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A GENERAL EQUILIBRIUM MODEL OF PRODUCTION WITH A RANDOM MARGINAL RATE OF SUBSTITUTION

by

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Recently, in this journal, Batra [1] has analyzed the qualitative effects of risk aversion in a two-sector general equilibrium model where uncertainty is due to random production in one of the sectors. By extending Rothenberg and Smith's [4] risk-neutral model, the effects of uncertainty on factor intensities, returns, and national income were explored invoking a small country assumption (i.e., price of output is fixed). Batra proves three basic results: (1) factors of production move from the uncertain industry to the certain industry as uncertainty is introduced; (2) if the uncertain sector is relatively capital intensive, then capital-labor ratios rise in both sectors; and (3) real returns to the intensive factor in the uncertain industry fall and returns to the intensive factor of the certain sector rise. His results are derived assuming risk aversion and a factor-neutral, multiplicative random disturbance in the production function of the uncertain sector,

$$X_1 = \varepsilon F_1(K_1, L_1) = \varepsilon L_1 f(k_1), \tag{1}$$

where X_1 is the output of industry 1; K_1 and L_1 are the quantities of capital and labor utilized in industry 1, respectively; F_1 denotes a transformation function; ε is a nonnegative random disturbance; and k_1 denotes the capital-labor ratio under the assumption of linear homogeneity. In this case the expected utility maximizer chooses K_1 and L_1 such that

$$\frac{w}{r} = \frac{F_{L1}}{F_{K1}} \tag{2}$$

where w/r is the labor-capital cost ratio, and alphabetic subscripts denote differentiation. Thus, the firm minimizes costs for each level of expected

output, and uncertainty does not intrinsically affect the choice of technique. In other words risk preferences do not affect the marginal rate of substitution at the optimum (other than through changes in expected output).

In this paper technology is generalized such that the marginal rate of substitution is uncertain. Moreover, the possibility of decreasing risk via capital- (or labor-) intensive technology is incorporated in the analysis. It is found that all three of the above propositions are sensitive to the manner in which factors of production affect the probability distribution of production. In general, the condition that both factors of production have a positive marginal risk effect is sufficient to obtain Batra's results when the marginal rate of substitution is random; but some alternative cases are developed which may also be reasonable. An implicit assumption of Batra's work which is continued in this paper is that contingent (futures) markets are not operating with respect to production risks.

1. A MORE GENERAL STOCHASTIC PRODUCTION FUNCTION

The production function for the uncertain sector examined in this paper is of the form

$$X_1 = F_1(K_1, L_1) + G_1(K_1, L_1)\varepsilon, \; E(\varepsilon) = 0, \; E(\varepsilon^2) < \infty \tag{3}$$

where, without loss of generality, it is assumed that $G_1 > 0$. Note that if $G_1 = F_1$, then (3) becomes $X_1 = F_1(K_1, L_1)(1 + \varepsilon)$ which is the special case examined by Batra.

A major advantage of (3) relates to the marginal impact of factor changes on risk. Focusing more specifically on the variance of output, $V(X_1)$, the marginal effect is

$$\frac{\partial V(X_1)}{\partial K_1} = 2G_1G_{K1}V(\varepsilon) \gtreqless 0 \text{ as } G_{K1} \gtreqless 0$$

$$\frac{\partial V(X_1)}{\partial L_1} = 2G_1G_{L1}V(\varepsilon) \gtreqless 0 \text{ as } G_{L1} \gtreqless 0.$$

Hence, unlike the usual specification in (1), factors of production may positively contribute to mean output while having a negative effect on the variance.[7]

Several reasonable situations suggest that this flexibility may be needed. For example, with random employee strikes in developing economies, firms may diminish risk by increasing the capital-labor ratio. Also, in developing countries, capital intensification in agriculture often has the effect of diminishing output variability due to adverse weather conditions (Just and Pope [3]). As may easily be shown, the general class of production functions with multiplicative disturbances preclude such cases.

2. COMPETITIVE BEHAVIOR OF THE UNCERTAIN SECTOR

The assumption of linear homogeneity in (3), i.e., constant stochastic returns to scale (see, for example, Stiglitz [6]), implies that (3) can be written as

$$X_1 = L_1f_1(k_1) + L_1g_1(k_1)\varepsilon.$$

It is assumed that marginal productivity is positive ($F_{K1}, F_{L1} > 0$) and that both F_1 and G_1 are concave; specifically, $F_{KK1} = f_1''/L_1 < 0$; $G_{KK1} = g_1''/L_1 < 0$; $F_{LL1} = \left(k_1^2/L_1\right) f_1'' < 0$; $G_{LL1} = \left(k_1^2/L_1\right) g_1'' < 0$.

