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Irreversibility, Uncertainty, and Global Warming: A Theoretical Analysis

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Abstract

In this paper we characterize the optimal rate of emission of greenhouse gases when investment in abatement capital is sunk, some part of the stock of greenhouse gases is non-degradable and there is an endogenous risk of catastrophic damages in the future. The agent wants to avoid two situations: (i) investing in sunk abatement capital today when the damages tomorrow turn out to be negligible; (ii) not reducing the stock of non-degradable greenhouse gases today when damages are revealed to be catastrophic tomorrow. Unfortunately, the stock of greenhouse gases cannot be reduced unless the agent invests in abatement capital. Given this trade-off, and the added feature that the probability of a catastrophe occurring may be endogenous, our paper asks what should be the optimal rate of emission of greenhouse gases. Previous studies have either relied on numerical simulations or failed to capture features of the environment we think important to global warming. Our paper fills these gaps in the literature by developing a stochastic dynamic programming model that allows for sunk capital, a non-degradable stock of greenhouse gases, and endogenous catastrophic damages, and yields analytical results. Loosely speaking, we find a stronger effect on the optimal rate of emissions associated with the accumulation of greenhouse gases than with investment in abatement capital, a result somewhat at odds with earlier findings.

1. Introduction

Climatologists report that, at current greenhouse gas emission levels, the stock of greenhouse gases in the atmosphere will double the preindustrial level in the next few decades. This will lead to an increase in global mean temperature by a best-guess estimate of 3.5°F (IPCC 1995b).[1] This is a large and sudden increase in mean temperature considering that the world is only 5–9°F warmer now than in the last ice age. The increase in global mean temperature is expected to lead to disruptions in the world's climate.[2] Whether these disruptions will cause economic damages and whether these damages will be catastrophic in nature is as yet uncertain.[3] There are those who believe that global warming will lead to sudden and catastrophic economic damages. Others believe that damages will occur slowly and continuously as the stock of greenhouse gases increases. Still others assert that damages due to global warming will be negligible.[4]

Given a threat of damages of an unknown magnitude, the question facing policymakers is whether they should change the rate at which greenhouse gases are being emitted today.[5] Four features of the economic environment bear on this decision and make the answer less than obvious: sunk or irreversible abatement capital; non-degradable or irreversible stocks of greenhouse gases; endogenous, and potentially catastrophic, damages; and future learning about the nature of damages.[6] Abatement capital is said to be sunk if resources once invested cannot be re-used for consumption or re-invested in other forms of capital. An obvious concern is whether the presence of irreversible capital alters optimal emission control decisions today. Given the uncertainty, should less be invested in abatement capital if that capital is irreversible?

A second important complicating factor is the irreversibility of the stock of greenhouse gases. The stock of greenhouse gases is said to be irreversible if it cannot be reduced through abatement and if it does not decay naturally. Climatologists claim that some part of the stock of greenhouse

[1]The range of predicted temperature increase is 2–6°F.

[2]Predictions about increases in global mean temperature and how disruptive this will be are based on simulations from computer models of the world's climate. While these models are improving over time they still remain rather crude and are plagued by incomplete knowledge of the atmosphere's functioning. One of the biggest limitations is that the net effect of clouds on the planet's temperature is still unknown. By and large the models are able to predict changes at the global level but perform poorly on a regional scale (Stevens 1997).

[3]IPCC (1995b) identifies possible catastrophic climate change events. A few among these are destabilization of methane clathrates, shutdown of major greenhouse gas sinks, disintegration of the west Antarctic ice sheet and collapse of the North Atlantic thermohaline circulation.

[4]For a discussion of the assessment of socio-economic impacts of climate change see IPCC (1995a).

[5]When policymakers change the rate of emission of greenhouse gases they in fact change the magnitude of damages that may be caused by global warming. This constitutes a policy of prevention. In addition or alternatively, policymakers could wait until after the damages have occurred and then institute policies that mitigate the effects of these damages. Building levies to control flooding due to an increase in sea level is one such policy in the general class of policies of cure. Chichilnisky and Heal (1993) discuss the applicability of insurance markets to mitigate damages after global warming has occurred.

[6]For the remainder of the paper we will use the terms sunk and irreversible interchangeably for abatement capital and the terms non-degradable and irreversible interchangeably for the stock of greenhouse gases.

gases will in fact be irreversible. The atmospheric concentration of carbon is not expected to return to its original (pre-industrial) level but instead is expected to reach a new equilibrium where about 13–18% of total carbon dioxide emitted will remain in the atmosphere for several thousand years (Maier-Raimer and Hasselmann 1987), and even the remainder will decay very slowly, over centuries.[7] Should policymakers reduce greenhouse gas emissions if, once emitted, gases remain in the atmosphere for hundreds and thousands of years?

A third important concern for policy makers is the extent to which the risk of future damages is endogenous and whether or not damages will be catastrophic in nature. If the probability of damages occurring depends on the behavior of economic agents, then the risk should be considered to be endogenous. In the context of global warming, since the probability of damages depends on the stock of greenhouse gases, the risk of damages is in fact endogenous.[8] Recent findings suggest that the possibility of damages being catastrophic in nature, in particular related to a disintegration of the West Antarctic ice sheet, is more serious than economists (and others) have realized (Kerr 1998). This suggestion is strengthened by the prospect that concentrations of greenhouse gases could, over the next couple of centuries, rise well beyond the conventionally assumed doubling of pre-industrial levels (Cline 1992). The implications of the potential for catastrophic impact are a major focus of our study.

A final issue that complicates policy decisions on global warming is how uncertainty is resolved over time. If uncertainty about the nature of damages due to global warming is resolved over time, then policymakers must decide whether they should wait to act until there is better information about the nature of damages. When time resolves uncertainty, Arrow and Fisher (1974) have shown that there is a premium or option value on policies that maintain flexibility.[9] Irreversibility of capital and the stock of greenhouse gases are two potential sources of inflexibility. Investment in irreversible capital today locks the economy into a particular use of resources which may turn out to be wasteful if tomorrow reveals that damages due to global warming are small. Kolstad (1996b, 1996a) has stressed this possibility. One then expects that investment in irreversible capital will be less than the investment that would be made if capital was reversible. With irreversible accumulation of greenhouse gases, on the other hand, emissions today lock the economy into a level of damages which may be revealed as catastrophic. To maintain the option of not having to bear large damages policymakers might increase investment in abatement today. Both Chichilnisky

[7]Farzin and Tahvonen (1996) were the first to incorporate this specification into the economic analysis of optimal carbon taxes.

[8]There has been a great deal of debate about the contribution of human activities to climate change. Two years ago the IPCC declared that human activities influence global climate through the increase in greenhouse gas concentrations.

[9]For other works on option value in the environment see Henry (1974) and Fisher and Hanemann (1990). For a more general treatment see Dixit and Pindyck (1994).

and Heal (1993) and Fisher and Hanemann (1993) have suggested that this approach may be appropriate.

