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Abstract

The problem of integrability of the Linear Approximate Almost Ideal Demand System is solved and implications for the associated expenditure function are derived. There are two possibilities: (i) the vector of logarithmic income coefficients vanishes and the matrix of logarithmic price coefficients is symmetric; or (ii) the matrix of logarithmic price coefficients has rank at most equal to one and is proportional to the outer product of the vector of logarithmic income coefficients. Case (i) is a flexible form with respect to price elasticities, but globally restricts all income elasticities to equal plus one. Case (ii) is flexible with respect to income elasticities, but highly restrictive with respect to price elasticities. There are $\frac{1}{2}(n-1)(n+2)$ and $2n-1$ independent parameters in case (i) and (ii), respectively, as opposed to $1+\frac{1}{2}(n-1)(n+4)$ parameters in the integrable AIDS model. Closed form expressions for the expenditure function are obtained in all cases.

Integrability of the Linear Approximate Almost Ideal Demand System

In the two decades since its introduction by Deaton and Muellbauer, the Almost Ideal Demand System (AIDS) has been extremely widely used in demand analysis. The vast majority of empirical applications follow Deaton and Muellbauer's suggestion and replace the translog price index that deflates income with Stone's index, which generates the Linear Approximate Almost Ideal Demand System (LA-AIDS). Although Deaton and Muellbauer (1980: 317-320) cautioned against and avoided the practice, most empirical applications of the LA-AIDS include tests for and the imposition of an approximate version of Slutsky symmetry by restricting the matrix of logarithmic price coefficients to be symmetric. Important examples include Anderson and Blundell (1983), Buse (1998), Moschini (1995), Moschini and Meilke (1989), and Pashardes (1993).[1]

The purpose of this short note is to clarify these practices by locally integrating the LA-AIDS and deriving the implications for the model parameters and expenditure function. We find two possible cases: (i) the vector of coefficients on logarithmic income vanishes and the matrix of coefficients on logarithmic prices is symmetric; or (ii) the rank of the matrix of logarithmic price coefficients is at most one and this matrix is proportional to the outer product of the vector of logarithmic income coefficients. Each case admits an exact closed form solution for the expenditure function. However, both cases are highly restrictive, although in different ways. The first case is a flexible form with respect to price elasticities, similar to the generalized Leontief (Diewert, 1971), but globally restricts all income elasticities to equal one. The second case is flexible with respect to income elasticities, but highly restrictive with respect to price elasticities. There are $\frac{1}{2}(n-1)(n+2)$ and $2n-1$ independent parameters in case (i) and (ii), respectively, as opposed to $1+\frac{1}{2}(n-1)(n+4)$ parameters in the integrable AIDS model.

To obtain these results, let $\boldsymbol{p}$ be the n-vector of market prices for goods, let u be the utility index, let $e(\boldsymbol{p},u)$ be the consumer's expenditure function, and let $\boldsymbol{w}$ be the n-vector of budget shares. The LA-AIDS model can be written in matrix notation as

$$\text{(1)} \qquad \boldsymbol{w} = \frac{\partial \ln e(\boldsymbol{p},u)}{\partial \ln \boldsymbol{p}} = \alpha + \boldsymbol{B}\ln \boldsymbol{p} + \gamma\left[\ln e(\boldsymbol{p},u) - (\ln \boldsymbol{p})'\frac{\partial \ln e(\boldsymbol{p},u)}{\partial \ln \boldsymbol{p}}\right]$$

where α and γ are n-vectors and $\boldsymbol{B}$ is an $n\times n$ matrix of parameters. It is convenient to change variables to logarithms of prices and total expenditure. Therefore, letting $\boldsymbol{x} \equiv \log(\boldsymbol{p})$ and $y(\boldsymbol{x},u) \equiv \log e(\boldsymbol{p}(\boldsymbol{x}),u)$, with $p_i(\boldsymbol{x}) \equiv e^{x_i}$, $i = 1,\cdots,n$, we can rewrite (1) in the form

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[1] However, see Browning and Meghir (1991) for an important application of estimating the integrable AIDS model, with the LA-AIDS model with a symmetric matrix of logarithmic price coefficients used in an initial step to obtain starting values for their nonlinear estimation procedure.