Section 1 maximizes the expected utility of profit, $E[U_1(\Pi_1)] \equiv E[U_1]$, by controlling input levels K_1 and L_1. Where $\Pi_1 = PX_1 - wL_1 - rK_1$, first-order conditions are:

$$E(U_1' \cdot \Pi_{K1}) = E(U_1' \cdot [P(F_{K1} + G_{K1}\varepsilon) - r]) = 0 \tag{4}$$

$$E(U_1' \cdot \Pi_{L1}) = E(U_1' \cdot [P(F_{L1} + G_{L1}\varepsilon) - w]) = 0. \tag{5}$$

From (4) and (5), one can deduce that

$$w = \frac{E[U_1' \cdot P(F_{L1} + G_{L1}\varepsilon)]}{E(U_1')} \tag{6}$$

$$r = \frac{E[U_1' \cdot P(F_{K1} + G_{K1}\varepsilon)]}{E(U_1')} \tag{7}$$

and

$$\frac{E(U_1' \cdot \varepsilon)}{E(U_1')} = \frac{r - PF_{K1}}{PG_{K1}} = \frac{w - PF_{L1}}{PG_{L1}}. \tag{8}$$

Finally, using (8), it can be shown that

$$\Pi_{L1} = \Pi_{K1}\delta \tag{9}$$

where

$$\delta \equiv \frac{w - PF_{L1}}{r - PF_{K1}} = \frac{G_{L1}}{G_{K1}} = \frac{g_1 - k_1g_1'}{g_1'}$$

and, thus, δ provides a convenient means of expressing one derivative of the profit function in terms of the other.

Second-order conditions for a unique maximum in section 1 require

$$D = \begin{vmatrix} E\left[U_1' \cdot \Pi_{LL1} + \Pi_{L1}^2 U_1''\right] & E\left[U_1' \cdot \Pi_{KL1} + \Pi_{L1}\Pi_{K1}U_1''\right] \\ E\left[U_1' \cdot \Pi_{KL1} + \Pi_{L1}\Pi_{K1}U_1''\right] & E\left[U_1' \cdot \Pi_{KK1} + \Pi_{K1}^2 U_1''\right] \end{vmatrix} > 0,$$

and

$$E\left[U_1' \cdot \Pi_{LL1} + \Pi_{L1}^2 U_1''\right] < 0.$$

Assuming risk aversion ($U_1'' < 0$) and concavity of all possible realizations (all ε) of the production function (which implies concavity of Π), it can be readily verified that second-order conditions must hold. Furthermore, it can be shown that second-order conditions place no restrictions on the sign of δ; and, hence, the case of risk-reducing inputs can be meaningfully investigated.

3. COMPLETION OF THE GENERAL EQUILIBRIUM MODEL

Suppose now that industry 2 has nonrandom production denoted by

$$X_2 = F_2(K_2, L_2) = L_2 f_2(k_2)$$

where $k_2 = K_2/L_2$. First-order conditions are

$$w = f_2 - k_2 f_2' \tag{10}$$

$$r = f_2' \tag{11}$$

and second-order conditions are guaranteed by the concavity of F_2. It may be further noted that concavity of F_2 implies F_{LL2}, $F_{KK2} < 0$, or $f_2'' < 0$.

Given a competitive equilibrium, equations (6), (7), (10), and (11) indicate that

$$E[U_1' \cdot P(F_{L1} + G_{L1}\varepsilon)] = F_{L2}\, E(U_1') \tag{12}$$

$$E[U_1' \cdot P(F_{K1} + G_{K1}\varepsilon)] = F_{K2}\, E(U_1'). \tag{13}$$

Finally, assuming full employment, two resource constraints complete the model of the economy:

$$L_1 + L_2 = L \tag{14}$$

$$K_1 + K_2 = K = L_1k_1 + L_2k_2 \tag{15}$$

where L and K are total resource endowments of labor and capital, respectively

4. THE IMPACT OF UNCERTAINTY ON THE ECONOMY

As in Batra, the major focus here is to determine the impact of marginal changes in risk on factor incomes, employment, and national income using the mean-preserving spread technique of Sandmo [5]. Defining

$$\varepsilon^* = \gamma\varepsilon,$$

any change in the parameter γ has the effect of changing the spread of the distribution while leaving $E(\varepsilon^*) = 0$. Hence, the marginal impact of uncertainty can be found by differentiating (14)-(17) with respect to γ (and evaluating at $\gamma = 1$). Under linear homogeneity, it is convenient to consider these equations as functions of L, k_1, and k_2. The resulting implicit differentiation yields

$$E\left[U_1'' \cdot P(F_{L1} + G_{L1}\varepsilon)\frac{d\Pi_1}{d\gamma} + U_1' \cdot PG_{L1}\varepsilon + PU_1' \cdot (F_{Lk1} + G_{Lk1}\varepsilon)\frac{dk_1}{d\gamma} - F_{L2}\,U_1'' \cdot \frac{d\Pi_1}{d\gamma} - U_1' \cdot F_{Lk2}\frac{dk_2}{d\gamma}\right] = 0 \tag{16}$$

$$E\left[U_1'' \cdot P(F_{K1} + G_{K1}\varepsilon)\frac{d\Pi_1}{d\gamma} + U_1' \cdot PG_{K1}\varepsilon + PU_1' \cdot (F_{Kk1} + G_{Kk1}\varepsilon)\frac{dk_1}{d\gamma} - F_{K2}\,U_1'' \cdot \frac{d\Pi_1}{d\gamma} - U_1' \cdot F_{Kk2}\frac{dk_2}{d\gamma}\right] = 0 \tag{17}$$

$$(k_1 - k_2)\frac{dL_1}{d\gamma} + L_1\frac{dk_1}{d\gamma} + L_2\frac{dk_2}{d\gamma} = 0.\text{[8]} \tag{18}$$