In this paper we develop a multi-period stochastic model that incorporates three of the four features of the decision making environment that we have drawn attention to—sunk capital, non-degradable stocks of greenhouse gases, and endogenous, and potentially catastrophic damages. Other contributions in this area have treated these features somewhat differently. Kolstad (1996a, 1996b) employs different definitions of sunk capital and irreversible accumulation of greenhouse gases that, as we shall show, have quite different implications for current policy on controlling emissions. He also implicitly assumes that the risk of catastrophic damages is exogenous in the sense that the probability of occurrence is not affected by behavior within the model. Ulph and Ulph (1997) similarly assume exogenous risk, and Kolstad's definition of irreversibility for greenhouse gases, but do not include abatement capital in their model. Conrad (1992) studies the effect of non-degradable stocks of greenhouse gases (employing a definition of irreversibility here similar to ours) on the optimal rate of emissions, but his model does not include capital, the potential for catastrophic damages, or endogenous risk. Clarke and Reed (1994) do consider what we have called endogenous risk, in a model of an accumulating pollutant that can trigger an irreversible environmental catastrophe, but both capital and the stock of the pollutant are assumed to be fully reversible. Finally, a similar model is developed by Aronsson, Johansson, and Lofgren (1997), with two types of capital, but no irreversibilities of the sort we emphasize here.

Our model does not allow agents to act on new information, but nonetheless generates changes in the optimal rate of emissions. Although we do find an "irreversibility effect" even in the absence of learning, we understand that a model with learning would perhaps be more realistic than one that does not allow for this. Adding learning into our multi-period model, which already has two state variables, capital and the stock of greenhouse gases, would however make the analytics intractable. In order to incorporate learning we would have to either restrict ourselves to a two period model or to numerical simulations, as has been done in the existing literature ((Kolstad 1996b) and (Ulph and Ulph 1997)). In another paper we develop such a numerical model, a parameterization of the model of this paper, that allows agents to act on improving information about the nature of damages (Narain and Fisher 1999). The numerical model is a modification and extension of one developed in this paper to aid in analyzing the stability properties of steady states and the monotonicity of optimal trajectories.

The rest of the paper is organized as follows. The next section contains a description of the theoretical model and analysis of the steady state. Section 3 considers the effect of irreversible capital and section 4 the effect of an irreversible stock of greenhouse gases on the optimal rate of investment. Section 5 repeats the analysis in section 3 and 4 with the added feature that the risk of a catastrophe is endogenous. Conclusions are presented in section 6.

2. Theoretical Model

Agents derive utility from consumption and disutility from the stock of greenhouse gases and from catastrophic damages, should a catastrophe occur.[10] The only source of uncertainty in the model is whether or not a catastrophe will occur. The extent of catastrophic damages, should a catastrophe occur, are known in advance.

The agent receives a fixed endowment of resources in every period which she allocates between consumption and investment in capital used to abate the flow of greenhouse gases. Abatement capital is either reversible or irreversible. Only reversible capital can be converted back into consumption, though at a cost. If not abated, emissions, a by-product of consumption, add to the stock of greenhouse gases which is either reversible or irreversible. Only reversible stocks decay naturally over time and neither type of stock can be abated.[11]

In addition to causing disutility in every period, the stock of greenhouse gases also affects the probability that a catastrophe will occur when the risk of a catastrophe is endogenous. Catastrophic damages, however, are themselves independent of the stock of greenhouse gases and drive utility to zero forever.[12] Catastrophic damages are therefore irreversible; once catastrophic damages have occurred they exist forever, and at the same magnitude as at the time of their occurrence.

Our characterization of a catastrophe does not allow agents to adjust their emission or investment levels after they have learned about the catastrophe. Thus even though uncertainty about the occurrence of a catastrophe is being resolved over time, there is no reason to wait for new information as there is no option value of delaying an irreversible decision. However, irreversible capital and stocks of greenhouse gases can still lead a change in the desired level of emissions.

2.1. **Primitives.** A representative agent derives utility from consumption, C, and disutility from the stock of greenhouse gases, M and from catastrophic damages, D. Let the momentary utility

[10] Cropper (1976) was the first to draw attention to the effect of catastrophic risks on optimal rate of emissions, though not in the context of global warming.

[11] We do not allow the stock of greenhouse gases to be abated in order to simplify our analysis. This assumption allows us to move from a model where the stock of greenhouse gases is reversible to one where the stock is irreversible by simply changing the rate of decay of greenhouse gases. In the absence of this assumption we would have to introduce a different equation of motion for reversible stocks of greenhouse gases and thus a new set of optimality conditions. Our results are not affected by this simplification.

[12] The assumption that the catastrophe drives utility to zero does not significantly affect our results. Any constant, nonzero level of utility could be substituted instead with no affect on results. In the appendix we show an example where a weaker, and perhaps more plausible, assumption, that utility after the catastrophe is a concave function of the levels of the stock of abatement capital and the stock of greenhouse gases at the time of the catastrophe, leads to a similar (though a bit messier) result. For simplicity of exposition we use the assumption of zero utility.

function $U = U(C, M, D)$ satisfy the conditions

$$U_1(C,M,D) > 0, \quad U_{11}(C,M,D) < 0, \quad U_{12}(C,M,D) < 0$$
$$U_2(C,M,D) < 0, \quad U_{22}(C,M,D) < 0$$

where subscripts denote differentiation. As long as there is no catastrophe, catastrophic damages are zero and the utility function is unaffected. After a catastrophe however, utility goes to zero forever so that $U(C, M, D > 0) = 0$. To simplify notation we drop catastrophic damages from the utility function and re-write the utility function as $U = U(C, M)$.

A fixed amount of output, R, is available each period for either consumption or investment, I, in abatement capital. Abatement capital, K, changes from one period to the next as a result of investment and depreciation according to

$$\dot{K} = I - \delta_K K \tag{1}$$

where δ_K is the rate of depreciation of capital. For the base model we assume that capital is reversible. This means that at any time consumption can be greater than the fixed amount of resource R. Specifically,

$$C \leq R + \Phi K \tag{2}$$

where Φ is a parameter that governs the cost of converting capital into consumption. When $\Phi = 0$ it is infinitely costly to convert capital into consumption and when $\Phi \geq 0$ capital can be converted into consumption, though at a cost.[13]

Greenhouse gas emissions, E, are a by-product of consumption. Let $g(C)$ be the emissions function where $g_1(C) > 0$ and $g_{11}(C) = 0$. If unabated, emissions increase the stock of greenhouse gases. Let $H(K)$ be the abatement function where $H_1(K) > 0$ and $H_{11}(K) \leq 0$. Capital abates only the flow, and not the stock of greenhouse gases, implying that the amount of greenhouse gas abated in a period cannot exceed the amount emitted in that period. The abatement function $H(K)$ thus lies between zero and one. The stock also decays naturally. Consequently, the law of motion for the stock of greenhouse gases is given by[14]

$$\dot{M} = g(C)\big(1 - H(K)\big) - \delta_M M \tag{3}$$

where $g(C)(1 - H(K))$ are net emissions and δ_M is the natural decay rate of greenhouse gases. If the rate of decay is close to zero, then the stock of greenhouse gases is considered to be irreversible.