$$(2) \qquad (\mathbb{I} + \gamma x')\frac{\partial y(x,u)}{\partial x} = \alpha + \mathbb{B}x + \gamma y(x,u)\,.$$

It can be shown that $|\mathbb{I} + \gamma x'| = 1 + \gamma' x$, so that $\mathbb{I} + \gamma x'$ is nonsingular and has an inverse defined by $\mathbb{I} - \gamma x'/(1 + \gamma' x)$ if and only if $\gamma' x \neq -1$ (Dhrymes, 1984: 38-39). Typically, p is a vector of price indices each normalized to unity in a base period, so that x vanishes in the base period. Also, the elements of γ, which measure the departure from homotheticity of the individual demand equations, often are quite small (e.g., the empirical results obtained by Deaton and Muellbauer). Moreover, 0° homogeneity requires that the elements of γ sum to zero. Therefore, in all cases where $x = \theta\iota$ with $\iota = [1\ 1\ \cdots\ 1]'$ for some $\theta \in \mathbb{R}$ (including $\theta = 0$) at a base point for the data, the matrix $\boldsymbol{I} + \gamma \boldsymbol{x}'$ is nonsingular in a neighborhood of that point. We therefore make the following assumption.

$$\textbf{A} \qquad 1 + \gamma' \boldsymbol{x} > 0 \ \forall \ \boldsymbol{x} \in \mathcal{N} \subset \mathbb{R}^n,$$

where $\mathcal{N}$ is open, convex, has a nonempty interior, and contains the line passing through $\mathbb{0}$ and ι.

Property A permits us to write the LA-AIDS as a system of linear partial differential equations,

$$(3) \qquad \frac{\partial y(x,u)}{\partial x} - \gamma\frac{y(x,u)}{(1+\gamma' x)} = \left[\boldsymbol{I} - \frac{\gamma x'}{(1+\gamma' x)}\right](\alpha + \boldsymbol{B}x),$$

where use has been made of $[\boldsymbol{I} - \gamma \boldsymbol{x}'/(1 + \gamma' \boldsymbol{x})]\gamma \equiv \gamma / (1 + \gamma' \boldsymbol{x})$. Then, by simply noting that

$$(4) \qquad \frac{\partial}{\partial \boldsymbol{x}}\left[\frac{y(\boldsymbol{x},u)}{1+\gamma' \boldsymbol{x}}\right] = \left[\frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} - \gamma\frac{y(\boldsymbol{x},u)}{(1+\gamma' \boldsymbol{x})}\right]\frac{1}{(1+\gamma' \boldsymbol{x})},$$

we can multiply (3) by $1/(1 + \gamma' x)$ to make the left-hand-side an exact differential. Consequently, Slutsky symmetry is equivalent to symmetry of the $n \times n$ matrix

$$(5) \qquad \frac{\partial}{\partial x'}\left\{\frac{1}{(1+\gamma' x)}\left[\boldsymbol{I} - \frac{\gamma \boldsymbol{x}'}{(1+\gamma' x)}\right](\alpha + \boldsymbol{B}\boldsymbol{x})\right\} =$$

$$\frac{\mathbb{B}}{(1+\gamma' x)} - \frac{[(\alpha + \boldsymbol{B}\boldsymbol{x})\gamma' + \gamma(\alpha' + \boldsymbol{x}'\boldsymbol{B}' + \boldsymbol{x}'\boldsymbol{B})]}{(1+\gamma' x)^2} - \frac{2(\alpha' \boldsymbol{x} + \boldsymbol{x}'\mathbb{B}\boldsymbol{x})\gamma\gamma'}{(1+\gamma' x)^3}.$$

Imposing symmetry on each of the terms associated with like powers of $(1 + \gamma' x)$ and ignoring terms that are automatically symmetric, we obtain $\mathbb{B} = \mathbb{B}'$ and $\gamma x'\mathbb{B} = \mathbb{B}' x \gamma'$. There are two ways that these conditions are satisfied simultaneously $\forall\ x \in \mathcal{N}$: (i) $\gamma \neq \mathbb{0}$ and $\mathbb{B} = \beta_0 \gamma\gamma'$ for some $\beta_0 \in \mathbb{R}$ (including $\beta_0 = 0$); and (ii) $\gamma = \mathbb{0}$ and $\mathbb{B} = \mathbb{B}'$.