Solution of (16)-(18) through use of Cramer's rule obtains

$$\frac{dL_1}{d\gamma} = \frac{L_2H_2P(k_1 + \delta)\,E(U_1' \cdot Y) + f_2''(k_2 + \delta)\,E(U_1')\,L_1H_2}{|A|} \tag{19}$$

$$\frac{dk_1}{d\gamma} = -\frac{(k_1 - k_2)\,H_2f_2''(k_2 + \delta)\,E(U_1')}{|A|} \tag{20}$$

$$\frac{dk_2}{d\gamma} = -\frac{(k_1 - k_2)\,H_2P(k_1 + \delta)\,E(U_1' \cdot Y)}{|A|} \tag{21}$$

where

$$H_2 = -E(PG_{K1}\,U_1' \cdot \varepsilon + U_1'' \cdot \Pi_{K1}\,G_1P\varepsilon) \tag{22}$$

and $|A| > 0$ (see the Appendix for details).

Several lemmas will be helpful in establishing the qualitative nature of (19)-(21). Lemma 1 can be obtained through application of lemmas derived by Feder [2]; however, since parts (a) and (b) are somewhat different than

explicit results which have appeared previously, their derivation is also presented in the Appendix.

LEMMA 1. If absolute risk aversion is nonincreasing (but positive), then

(a) $\operatorname{sgn} E[U_1'' \cdot \Pi_{K1}] = \operatorname{sgn} G_{K1}$

$\operatorname{sgn} E[U_1'' \cdot \Pi_{L1}] = \operatorname{sgn} G_{L1}$

(b) $E[U_1'' \cdot \Pi_{K1}\,G_{K1}\varepsilon],\ E[U_1'' \cdot \Pi_{L1}\,G_{L1}\varepsilon] < 0$

(c) $E[U_1' \cdot \varepsilon] < 0$.

LEMMA 2. If every realization of the production function satisfies concavity, then

$E[U_1' \cdot Y] < 0$

where $Y \equiv f_1'' + g_1''\varepsilon = L_1\,(F_{KK1} + G_{KK1}\varepsilon)$.

Proof. If $F_1 + G_1\varepsilon$ is concave for all ε, then

$$Y \equiv f_1'' + g_1''\,\varepsilon < 0 \quad \text{for all } \varepsilon.$$

Since $U_1' > 0$, the conclusion of the lemma follows.

LEMMA 3. Assuming nonincreasing absolute risk aversion,

$\operatorname{sgn} H_2 = \operatorname{sgn} g_1'$.

Proof. Multiplying and dividing (22) by G_{K1} or, equivalently, by g_1' yields

$$H_\gamma = -\frac{PG_{K1}^{2} E(U_1'' \cdot \epsilon) + PG_1 G_{K1} E(U_1'' \cdot \Pi_{K1}\epsilon)}{g_1'} .$$

Assuming risk aversion, the first term in parentheses is negative by Lemma 1(c); the second term in parentheses is negative by Lemma 1(b). Therefore, $\operatorname{sgn} H_\gamma = \operatorname{sgn} g_1'$.

LEMMA 4. With concavity of f_1 and g_1 and nonincreasing absolute risk aversion, $|A|' > 0$.

Proof. See the Appendix.

Turning now to equation (20), it is clear from concavity and Lemma 3 that

$$\operatorname{sgn} \frac{dk_1}{d\gamma} = \operatorname{sgn} [(k_1 - k_2)\, g_1'(k_2 + \delta)]. \tag{23}$$

Using this equation and the above lemmas, the following theorem can be derived.

THEOREM 1. Suppose there is an increase in production uncertainty in the uncertain sector. If the uncertain sector is capital intensive, then (i) the capital-labor ratio in the uncertain sector will unambiguously increase if labor marginally increases production risk or capital marginally reduces production risk; but (ii) if labor marginally reduces production risk, the capital-labor ratio will decrease if the marginal risk reduction is sufficiently large, specifically,

$$\frac{dk_1}{d\gamma} \gtreqless 0 \qquad \text{as } G_{L1} \gtreqless -\frac{k_2}{k_1 - k_2}\frac{G_1}{L_1}. \tag{24}$$

Conversely, if the uncertain sector is labor intensive, then (iii) the capital-labor ratio in the uncertain sector will unambiguously decrease if capital marginally increases risk or labor marginally reduces risk; but (iv) the capital-labor ratio will increase if the marginal risk effect of capital is sufficiently large negatively, specifically,

$$\frac{dk_1}{d\gamma} \gtreqless 0 \qquad \text{as } G_{K1} \lesseqgtr \frac{1}{k_1 - k_2}\frac{G_1}{L_1}. \tag{25}$$