[13] For a discrete time model $\Phi = 1$ denotes costless conversion.

[14] By restricting the decay rate to be a linear function of the stock of greenhouse gases we are in fact assuming that there is a unique steady state for the stock of greenhouse gases. See Tahvonen (1995) for a discussion of multiple steady states with non-convex decay functions.

Finally, there always exists the possibility of a catastrophe occurring. The possibility of a catastrophe is captured by a damage function that follows a jump process. The law of motion for catastrophic damages is given by

$$\dot{D} = \begin{cases} a & \text{with probability } p, \\ 0 & \text{with probability } (1-p). \end{cases} \tag{4}$$

where p is the probability of catastrophic damages occurring and a is the magnitude of the catastrophic jump. If the catastrophe is exogenous then p is a constant. With endogenous catastrophic risk the probability of a catastrophe occurring is an increasing and convex function of the stock of greenhouse gases. That is, $p = p(M)$ with $p_1(M) > 0$ and $p_{11}(M) > 0$.[15]

2.2. **Objective.** The agent chooses a stream of consumption to maximize expected intertemporal utility subject to equations (1)–(4). The only source of uncertainty is whether or not a catastrophe will occur.

$$\max_C E_t \int_t^\infty U(C, M, \tau) d\tau \tag{5}$$

The Bellman-Hamilton-Jacobi equation[16] for this problem is

$$(r+p)V(K,M) = \max_{C \leq R+\Phi K} \left[U(C,M) + \mu_1 \dot{K} + \mu_2 \dot{M} \right] \tag{6}$$

where $V(K,M)$ is the value function, r is the discount rate, μ_1 is the co-state variable associated with the stock of abatement capital and μ_2 is the co-state variable associated with the stock of greenhouse gases.[17]

2.3. **Optimality Conditions.** In this subsection we establish optimality conditions for consumption, capital and the stock of greenhouse gases. For now we assume that risk is exogenous and so p is a constant. We begin by differentiating equation (6) with respect to the choice variable—consumption. This gives the following first order condition

$$U_1(C,M) - \lambda - \mu_1 + \mu_2 g_1(C)(1 - H(K)) = 0 \tag{7}$$

where λ is the Lagrange multiplier on the consumption constraint given by equation (2). The co-state equations of motion, obtained by differentiating equation (6) with respect to the state

[15] Tsur and Zemel (1996) consider the effect of a different type of catastrophic risk on the optimal rate of emissions. In their model uncertainty stems from not knowing what is the exact level of stock needed to trigger a catastrophe. They call this risk endogenous and contrast it with the stochastic process we consider which they refer to as an exogenous risk. Since in the context of global warming there exist a lag between stock build up and the time when the effects of that level of stock are felt we believe that modeling the risk as a stochastic process is appropriate.

[16] The Bellman-Hamilton-Jacobi equation is derived in the appendix.

[17] μ_1 is in fact equal to $V_1(K,M)$ while $\mu_2 = V_2(K,M)$.

variables, K and M, are

$$\dot{\mu}_1 = \mu_1(r + \delta_K + p) - \lambda\Phi + \mu_2 g(C) H_1(K) \tag{8}$$

$$\dot{\mu}_2 = \mu_2(r + \delta_M + p) - U_2(C, M) \tag{9}$$

We are now in a position to characterize the steady state and analyze its stability properties.

2.4. **Steady State.** Equation (7), and equations (8) and (9) at the steady state, combine to give the Euler equation, which in turn with the steady state laws of motion gives a system of equations which determines the steady state level of consumption, capital and stock of greenhouse gases (the arguments are suppressed for compactness)

$$U_1 = -\frac{U_2}{(r + \delta_M + p)} \left(g_1(1 - H) + \frac{gH_1}{r + \delta_K + p} \right) + \lambda \left(\frac{\Phi}{(r + \delta_K + p)} + 1 \right) \tag{10}$$

$$K^* = \frac{R - C^*}{\delta_K} \tag{11}$$

$$M^* = \frac{g(1 - H)}{\delta_M} \tag{12}$$

where stars denote steady state levels. When the constraint on consumption is not binding, $\lambda = 0$ and the Euler equation states that along the steady state consumption trajectory there is nothing to gain by increasing consumption. Equation (11) states that at the steady state, investment is equal to capital depreciation while Equation (12) states that net emissions are equal to the decay in the steady state stock of greenhouse gases.

When the constraint on consumption is binding, $\lambda > 0$ and steady state consumption is equal to $C^* = R + \Phi K^*$. Since negative investment period after period drives the steady state capital stock to zero, steady state consumption is in fact equal to R.

2.5. **Stability of the Steady State.** To analyze the stability properties of the steady state we use the necessary and sufficient conditions identified by Dockner (1985). The first step in this process is to express the necessary conditions for optimality (equations (1), (3), (7), (8) and (9)) in terms of the state and the co-state variables only. From equation (7) and the assumption that the second derivative of equation (6) with respect to the control is nonzero, we can, using the implicit function theorem, express consumption as a function of the state and co-state variables. Substituting this expression into the co-state and state variables yields the necessary modified dynamic system (equations of motion for the state and the co-state variables).

By a theorem due to Ljapunow (Coddington and Levinson 1955), the stability behavior of the non-linear dynamic system can be studied by analyzing the stability properties of the first approximation of the system around the steady state. Let $J(K^*, M^*, \mu_1^*, \mu_2^*)$ denote the Jacobian matrix of the modified system, evaluated at the steady state.

(13)
$$J = \begin{bmatrix} \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial M} & \frac{\partial \dot{K}}{\partial \mu_1} & \frac{\partial \dot{K}}{\partial \mu_2} \\ \frac{\partial \dot{M}}{\partial K} & \frac{\partial \dot{M}}{\partial M} & \frac{\partial \dot{M}}{\partial \mu_1} & \frac{\partial \dot{M}}{\partial \mu_2} \\ \frac{\partial \dot{\mu}_1}{\partial K} & \frac{\partial \dot{\mu}_1}{\partial M} & \frac{\partial \dot{\mu}_1}{\partial \mu_1} & \frac{\partial \dot{\mu}_1}{\partial \mu_2} \\ \frac{\partial \dot{\mu}_2}{\partial K} & \frac{\partial \dot{\mu}_2}{\partial M} & \frac{\partial \dot{\mu}_2}{\partial \mu_1} & \frac{\partial \dot{\mu}_2}{\partial \mu_2} \end{bmatrix}$$

2.5.1. *Saddle Point.* We are interested in establishing conditions under which the steady state is a saddle point. If the steady state is a saddle point then there exists a two-dimensional (local) manifold consisting of the steady state and all solutions that converge towards the steady state. This implies that given the initial stock of capital and greenhouse gases, it is possible to pick the initial co-state variables for both capital and greenhouse gases to ensure that the system converges to the steady state.

