

[2]Independence between output and the error term would exist if the amount of output supplied is determined exogenously (demand conditions). In general, estimation of aggregate production relationships using time-series data would have to account for endogenously determined prices.

[3]Although the analysis here is partial in that prices of inputs are not allowed to change as a result of changes in their quantity demanded, this can be remedied by considering more than one production process. If the supply function of the input- producing industry is added, an estimate of the own-price flexibility of input supply can be obtained as

(1') $\kappa_i = \partial p_i/\partial x_i \, x_i p_i^{-1}$.

By differentiating the profit-maximizing relationship, an additional term representing the partial derivatives with respect to the variable input prices is obtained.

[4]Using a simple cost-minimization approach, the aggregate-supply function (Pagoulatos and Debertin) in log-linear form is

(2') $lny = (1 - \zeta)^{-1} \, lnA \prod_i \delta_i^{\delta i} - \sum_i (1-\zeta)^{-1} \delta_i lnp_i + \zeta(1-\zeta)^{-1} \, lnp.$

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