Proof. Theorem 1 is proved by determining the signs of terms on the right-hand side of equation (23). To do this, note that

$$k_2 + \delta = k_2 + \frac{G_{L1}}{G_{K1}} = (k_2 - k_1) + \frac{g_1}{g_1'} \tag{26}$$

and that $G_{L1} = g_1 - k_1 g_1' = g_1 - k_1 G_{K1}$; therefore, $G_{L1} < 0$ implies $G_{K1} > 0$ and $G_{K1} < 0$ implies $G_{L1} > 0$. Consider case (i) where either both G_{L1}, $G_{K1} > 0$, or $G_{L1} > 0$, $G_{K1} < 0$. Using the middle expression in (26), it is clear that G_{L1}, $G_{K1} > 0$ implies $k_2 + \delta > 0$; and from (23) $\operatorname{sgn} dk_1/d\gamma = \operatorname{sgn}(k_1 - k_2)$. If $G_{L1} > 0$ and $g_1' < 0$, then $k_2 + \delta$ and g_1' always have the same sign when $k_1 - k_2 > 0$, thus verifying case (i). Case (iii) is established in a similar manner from (26) by noting that G_{K1}, $G_{L1} > 0$ implies $dk_1/d\gamma < 0$ when $k_1 - k_2 < 0$. If $G_{K1} > 0$ and $G_{L1} < 0$, then $k_2 + \delta > 0$ from (26) if $k_1 - k_2 < 0$, thus verifying the result. Finally, from (23) and (26), note that $dk_1/d\gamma < 0$ if $g_1(k_1 - k_2) - (k_1 - k_2)^2 g_1' < 0$. Case (ii) follows from (23) when $k_1 - k_2 > 0$ by noting that $g_1' = (g_1 - G_{L1})/k_1$ and $g_1 = G_1/L_1$. Case (iv) is established from (23) using (26) where $k_1 - k_2 < 0$ and $g_1' = G_{K1}$.

Remark 1. The implication of the multiplicative disturbance which has been commonly used for simplicity in previous literature implies, in the context of the model of this paper, that $G_{L1}, G_{K1} > 0$. By (26) and Lemma 3, this is a special case where (23) reduces to

$$\operatorname{sgn} \frac{dk_1}{d\gamma} = \operatorname{sgn} (k_1 - k_2)$$

which is the result noted by Batra. Theorem 1 shows that similar results are also obtained in other general cases of qualitative marginal risk effects. Ambiguous results remain only in cases where $G_{L1} < 0$, $k_1 - k_2 > 0$ and $G_{K1} < 0$, $k_1 - k_2 < 0$, i.e., where the marginal risk effects "work against" the relative factor intensities. Nevertheless, Theorem 1 shows that results contrary to Batra's results are possible. For example, if labor sufficiently reduces risk at the margin, then an increase in uncertainty can cause a decline in the capital-labor ratio of a capital-intensive uncertain sector.

Remark 2. Intuition indicates that, as uncertainty is introduced, an input which reduces (increases) uncertainty should become relatively more (less) valuable, *ceteris paribus*, to the sector facing uncertainty. Indeed, this is suggested by (6) and (7). Furthermore, one can note that if risk is increasing in the capital-labor ratio (which implies risk is increasing in capital), then risk must be decreasing in the labor-capital ratio (and thus also be decreasing in labor) and conversely. Thus, if capital marginally reduces risk in case (i) of Theorem 1, then capital becomes relatively more valuable while labor becomes less valuable and, hence, the capital-labor ratio should unambiguously increase since the marginal risk effects tend to strengthen the relative factor intensities. Similar reasoning applies where labor marginally reduces risk in case (iii) of Theorem 1. The intuition

where both inputs marginally increase risk follows along earlier lines suggested by Batra. That is, if both inputs increase risk, then introduction of uncertainty causes resources to shift away from the uncertain sector; thus, expected output is reduced along the (concave) contract curve in the Edgeworth-Bowley box causing the capital-labor ratio to increase (decrease) in the capital- (labor-) intensive case.

The results in Theorem 1 and intuition of Remark 2 suggests that the absolute use of risk-reducing inputs may actually increase in the uncertain sector as uncertainty is introduced. In the following theorem it is indeed found that the conditions which govern directional input changes in response to risk are closely tied to cases (i)-(iv) of Theorem 1.

THEOREM 2.

(a) With a marginal increase in uncertainty in cases (i) and (iii) of Theorem 1, both labor input and capital input in the uncertain sector are reduced, while both inputs increase in the certain sector.

(b) For cases (ii) and (iv) of Theorem 1, however, an increase in uncertainty leads to an increase in capital or labor in the uncertain sector if the associated marginal risk reduction of the relevant input is sufficiently large.