Under the assumption that $(r + p) \geq 0$, the necessary and sufficient conditions for the steady state to be a saddle point are: (i) $L < 0$; (ii) $\det J > 0$; and (iii) $\det J < (\frac{L}{2})^2$, where

(14)
$$L = \begin{vmatrix} \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial \mu_1} \\ \frac{\partial \dot{\mu}_1}{\partial K} & \frac{\partial \dot{\mu}_1}{\partial \mu_1} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{M}}{\partial M} & \frac{\partial \dot{M}}{\partial \mu_2} \\ \frac{\partial \dot{\mu}_2}{\partial M} & \frac{\partial \dot{\mu}_2}{\partial \mu_2} \end{vmatrix} + 2 \begin{vmatrix} \frac{\partial \dot{K}}{\partial M} & \frac{\partial \dot{K}}{\partial \mu_2} \\ \frac{\partial \dot{\mu}_1}{\partial M} & \frac{\partial \dot{\mu}_1}{\partial \mu_2} \end{vmatrix}$$

Since we have a four dimensional system, two state variables and two co-state variables, it is hard to further study these stability properties analytically. Instead, we impose functional forms for the utility, emissions and abatement functions, and analyze the stability properties numerically.[18]

2.5.2. *Functional Forms.* We assume that agents have quadratic preferences over consumption, and the stock of greenhouse gases.

(15)
$$U(C_t, M_t) = \frac{-1}{2}\left((C_t - b)^2 + M_t^2\right)$$

where b is the bliss point. Greenhouse gases are produced as a result of consumption with a linear technology $E_t = \sigma C_t$, where E_t denotes emissions and σ is a constant denoting the emissions to consumption ratio. Finally, the abatement function is given by

(16)
$$A_t = H(K_t) = \frac{2}{1 + \exp(-\rho K_t)} - 1$$

where A_t is the amount of abatement in period t and ρ is the slope of the modified logistic function. We use a modified logistic function to limit abatement to be a fraction that lies between 0 and 1.

2.5.3. *Results.* We now look for parameters values that ensure that the steady state is a saddle point. The results of a five dimensional grid search, using the parameters δ_k, δ_m, σ, ρ and p, are

[18] With the added assumption that $U_{12}(C, M) = 0$, the first two necessary and sufficient conditions for a saddle point can in fact be established analytically. It is hard to establish condition (iii) and for this reason we move to a numerical analysis.

Parameter	*Minimum Value*	*Maximum Value*
δ_k	0.1	0.9
δ_m	0.01	0.5
σ	0.5	5
ρ	0.1	0.9
p	0.002	0.02

TABLE 1. Parameter Values Used for Saddle Point Search

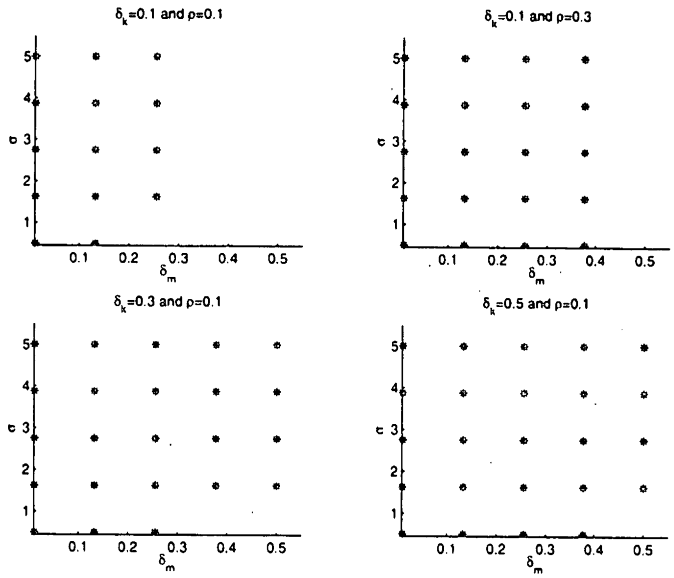

FIGURE 1. Parameter Values for Saddle Point Steady State

summarized in Figure 1. The stars signify parameter values for which the steady state is a saddle point. The maximum and minimum values for each of these parameters over which the search was conducted are given in Table 1.[19] Each of these subplots hold irrespective of the value of p.

The first subplot shows that for low values of the rate of capital depreciation, δ_k, and capital productivity, ρ, if the rate of decay of greenhouse gases is high (between 0.3 and 0.5), then, irrespective of the value of the emission to consumption ratio, the steady state is not a saddle point. Increasing capital productivity; moving from subplot 1 to subplot 2, gives the result that higher values of the rate of decay now support a saddle point. If the rate of capital depreciation is increased instead, say moving from subplot 1 to either subplot 3 or 4, still larger values of the rate of decay support a saddle point. Consequently, the higher the values of δ_k, σ and ρ and the lower the value of δ_m, the more likely is the steady state to be a saddle point.

[19]The other parameters used in the simulations took on the following values: (i) $r = 0.03$; (ii) $R = 20$; and (iii) $b = 50$.

2.5.4. *Monotonicity.* If the steady state is a saddle point then it is easy to show that the state variables, K and M, evolve monotonically. For this we first restrict ourselves to the state plane of the linearized stable manifold, that is, to the submatrix of the Jacobian J

$$J_s = \begin{bmatrix} \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial M} \\ \frac{\partial \dot{M}}{\partial K} & \frac{\partial \dot{M}}{\partial M} \end{bmatrix} \tag{17}$$

For the analytical model described in this section, the J_s matrix is given by (the arguments are suppressed),

$$J_s = \begin{bmatrix} -\frac{U_2 H_1 g_1}{U_{11}(r+p+\delta_m)} - \delta_k & \frac{U_{12}}{U_{11}} \\ \frac{(1-H)g_1^2 U_2 H_1}{U_{11}(r+p+\delta_m)} - gH_1 & -\frac{(1-H)g_1 U_{12}}{U_{11}} - \delta_m \end{bmatrix} \tag{18}$$

The trace of this matrix is negative, its determinant is positive and its discriminant is positive. This implies that the related phase-plane diagram is characterized by an improper stable node. Thus the optimal trajectory is locally monotonic (Dockner 1985).

3. Irreversible Capital

In this section we explore the implications of capital being irreversible. We find that these are weaker under our suggested definition of irreversibility than they are under an alternative definition employed in the literature.