Case (i) gives the LA-AIDS model in the form

$$(6)\qquad \frac{\partial y}{\partial x} = \alpha + \beta_0\gamma\gamma'x + \gamma\left(\frac{y - \alpha'x - \beta_0(\gamma'x)^2}{1+\gamma'x}\right) = \alpha + \gamma\left(\frac{y - \alpha'x + \beta_0\gamma'x}{1+\gamma'x}\right).$$

This is a very simple system of linear first-order partial differential equations. Noting that

$$(7)\qquad \frac{\partial}{\partial x}\left(\frac{\alpha'x}{1+\gamma'x}\right) = \left[\alpha - \gamma\left(\frac{\alpha'x}{1+\gamma'x}\right)\right]\frac{1}{(1+\gamma'x)},$$

and that

$$(8)\qquad \frac{\partial}{\partial x}\left\{\beta_0\left[\ln(1+\gamma'x) - \left(\frac{\gamma'x}{1+\gamma'x}\right)\right]\right\} = \frac{\beta_0\gamma\gamma'x}{(1+\gamma'x)^2},$$

combining these two equations with equation (4), and integrating with respect to x, we obtain the logarithmic expenditure function as

$$(9)\qquad y(x,u) = \alpha'x + \beta_0\left[(1+\gamma'x)\ln(1+\gamma'x) - \frac{\gamma'x}{(1+\gamma'x)}\right] + (1+\gamma'x)u,$$

with an obvious normalization for the utility index.

Case (ii) generates the homothetic LA-AIDS demand model,

$$(10)\qquad \frac{\partial y}{\partial x} = \alpha + Bx,$$

which gives the logarithmic expenditure function as

$$(11)\qquad y(x,u) = \alpha'x + \tfrac{1}{2}x'Bx + u,$$

again with an obvious normalization.

This establishes the necessity of the demand structures (6) or (10) for integrability of the LA-AIDS model.

On the other hand, (10) trivially has the LA-AIDS form, with $\gamma = \mathbf{0}$. To show sufficiency for (6), write

$$(12)\qquad y - x'\frac{\partial y}{\partial x} = y - x'\left[\alpha + \beta_0\gamma\gamma'x + \gamma\left(\frac{y-\alpha'x-\beta_0(\gamma'x)^2}{1+\gamma'x}\right)\right] = \frac{y-\alpha'x-\beta_0(\gamma'x)^2}{1+\gamma'x}.$$

Hence, equation (6) can be rewritten in the equivalent LA-AIDS form,

$$(13)\qquad \frac{\partial y}{\partial x} = \alpha + \beta_0\gamma\gamma'x + \gamma\left(y - x'\frac{\partial y}{\partial x}\right).$$

Remarks:

1. Surprisingly, the homothetic integrable LA-AIDS model (10) has the same structure as the homothetic Linear Incomplete Demand System (LIDS) in LaFrance (1985). If one is willing to forgo symmetric functional forms for the demands of all goods, which may be a minor concern in many cases, this suggests a simple way to nest the LA-AIDS and LIDS models through a Box-Cox transformation. To see this, let total expenditure on all goods be m, let the model apply to n of $N \geq n+1$ goods and define $m(\lambda) \equiv (m^{\lambda}-1)/\lambda$, $p_i(\lambda) \equiv (p_i^{\lambda}-1)/\lambda$, and $\boldsymbol{p}(\lambda) \equiv [p_1(\lambda) \cdots p_n(\lambda)]'$. Also suppose that m and $\boldsymbol{p}$ are deflated prices, where the common deflator is any known, positive valued, and 1° homogeneous function of (at least some of) the prices of all other goods, say $\pi(\tilde{\boldsymbol{p}})$. Then we can write an integrable and homothetic Price Independent Generalized Linear (PIGL) incomplete demand system in budget share form as

(14) $$\boldsymbol{w} = m^{-\lambda}\boldsymbol{P}^{\lambda}[\alpha + \boldsymbol{B}\boldsymbol{p}(\lambda)],$$

where $\boldsymbol{P}^{\lambda} \equiv \mathrm{diag}[p_i^{\lambda}]$. It is easy to show that the deflated expenditure function for this incomplete demand system satisfies