Proof. From (19), one finds that

$$\operatorname{sgn} \frac{dL_1}{d\gamma} = \operatorname{sgn} \frac{\left\{\left[L_2 PE(U_1' \cdot Y)\, g_1/g_1' + L_1 f_2''(k_2 + \delta)\, E(U_1')\right]\right\} H_2}{|A|} \tag{27}$$

Utilizing Lemma 4 and the fact that $\delta = g_1/g_1' - k_1$, equation (27) becomes

$$\operatorname{sgn} \frac{dL_1}{d\gamma} = \operatorname{sgn} \frac{\left\{L_2 PE(U_1' \cdot Y)\, g_1 + f_2'' L_1 [(k_2 - k_1)\, g_1' + g_1]\, E(U_1')\right\} H_2}{g_1'}. \tag{28}$$

Since $H_2/g_1'' > 0$ (by Lemma 3), Lemmas 2 and 4 and concavity of F_2 imply $dL_1/d\gamma < 0$ iff $(k_2 - k_1)\, g_1'' > 0$. Thus, $g_1'' > 0$ and $(k_2 - k_1) > 0$ establishes (a) for case (iii) of Theorem 1. Alternatively, if $G_{L1} > 0$ and $G_{K1} < 0$, then $(k_2 - k_1)\, g_1'' > 0$ when $(k_2 - k_1) < 0$; or if $G_{L1} > 0$ and $G_{K1} > 0$, then $k_2 + \delta > 0$ from (27). Thus, (a) follows for case (i) of Theorem 1. To see that capital also decreases under similar conditions, note that $K_1 = L_1 k_1$; hence, using equations (19)-(21),

$$\frac{dK_1}{d\gamma} = L_1 \frac{dk_1}{d\gamma} + k_1 \frac{dL_1}{d\gamma}$$

$$= \frac{\{L_1 k_2(k_2 + \delta) f_2'' E(U_1') + k_1 L_2 P E(U_1' \cdot Y)\ (g_1/g_1')\}\ H_2}{|A|}.$$

Again, use of $\delta = g_1/g_1' - k_1$, along with Lemmas 2 and 3, implies that

$$\frac{dK_1}{d\gamma} = \frac{L_1 k_2[(k_2 - k_1)\ g_1' + g_1]\ f_2'' E(U_1') + k_1 L_2 P E(U_1' \cdot Y)\ g_1\ H_2}{g_1' |A|} < 0$$

if $(k_2 - k_1)\, g_1' > 0$ which verifies the assertions of (a) relating to capital. The remaining assertions of part (a) thus follow upon additionally using (14) and (15). Finally, part (b) is obtained when $(k_2 - k_1)\, g_1' < 0$ [cases (ii) and (iv) of Theorem 1] such that, for labor,

$$E(U')f_2'' L_1[(k_2 - k_1)\ g_1'] > |f_2'' L_1 g_1 E(U_1') + L_2 P E(U_1' \cdot Y)\ g_1| \qquad (29)$$

or, for capital,

$$L_1 k_2(k_2 - k_1)\ g_1' f_2'' E(U_1') > |[L_1 k_2 g_1 f_2'' E(U_1') + k_1 L_2 P E(U_1' \cdot Y)\ g_1]|. \qquad (30)$$

Remark 3. Hence, when the marginal risk effects do not tend to strengthen the relative factor intensity of the uncertain sector [cases (ii) and (iv)], a marginal increase in uncertainty may indeed lead to increased use of the

risk-reducing factor. Comparing (29) and (30) with (24) and (25), however, it is clear that a reversal of the effects of uncertainty on absolute factor use (Theorem 2) requires a stronger negative marginal risk effect than is necessary to reverse the effects of uncertainty on relative intensities in the uncertain sector (Theorem 1). Nevertheless, only when both inputs are risk increasing or where the risk-reducing factor is already used intensively does a marginal increase in uncertainty unambiguously lead to a reduction in both input uses. Thus, only in the latter case [corresponding to (i) and (iii) of Theorem 1] is it clear that an increase in uncertainty causes a decrease in expected output in the uncertain sector and an increase in output in the certain sector.

Given the ambiguity of Theorems 1 and 2 relating to the effects of uncertainty on relative factor intensities in the uncertain sector and on absolute factor employment levels in both sectors, it may seem reasonable in a general equilibrium context to expect similar ambiguities with respect to relative factor intensities in the certain sector as well. However, Theorem 3 determines that this is not the case and, in fact, Batra's results relating to certain sector-relative intensities can be generalized.

THEOREM 3. An increase in uncertainty lowers (raises) the capital-labor ratio in the certain sector if it is relatively more (less) capital intensive regardless of whether factors increase or decrease risk.

Proof. Using equation (21), it is readily verified that sgn $[-P(k_1 + \delta)\ E(U_1' \cdot Y)]$ = sgn g_1' and, by Lemma 3, that sgn H_2 = sgn g_1'. Hence, using Lemma 4, one finds sgn $dk_2/d\gamma$ = sgn $(k_1 - k_2)$.

In the standard case of risk-increasing inputs, intuition suggests that a factor's real reward declines as uncertainty increases. But given the above cases where use of a risk-reducing input in the uncertain sector can increase with increased uncertainty, the effects of uncertainty on such a factor's real reward are no longer clear. Theorem 4 shows, however, that real rewards decline with increased uncertainty in any case.