3.1. **Defining Irreversible Capital.** The key previous work here is by Kolstad (1996b), who equates irreversibility of capital with durability, arguing that capital is sunk if it has a low rate of depreciation. In our judgment this definition fails to capture the essential problem faced by policymakers, who we assume wish to avoid a situation where valuable resources invested today in abatement capital cannot be converted back into some productive use in the future, or into consumption, if damages turn out to be negligible. What matters is the adjustment cost of conversion, not depreciation. Durable capital may still have a low conversion cost. We therefore prefer to define irreversible capital as capital that is prohibitively costly to convert into consumption.[20] Or, capital is considered to be irreversible if investment is constrained to be positive in every period.

3.2. **Durable Capital.** We begin by writing the Euler equation, equation (10), as a function of steady state consumption and system parameters (steady state capital and stock of greenhouse gases are both functions of steady state consumption and system parameters from equations (11) and

[20] Pindyck (1991) defines capital to be irreversible if it cannot be used productively by a different industry. To avoid having to add another state variable we define irreversibility in terms of the ability to switch between capital and consumption. Otherwise our definition matches that of Pindyck (1991).

(12), respectively).[21] With this simplification the effect of a change in the degree of irreversibility of capital on consumption is given by differentiating the Euler equation with respect to the rate of depreciation. The result is formalized in the following proposition.[22]

Proposition 1. *If* $-H_{11}(K^*)\frac{K^*}{H_1(K^*)} \geq \frac{\delta_K}{(r+\delta_k+p)}$ *then* $\frac{dC^*}{d\delta_K} < 0$.

In words the proposition states that steady state consumption will increase (and steady state investment will decrease) as capital becomes more irreversible if the gain from the increase in capital, caused by the decline in the rate of depreciation, is greater than the loss caused by the decline in the marginal product of capital. This result is fairly straightforward since the presence of durable capital reduces the need for new investment.

The effect of capital becoming more irreversible on steady state stocks of capital and greenhouse gases is ambiguous (see corollary 1 in the appendix for a formal statement). Consequently, the effect on the optimal rate of net emissions, $g(C^*)(1 - H(K^*))$, of a change in the irreversibility of capital is unknown.

3.3. **Irreversible Capital.** We now show that, under our definition, an increase in capital irreversibility has no effect the optimal rate of investment. We capture irreversibility through the parameter Φ, with a decrease in Φ implying greater irreversibility. Capital is perfectly irreversible when $\Phi = 0$.

The parameter Φ enters the steady state system of equation only as part of the coefficient on the consumption multiplier, α. This implies that if the constraint on consumption is not binding then Φ does not affect the steady state level of consumption or investment. This result holds even when the constraint on consumption is binding. Recall that when the constraint binds the steady state level of consumption is in fact equal to R and is independent of Φ. The constraint on consumption is less likely to bind the greater is the disutility associated with the stock of greenhouse gases because agents will want to undertake investment to reduce the disutility from the stock of greenhouse gases.

When the consumption constraint binds, and so long as $\Phi > 0$, agents will drive the inherited capital stock to zero very quickly. The level of investment, while the system is out of steady state, remains independent of the level of Φ, at zero. Out of steady state, the level of consumption (and thus emissions) however does not remain independent of Φ. This is because the level of Φ determines how much consumption can be had from a given amount of capital. When the consumption constraint binds and $\Phi = 0$, agents have no option but to wait for capital depreciation to drive the stock of capital to zero. Once again, there is no effect on the level of investment out of

[21] We consider only an interior solution.

[22] Proofs for all the propositions are in the appendix.

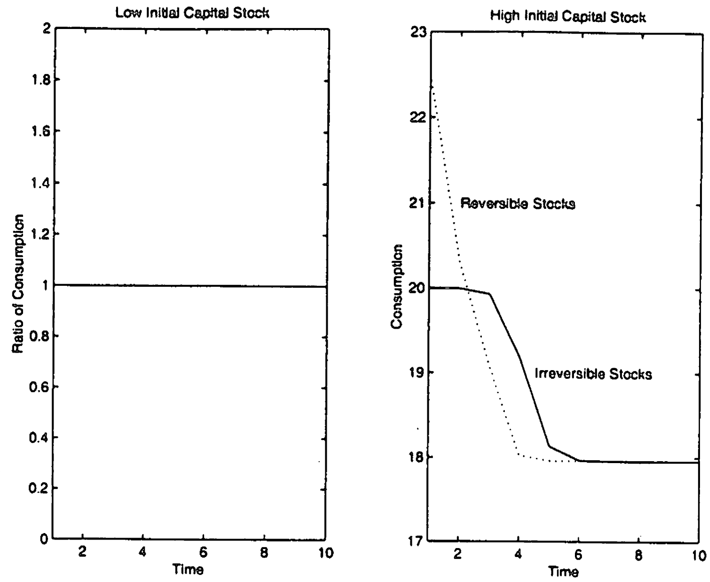

FIGURE 2. Consumption Paths for Reversible and Irreversible Stocks of Capital with Exogenous Risk

steady state. Even the level of consumption out of steady state is unaffected. Capital cannot add to consumption since it is perfectly irreversible.

Simulations based on the numerical model presented in section 2 confirm these results. The first subplot in Figure 2 shows that when the initial stock of abatement capital is low, the ratio of consumption under reversible capital to consumption under irreversible capital is one during the transition to and at the steady state. However, if the initial stock of capital is high then consumption is at first higher and then lower under reversible capital as compared to consumption under irreversible capital (see subplot 2 of Figure 2). This is because consumption is constrained to be less than or equal to $R = 20$ when capital is irreversible. The steady state level of consumption is the same for both types of capital.

4. IRREVERSIBLE STOCK OF GREENHOUSE GASES

We next explore whether or not there is an irreversibility effect associated with the stock of greenhouse gases. Contrary to some earlier results, we find that increasing non-degradability of the stock of greenhouse gases does lead to a decline in consumption even in a multi-period model.

4.1. **Defining Irreversible Stock.** The literature defines the stock of greenhouse gases to be irreversible if emissions in a given period are restricted to be non-negative ((Kolstad 1996b) and (Ulph and Ulph 1997)). No restriction is placed on the rate of decay of the stock of greenhouse gases. In contrast, we additionally require a near-zero decay rate for the stock of gases to be

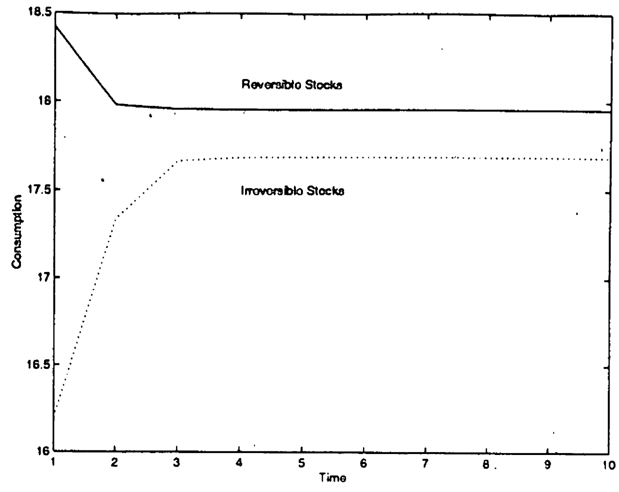

FIGURE 3. Consumption Paths for Reversible and Irreversible Stocks of Greenhouse Gases with Exogenous Risk

considered irreversible. If the stock decays or if emissions are permitted to be negative, then the stock will dissipate over time and cannot be considered irreversible.