(15) $$\frac{\mathrm{e}(\boldsymbol{p},\tilde{\boldsymbol{p}},u)^{\lambda}-1}{\lambda} \equiv \alpha'\boldsymbol{p}(\lambda) + \tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda) + \theta(\tilde{\boldsymbol{p}},u),$$

where $\theta(\tilde{\boldsymbol{p}},u)$ is 0° homogeneous in the prices of all other goods and increasing in u, but otherwise can not be identified from the incomplete demand system (LaFrance (1985); LaFrance and Hanemann (1989)). It also is easy to show that the demands (14) are homothetic, with income elasticities equal to $1-\lambda \ \forall \ \lambda \in \mathbb{R}$.

2. The above procedure for nesting the homothetic LA-AIDS, PIGL and LIDS models generalizes readily to the non-homothetic integrable AIDS model, which is characterized by a quadratic form in logarithmic prices,

(16) $$\boldsymbol{w} = \alpha + \boldsymbol{B}\ln(\boldsymbol{p}) + \gamma\left[\ln(m) - \alpha_0 - \alpha'\ln(\boldsymbol{p}) - \tfrac{1}{2}\ln(\boldsymbol{p})'\boldsymbol{B}\ln(\boldsymbol{p})\right].$$

With the above definitions for $m(\lambda)$ and $\boldsymbol{p}(\lambda)$, we can write an integrable nonhomothetic PIGL model that is linear in the Box-Cox expenditure term and linear and quadratic in the Box-Cox price vector as,

(17) $$\boldsymbol{w} = m^{-\lambda}\boldsymbol{P}^{\lambda}\left\{\alpha + \boldsymbol{B}\boldsymbol{p}(\lambda) + \gamma\left[m(\lambda) - \alpha_0 - \alpha'\boldsymbol{p}(\lambda) - \tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)\right]\right\}.$$

Unlike the homothetic case above, for all values of λ this is a fully flexible functional form that allows one to econometrically choose the income aggregation function through estimation of the Box-Cox parameter λ. For $\lambda = 0$ we obtain the integrable AIDS model, while for $\lambda = 1$ we obtain the LINQUAD incomplete demand system of LaFrance (1990). For all other values of λ we obtain an integrable PIGL model that nests AIDS and LINQUAD. It is straightforward to show that the deflated expenditure function in this case satisfies

$$\frac{e(\boldsymbol{p},\tilde{\boldsymbol{p}},u)^{\lambda}-1}{\lambda}\equiv\theta(\tilde{\boldsymbol{p}},u)e^{\gamma'\boldsymbol{p}(\lambda)}+\left[\alpha_0+\alpha'\boldsymbol{p}(\lambda)+\tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)\right]. \tag{18}$$

3. When the coefficients on the logarithm of total expenditure do not all vanish, the LA-AIDS and LIDS models can not be nested with the above methods. However, a structural relationship between these two models continues to exist, and this relationship also is shared with the Linear Expenditure System (LES) in this case. In particular, the integrable LA-AIDS model with at least one non-zero coefficient on the logarithm of total expenditure can be characterized by a linear relationship among *budget shares*,

$$\boldsymbol{w}\equiv\alpha+\gamma(w_1-\alpha_1)/\gamma_1\ \forall\ (\boldsymbol{p},m), \tag{19}$$

where, without loss in generality, we have assumed that $\gamma_1 \neq 0$. By the same token, the LES system can be characterized by a linear relationship among *expenditures*,

$$\boldsymbol{e}\equiv\alpha+\gamma(e_1-\alpha_1)/\gamma_1\ \forall\ (\boldsymbol{p},m), \tag{20}$$

where $e_i \equiv p_iq_i$ and $\boldsymbol{e} \equiv [e_1 \cdots e_n]'$ is the n-vector of expenditures on the goods $\boldsymbol{q}$. Similarly, the integrable LIDS model can be characterized by a linear relationship among *quantities* (see LaFrance (1985)),

$$\boldsymbol{q}\equiv\alpha+\gamma(q_1-\alpha_1)/\gamma_1\ \forall\ (\boldsymbol{p},m). \tag{21}$$

In this context, the *linear* part of the LA-AIDS acronym achieves a new meaning.

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