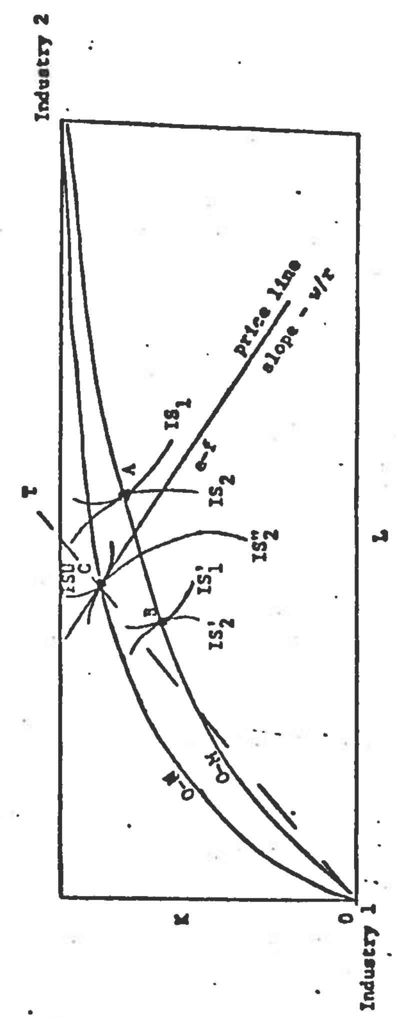
THEOREM 4. An increase in uncertainty causes a decline in the real reward of the factor used intensively by the industry facing uncertainty and a rise in the real reward of the factor used intensively by the certain industry regardless of whether factors marginally reduce or increase risk.

Proof. From Theorem 3, if $k_2 > k_1$, then $dk_2/d\gamma < 0$ which by concavity implies that f_2' and r rise. Similarly, if f_2' rises, then $F_{L2} = f_2 - k_2 f_2'$ falls and w falls also. If, on the other hand, $k_1 > k_2$, then r falls and w rises by similar reasoning.

Remark 4. Theorem 4 further emphasizes the fact that risk is costly. That is, for example, an increase in risk reduces the social value of each factor used in the uncertain sector no matter how strong its risk-reducing marginal effect may be.

5. AN EXPOSITORY EXAMPLE

The results of Theorems 1-4 indicate a somewhat surprising result: the directional impact of risk on the capital-labor ratio in the certain sector is unaltered from Batra's case of increasing risk even though the directional impacts on the capital-labor ratio for the uncertain sector may be altered. It is perhaps helpful to consider the intuition of these results in the context of an Edgeworth-Bowley Box. For expositional purposes, assume industry 1 is relatively capital intensive. In the figure the risk-neutral contract curve is labeled OM. Similarly, in Batra's case



Edgeworth-Bowley Box

of risk aversion each firm equates the marginal rate of substitution (based on expected output) with the factor-price ratio. As risk is increased in this case, production in sector 1 declines while the output of sector 2 expands. This takes place along the risk-neutral contract curve in Batra's case since the relative effects of the inputs on output variability are the same as on expected output. Thus, equilibrium changes from points A to B. Since industry 1 uses relatively more capital (and less labor), the return to capital must fall (and returns to labor rise) to accommodate the contraction of sector 1 (and expansion of sector 2).

In the case considered in this paper, on the other hand, the ratio of derivatives of expected utility is equated to the factor-price ratio for industry 1. From (7) and (8), one obtains[3]

$$\frac{F_{L1}}{F_{K1}} = \frac{w - G_{L1}P\sigma}{r - G_{K1}P\sigma} = \frac{w - (g_1 - k_1 g_1')P\sigma}{r - g_1'P\sigma} = -\frac{dK_1}{dL_1} \tag{31}$$

or

$$\frac{w}{r} = \frac{(f_1 - k_1 f_1') + (g_1 - k_1 g')\sigma}{f_1' + g_1'\sigma} \tag{32}$$

where $\sigma = \mathrm{cov}[U_1'(\Pi), \varepsilon]/E[U_1'] < 0$.

For exemplary purposes, assume $g_1' < 0$ [i.e., where capital marginally reduces risk in case (i) of Theorem 1]. Contrasting the risk-neutral case with (31) and (32), it is clear that $F_{L1}/F_{K1} > w/r$ under risk aversion. Thus, equilibrium under risk is at some point C where the expected utility contour (ISU) is tangent to the isoquant of industry 2 (IS_2''). Let e-f denote the corresponding factor-price line. Also, note that the marginal rate of substitution in production for sector 1 (F_{L1}/F_{K1}) is larger than the factor-price ratio (i.e., the price line must be steeper to equilibrate the two marginal rates of substitutions, based upon expected output).

Now, as risk is increased, equilibrium output falls in sector 1 and rises in sector 2. This necessitates a rise in the capital-labor ratio, not only due to relative factor intensities but because capital marginally decreases risk. Hence, the new equilibrium lies above the ray OT (the risk averse contract curve shifts upward), and production in section 1 declines. Because it is relatively capital intensive, this sector releases more capital (and less labor) than industry 2 can absorb at the prerisk equilibrium. Hence, returns to capital fall and returns to labor rise. Othe cases may be reasoned similarly.