4.2. **Rate of Decay.** We capture greenhouse gas stock irreversibility through the parameter δ_m with a decline in δ_m implying an increase in stock irreversibility. The effect, then, of a change in the degree of stock irreversibility can be studied by differentiating the Euler equation (expressed in terms of consumption and system parameters) with respect to the rate of decay. This yields the following proposition,

Proposition 2. *Steady state consumption is an increasing function of the rate of decay of greenhouse gases.*

As the stock of greenhouse gases becomes irreversible, consumption decreases while investment in abatement capital increases. A lower rate of decay implies that any emissions that are released into the atmosphere remain for a longer period of time. This in turn implies that agents have to incur the disutility of these emissions for a longer period. Agents thus choose to reduce consumption, the source of these emissions.

Along with investment the stock of capital increases and the level of net emissions decreases. However, the effect of an increase in irreversibility on the stock of greenhouse gases itself is ambiguous (this is formalized in corollary 2 in the appendix).

Figure 3 shows that under irreversible stocks of greenhouse gases, not only is the steady state level of consumption lower, but so is consumption along the approach path, as compared to that under reversible stocks of greenhouse gases. Our results thus hold even away from the steady state.

4.3. **Non-Negative Emissions.** If instead stock irreversibility is defined in terms of non-negative emissions with no restriction on the rate of decay, then an increase in irreversibility has no effect on the steady state level of consumption or investment. The constraint on emissions simply does not bind at the steady state. This can be seen from the following equation for the steady state stock of greenhouse gases (assuming an interior solution).

$$M^* = \frac{g(C^*)(1 - H(K^*))}{\delta_M} \geq 0 \tag{19}$$

The steady stock of greenhouse gases is restricted to be non-negative which in turn implies that steady state emissions will be non-negative (positive if the stock is positive and zero if the stock is zero). The constraint on emissions simply does not bind. A change in the degree of irreversibility does not affect steady state consumption or investment. A similar argument holds when the constraint on consumption is binding.

If agents choose to reduce the stock of greenhouse gases to zero, and want to do so quickly, then in the transition to steady state the non-negativity constraint will bind. This situation will also arise when the agents inherit a large stock of greenhouse gases and want to reduce the stock quickly. In both these situations agents will prefer to emit negative amounts of greenhouse gases to reduce the stock as fast as possible. This will not be possible if the stock is irreversible. Consequently, stock irreversibility may have an effect away from the steady state. However, for this effect to hold it must be true that a drastic reduction of the stock is optimal or that agents begin with a large endowment of the stock of greenhouse gases.

5. Endogenous Risk

If it is true that global warming is triggered by an increase in the stock of greenhouse gases, then the threat can be mitigated by reducing the stock, that is, by economic agents changing their behavior. In other words, the risk of a catastrophe is avoidable, or endogenous. We now explore the implications of adding endogenous risk to the model.

5.1. **Optimality Conditions.** As with exogenous risk, we begin by differentiating equation (6) with respect to the choice variable—consumption. This gives the first order condition

$$U_1(C, M) - \lambda - \mu_1 + \mu_2 g_1(C)(1 - H(K)) = 0 \tag{20}$$

The equations of motion for the co-state equations, obtained by differentiating equation (6) with respect to the state variables, K and M, are

$$\dot{\mu}_1 = \mu_1\big(r + \delta_K + p(M)\big) - \lambda\Phi + \mu_2 g(C) H_1(K) \tag{21}$$

$$\dot{\mu}_2 = \mu_2\big(r + \delta_M + p(M)\big) - U_2(C, M) - V(K, M) p_1(M) \tag{22}$$

To allow for endogenous risk, the probability of catastrophe, p, is now a function of the stock of greenhouse gases.

5.2. **Steady State.** Since equation (22) contains $V(K, M)$, to obtain the Euler equation we need an additional equation that relates the value function to the primitives of the economy. This additional equation is obtained by evaluating the Bellman-Hamilton-Jacobi equation, equation (6), at the steady state.

$$(r + p(M))V(K, M) = U(C, M) + \lambda(R + \Phi K - C) \tag{23}$$

Now equations (20) and (23), and equations (21) and (22), evaluated at the steady state, combine to give the Euler equation, which in turn with the steady state laws of motion gives a system of equations which determines the steady state level of consumption, capital and stock of greenhouse gases.

(24)

$$U_1 = \frac{(U + \lambda(R + \Phi K - C))\,p_1 - (r + p)U_2}{(r + p)(r + \delta_M + p)} \left(g_1(1 - H) + \frac{gH_1}{(r + \delta_K + p)}\right) + \lambda\left(\frac{\Phi}{(r + \delta_K + p)} + 1\right)$$

(25)

$$K^* = \frac{R - C^*}{\delta_K}$$

(26)

$$M^* = \frac{g(1 - H)}{\delta_M}$$

When $\lambda = 0$ (the constraint on consumption is not binding) the Euler equation states that, along the optimal consumption path, net utility from an increase in consumption is zero. Equations (25) and (26) give arbitrage conditions for optimal stocks of capital and greenhouse gases. When the consumption constraint is binding ($\lambda > 0$), steady state consumption is once again equal to R.

5.3. **Saddle Point.** Once again we use the necessary and sufficient conditions outlined by Dockner (1985) to establish the conditions for the steady state to be a saddle point. The complexity of the dynamic system limits us to a numerical analysis. Before we do that we specify one additional functional form for the probability of catastrophe function.

5.3.1. *Functional Forms.* The probability function is assumed to be the modified logistic function

$$p_t = \frac{2}{(1 + \exp(-\omega M_t))} - 1 \tag{27}$$

where ω is a parameter that captures the sensitivity of p_t to the stock of greenhouse gases. The modified logistic restricts p_t to lie between zero and one.