6. APPENDIX

Derivation of Equations (19)-(21). Substitution of $d\Pi_1/d\gamma$ into (16)-(18) and collecting terms give the matrical equation,[4]

$$AX = H \tag{A.1}$$

where

$$H = (H_1\ H_2\ 0)^T$$

$$X = \left(\frac{dL_1}{d\gamma}\ \frac{dk_1}{d\gamma}\ \frac{dk_2}{d\gamma}\right)^T \quad \text{(T denotes transpose)}$$

$$A = (A_{ij})_{3x3}$$

$$\frac{A_{11}}{\delta} = A_{21} = E\left(U_1'' \cdot \Pi_{KL}^2 \frac{g_1}{g_1'}\right)$$

$$A_{12} = E\left[\delta L_1 U_1'' \cdot \Pi_{K1}^2 - k_1 P(f_1'' + h_1''\varepsilon)U_1'\right]$$

$$A_{22} = E\left[L_1 U_1'' \cdot \Pi_{K1}^2 + P(f_1'' + h_1''\varepsilon)\ U_1'\right]$$

$$A_{13} = E\left[\delta L_1 f_2''(k_2 - k_1)\ U_1'' \cdot \Pi_{K1} + k_2 f_2'' U_1'\right]$$

$$A_{23} = E\left[L_1 f_2''(k_2 - k_1)\ U_1'' \cdot \Pi_{K1} - f_2''\ U_1'\right]$$

$$A_{31} = k_1 - k_2$$

$$A_{32} = L_1$$
$$A_{33} = L_2$$

and

$$\frac{B_1}{\delta} = B_2 = -E[PG_{K1}\,\varepsilon U_1' + U_1'' \cdot \Pi_{K1}\,G_1 P\varepsilon].$$

If $|A| > 0$ (as is shown below), use of Cramer's rule in equation (A.1) thus yields

$$\frac{dL_1}{d\gamma} = -\frac{L_2 B_2(A_{12} - \delta A_{22}) + (\delta A_{23} - A_{13})\,L_1 B_2}{|A|}$$

$$\frac{dk_1}{d\gamma} = -\frac{(k_1 - k_2)\,B_2(\delta A_{23} - A_{13})}{|A|}$$

$$\frac{dk_2}{d\gamma} = -\frac{(k_1 - k_2)\,B_2(A_{12} - \delta A_{22})}{|A|}\,.$$

Equations (19)-(21) follow immediately.

Proof of Lemma 1(a). Profits in industry 1 can be written as

$$\Pi_1(\varepsilon) = E(\Pi_1) + PG_1\varepsilon.$$

Let ε^* be such that $\Pi_{K1}(\varepsilon^*) = 0$. Then, note that

$$U_1'' \gtreqless -R_a(\Pi^*)\,U_1' \text{ as } \varepsilon \gtreqless \varepsilon^* \qquad \text{(A.2)}$$

where absolute risk aversion, $R_a(\Pi) \equiv -U_1''/U_1'$, is nonincreasing and $\Pi^* \equiv \Pi(\varepsilon^*)$ (since $d\Pi/d\varepsilon > 0$). Multiplying (A.2) by Π_{K1} thus obtains

$$U_1'' \cdot \Pi_{K1} \gtreqless -R_a(\Pi^*)\,U_1' \cdot \Pi_{K1} \text{ as } G_{K1} \gtreqless 0 \qquad \text{(A.3)}$$

since

$$\Pi_{K1}(\varepsilon) \gtreqless \Pi_{K1}(\varepsilon^*) = 0 \left| \begin{array}{l} \text{as } \varepsilon \gtreqless 0 \text{ if } G_{K1} \geq 0 \\ \text{as } \varepsilon \lesseqgtr 0 \text{ if } G_{K1} \leq 0. \end{array} \right.$$

Taking expectations of (A.3) and using first-order conditions in (5) yield the result

$$E(U_1'' \cdot \Pi_{K1}) \gtreqless -R_a(\Pi^*)\,E(U_1' \cdot \Pi_{K1}) = 0 \text{ as } G_{K1} \gtreqless 0.$$

The proof that

$$E(U_1'' \cdot \Pi_{L1}) \gtreqless 0 \text{ as } G_{L1} \lesseqgtr 0\,.$$

follows in an identical manner.