Parameter	*Minimum Value*	*Maximum Value*
δ_k	0.1	0.5
δ_m	0.05	0.3
σ	0.5	5
ρ	0.01	0.05
ω	0.0001	0.0005

TABLE 2. Parameter Values Used for Saddle Point Search

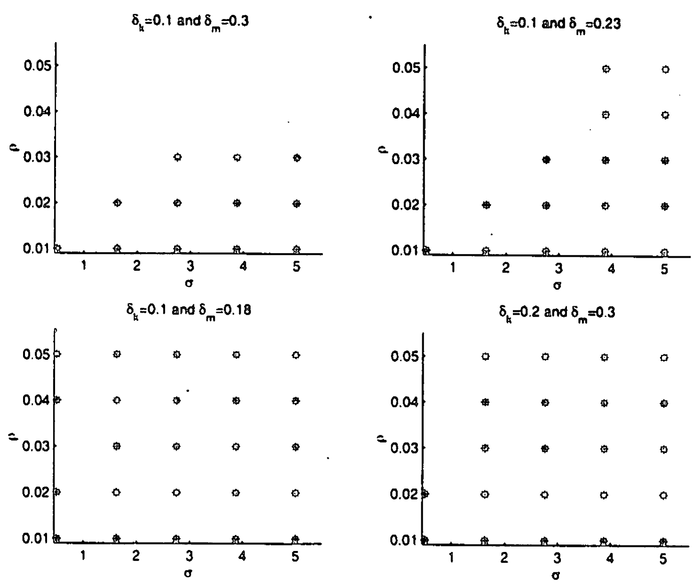

FIGURE 4. Parameter Values for Saddle Point Steady State

5.3.2. *Results.* We now look for parameters values that ensure that the steady state is a saddle point. The results of a five dimensional grid search, using the parameters δ_k, δ_m, σ, ρ and ω, are summarized in Figure 4. The maximum and minimum values for each of these parameters over which the search was conducted are given in Table 2.[23] Each of these subplots hold irrespective of the value of ω and the stars signify parameter values for which the steady state is a saddle point.

The first subplot shows that for a low value of the rate of capital depreciation, δ_k, and a high value for the rate of decay of greenhouse gases, δ_m, if the parameter affecting capital productivity, ρ, is high (between 0.03 and 0.05), then, for most values of the emission to consumption ratio, σ, the steady state is not a saddle point. Increasing the rate of capital depreciation, moving from subplot 1 to subplot 4, gives the result that higher values of ρ now support a saddle point. If the rate of decay of greenhouse gases is decreased instead, say moving from subplot 1 to either subplot 2 or 3, still larger values of ρ support support a saddle point. Consequently, the higher the values

[23]The other parameters used in the simulations took on the following values: (i) $r = 0.03$; (ii) $R = 20$; and (iii) $b = 50$.

of δ_k and σ, and the lower the values of δ_m and ρ, the more likely is the steady state to be a saddle point.

5.3.3. *Monotonicity.* Having established that the steady state is a saddle point we now show that the state variables, K and M, evolve monotonically. As was done in 2.5.4, we establish monotonicity by looking at the trace, determinant and discriminant of the submatrix of the Jacobian, specifically at,

$$J_s = \begin{bmatrix} -\frac{\mu_2^* H_1 g_1}{U_{11}} - \delta_k & \frac{U_{12}}{U_{11}} \\ \frac{(1-H)g_1^2 \mu_2^* H_1}{U_{11}} - gH_1 & -\frac{(1-H)g_1 U_{12}}{U_{11}} - \delta_m \end{bmatrix} \tag{28}$$

The trace of this matrix is negative and its determinant and discriminant are both positive. This implies that related phase-plane diagram is characterized by an improper stable node and that the optimal trajectory is locally monotonic.

5.4. **Irreversible Capital.** With or without a binding constraint on consumption, Φ, the parameter governing the cost of converting capital to consumption, does not affect steady state consumption, capital or stock of greenhouse gases. Away from steady state too, Φ does not affect investment decisions. If agents choose to consume all their endowment and devote nothing to investment then they will run down the capital stock at a rate dictated by Φ (quickly if Φ is high and slowly if Φ is low). However, Φ does not affect an agent's decision to invest nothing and consume all. As with exogenous risk, the degree of irreversibility of abatement capital does not affect decisions to consume or emit greenhouse gases.

Figure 5 confirms this result. The ratio of consumption under reversible stocks of abatement capital to consumption under irreversible stocks is one. Thus at the steady state and away from it, a change in the degree of capital irreversibility does not affect the level of consumption or investment.

5.5. **Durable Capital.** Now let us consider the case where risk is endogenous and the rate of depreciation is used to capture capital irreversibility. Once again steady state capital can be expressed as a linearly decreasing function of steady state consumption, equation (25), and the steady state stock of greenhouse gases as an increasing function, equation (26). This in turn implies that we can write the steady state Euler equation as a function of steady state consumption and some parameters.

To analyze the effect on the optimal rate of consumption of a change in the degree of durability of capital, we differentiate the Euler equation, equation (24), with respect to the rate of depreciation. The differentiation yields a complicated expression with an ambiguous sign. Adding endogenous risk thus dilutes the result that durable capital leads to a decrease in investment. Individuals may choose to increase investment in order to reduce the risk of the catastrophe. This counters the

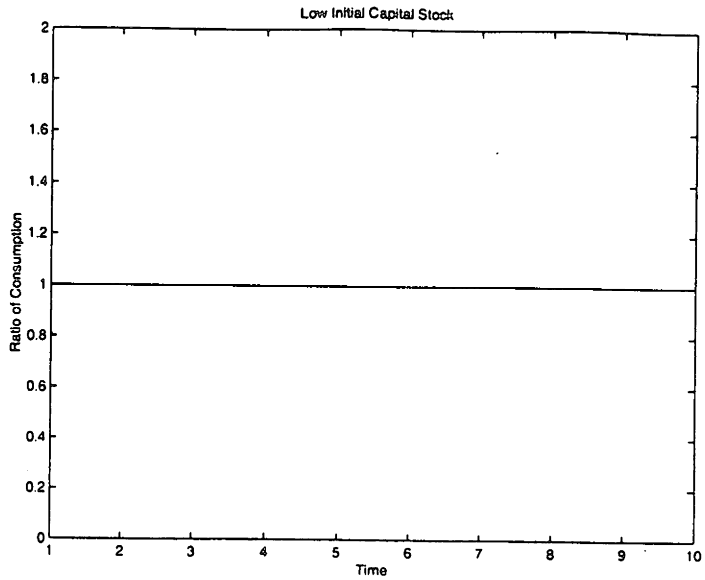

FIGURE 5. Consumption Paths for Reversible and Irreversible Stocks of Capital with Endogenous Risk

need to decrease investment as capital becomes more durable. Models that do not account for endogenous risk will find a stronger irreversibility effect for capital.

5.6. **Decay Rate.** Now let us consider how consumption at the steady state changes with a change in the degree of reversibility of the stock of greenhouse gases.

Proposition 3. *If* $\frac{-\partial p}{\partial M}\frac{\partial M}{\partial \delta_M}\left(\frac{(r+\delta_M+p)}{(r+\delta_K+p)}+\frac{(2r+2p+\delta_M)}{(r+p)}\right) < 1$ *then* $\frac{dC^*}{d\delta_M} > 0$

In words, as the stock of greenhouse gases become irreversible, consumption decreases while investment increases if a reduction in the rate of decay leads to a relatively small increase in the probability of a catastrophe. Consequently, there an increase in the degree of irreversibility of the stock of greenhouse gases leads to lower consumption when the risk is endogenous so long as the increase in irreversibility does not cause a large increase in the probability of catastrophe. If risk does increase rapidly, then it may be optimal to increase consumption today rather than wait for a tomorrow that may never come.