Proof of Lemma 1(b). Note that

$$G_{K1}\varepsilon = \frac{1}{P}\,(PF_{K1} + PG_{K1}\varepsilon - r) + \frac{1}{P}\,(r - PF_{K1}) = \frac{1}{P}\,(\Pi_{K1} + r - PF_{K1});$$

hence,

$$E(U_1'' \cdot \Pi_{K1}\,G_{K1}\varepsilon) = \frac{1}{P}\left[E\left(U_1'' \cdot \Pi_{K1}^2\right) + (r - PF_{K1})\,E(U_1'' \cdot \Pi_{K1})\right]. \qquad \text{(A.4)}$$

With risk aversion, $E\left(U_1'' \cdot \Pi_{K1}^2\right) < 0$. Also, using Lemma 1(c) and equation (9) imply $\operatorname{sgn} G_{K1} = \operatorname{sgn} (PF_{K1} - r)$ with risk aversion. Additionally, using Lemma 1(a) in (A.4) thus obtains

$$E(U_1'' \cdot \Pi_{K1}\,G_{K1}\varepsilon) < 0.$$

The result for $E(U_1'' \cdot \Pi_{L1}\,G_{L1}\varepsilon) < 0$ is obtained similarly.

Proof of Lemma 4. From (A.1), one finds

$$|A| = L_1A_{21}(A_{13} - \delta A_{23}) + L_2A_{21}(\delta A_{22} - A_{12}) \\ + (k_1 - k_2)(A_{23}A_{12} - A_{13}A_{22}). \tag{A.5}$$

Substituting further from the definition of A into (A.5), the first two terms in (A.5) become

$$(k_1 + \delta)\, E\left(U_1'' \cdot \Pi_{K1}^2\right)\left[L_1f_2''(k_2 + \delta)\, E(U_1') + L_2P\left(\frac{g_1}{g_1'}\right) E(U_1' \cdot Y)\right] \tag{A.6}$$

while the last term in (A.5) becomes

$$\left(\frac{g_1}{g_1'}\right)L_1f_2''(k_1 - k_2)^2\, PE(U_1' \cdot Y)\, E(U_1' \cdot \Pi_{K1}) + E(U_1' \cdot Y)\, E(U_1') \\ f_2''(k_1 - k_2)^2\, P + L_1f_2''(k_2 + \delta)\, E(U_1')\, E\left(U_1'' \cdot \Pi_{K1}^2\right)(k_2 - k_1). \tag{A.7}$$

Combining the first term in (A.6) with the last term in (A.7) yields

$$L_1f_2''(k_2 + \delta)^2\, E(U_1')\, E\left(U_1'' \cdot \Pi_{K1}^2\right) > 0$$

assuming risk aversion and concavity of F_2. Also, noting that $E\left(U'' \cdot \Pi_{K1}^2\right) < 0$, the second term of (A.6) and the second term of (A.7) are positive by Lemma 2. Thus, $|A| > 0$ if

$$(k_1 + \delta)\, L_1f_2''(k_1 - k_2)^2\, PE(U_1' \cdot Y)\, E(U_1'' \cdot \Pi_{K1}) \geq 0. \tag{A.8}$$

But the inequality in (A.8) is established by Lemmas 1 and 2 and concavity of f_2 since $k_1 + \delta = g_1/g_1'$ and, hence, sgn $k_1 + \delta$ = sgn g_1'.

FOOTNOTES

[1] For example, $\dfrac{\partial V(X_1)}{\partial K_1}$ in (1) is $2F_1F_{K1}V(\varepsilon) > 0$ if $F_1, F_{K1} > 0$.

[2] Note that $F_{LK1} + G_{Lk1} = -k_1(f_1'' + g_1'')$; $F_Lk_2 = -k_2f_2''$; $F_{Kk1} + G_{Kk1} = f_1'' + g_1''$ and $F_{Kk2} = f_2''$.

[3] Here $\dfrac{dK_1}{dL_1}$ is evaluated at optimal expected utility. Note that $dE[U(\Pi)] = 0$ implies

$$\frac{dK_1}{dL_1} = \frac{-\partial E[U(\Pi)]/\partial L_1}{\partial E[U(\Pi)]/\partial K_1}.$$

Also, $\dfrac{\partial E[U(\Pi)]}{\partial L_1} = \dfrac{(w/r)\, \partial E[U(\Pi)]}{\partial K_1}$ since

$$\frac{w}{r} = \frac{E[U'(\Pi)\; P(F_{L1} + G_{L1}\varepsilon)]}{E[U'(\Pi)\; P(F_{K1} + G_{K1}\varepsilon)]}$$

from (6) and (7).

[4] Here Π_1 can be written as

$$\Pi_1 = L_1[P(f_1 + g_1\varepsilon) - (f_2 - k_2f_2') - k_1f_2'] = L_1\Pi_{K1}(k_1 + \delta).$$

Hence,

$$\frac{d\Pi_1}{dY} = L_1\left[\Pi_{K1}\left(\frac{dk_1}{dY}\right) + F_{K2}(k_2 - k_1)\left(\frac{dk_2}{dY}\right) + Pg_1\varepsilon \right. \\ \left. + (k_1 + \delta)\, \Pi_{K1}\left(\frac{dL_1}{dY}\right)\right].$$

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LIST OF SYMBOLS

ϵ	epsilon
∞	infinity
∂	differential
δ	delta
Π	pi
γ	gamma
$>$	greater than
$<$	less than
σ	sigma
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\lesseqgtr$	less than, equal to, or greater than
$\gtreqless$	greater than, equal to, or less than