Figure 6 shows that a change in the degree of stock irreversibility can affect the level of consumption at as well as away from the steady state. An increase in the degree of irreversibility decreases the amount of resources devoted to consumption and thereby increases the amount devoted to investment.

5.7. **Non-negative Emissions.** Once again the non-negativity constraint does not bind at the steady state or during the transition to the steady state unless the agents begin with a large stock

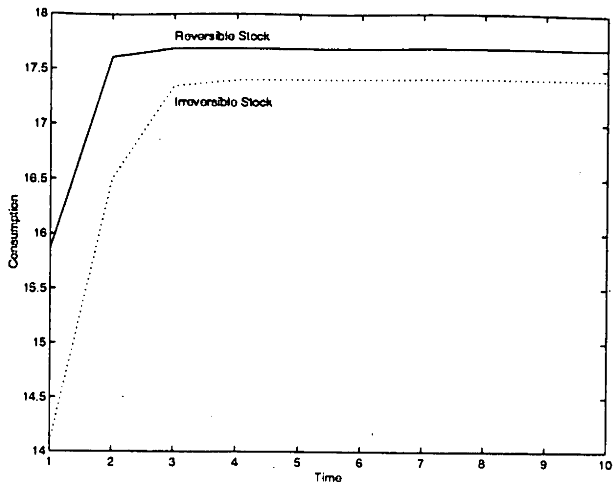

FIGURE 6. Consumption Paths for Reversible and Irreversible Stocks of Greenhouse Gases with Endogenous Risk

of greenhouse gases that they want to reduce drastically. Consequently, defining irreversibility of the stock of greenhouse gases in terms of non-negative emissions weakens the effect of the stock of greenhouse gases on the optimal rate of emissions.

6. SUMMARY AND CONCLUSIONS

Previous studies have suggested that irreversible stocks of greenhouse gases do not lead to a reduction in emissions but irreversible capital does lead to an increase in emissions. We have shown that this result depends on the definition of irreversible capital—capital is irreversible if it does not decay. The result is weakened under what we suggest as a more intuitive definition for irreversibility—capital is irreversible if it cannot be converted into consumption (or other capital). Our definition of irreversible stocks of greenhouse gases, on the other hand, which requires that some part of the stock does not decay, along with nonnegative emissions, does imply an effect of irreversible greenhouse gas stocks on the optimal rate of investment. Further, with endogenous risk stocks of greenhouse gases still affect the optimal rate of consumption. No such effect exists with irreversible abatement capital and endogenous risk.

These results have clear policy implications. Results in the literature to date have implied a reduction in desired investment in abatement capital, since only capital is found to affect the optimal rate of investment. Our results suggest that this may not be an optimal policy. It may be optimal intead to increase the level of desired investment.

Appendix A. Derivation of the Bellman-Hamilton-Jacobi Equation

An agent chooses a stream of consumption to maximize expected intertemporal utility subject to equations (1)–(4).

$$\max_C E_t \int_t^\infty U(C, M, D, \tau) d\tau$$

Let $J(K, M, D, t)$ denote the corresponding value function. To derive the appropriate Bellman-Hamilton-Jacobi equation we begin by splitting the dynamic program into two parts[24]

$$J(K, M, D, t) = \max_C E_t \left[\int_t^{t+dt} U(C, M, D, \tau) d\tau + \int_{t+dt}^\infty U(C, M, D, \tau) d\tau \right] \tag{29}$$

Since

$$E_{t+dt} \int_{t+dt}^\infty U(C, M, D, \tau) d\tau = E_{t+dt} J(K + dK, M + dM, D + dD, t + dt)$$

equation (29) simplifies to

$$\begin{aligned} J(K, M, D, t) = \max_C \Big[& U(C, M, D, t)dt + J\big(K + dK, M + dM, D + a, t + dt\big)pdt \\ & + J\big(K + dK, M + dM, D, t + dt\big)\big(1 - pdt\big) \Big] \end{aligned} \tag{30}$$

Next we take a first order Taylor series expansion of the last two terms on the right hand side of equation (30) around $dt = 0$. This gives the following expression

$$\begin{aligned} J(K, M, D, t) = \max_C \Big[& U(C, M, D, t)dt + J(K, M, D + a, t)pdt + J(K, M, D, t) \\ & + J_1(K, M, D, t)(R - C - \delta_K K)dt \\ & + J_2(K, M, D, t)\big(g(C)(1 - H(K)) - \delta_M M\big)dt \\ & + J_4(K, M, D, t)dt - J(K, M, D, t)dt + h.o.t. \Big] \end{aligned}$$

where $J_1(K, M, D, t)$ is the derivative of the value function with respect to its first argument. $J_2(K, M, D, t)$ and $J_4(K, M, D, t)$ are similarly defined and h.o.t. denotes higher order terms in the Taylor expansion.[25] Subtracting $J(K, M, D, t)$ from both sides, dividing through by dt and letting

[24] This derivation draws heavily on Mangel (1985) and Karp (1997).

[25] Note that because damages take on integer values we do not differentiate the value function with respect to damages.

dt approach zero with the added assumption that $lim_{dt \to 0} \frac{h.o.t.}{dt} = 0$ gives

$$0 = \max_C \Big[U(C,M,D,t) + \big(J(K,M,D+a,t) - J(K,M,D,t)\big)p$$

(31) $$+ J_1(K,M,D,t)(R - C - \delta_K K)$$

$$+ J_2(K,M,D,t)\big(g(C)(1 - H(K)) - \delta_M M\big) + J_4(K,M,D,t)\Big]$$

For the autonomous problem the value function $J(K,M,D,t)$ can be written as $e^{-rt}W(K,M,D)$. Making this substitution into equation (31) and multiplying through by e^{rt} gives the following version of the Bellman-Hamilton-Jacobi equation

$$rW(K,M,D) = \max_C \Big[U(C,M,D) + \Big(W(K,M,D+a) - W(K,M,D) \Big) p$$

$$+ W_1(K,M,D)\big(R - C - \delta_K K\big)$$

$$+ W_2(K,M,D)\big(g(C)(1 - H(K)) - \delta_M M\big)\Big]$$

Up until the time when the catastrophe occurs $D = 0$ and once the catastrophe has occurred utility goes to zero forever, or that, $W(K,M,a) = 0$. With these and the final simplification that $W(K,M,0) = V(K,M)$ and $U(C,M,D) = U(C,M)$ the Bellman-Hamilton-Jacobi equation can be written as

$$rV(K,M) = \max_C \Big[U(C,M) + V_1(K,M)(R - C - \delta_K K)$$

$$+ V_2(K,M)\big(g(C)(1 - H(K)) - \delta_M M\big) - V(K,M)p\Big]$$

Appendix B. Proofs for Propositions and Corollaries

B.1. **Proof for Proposition 1.** Differentiating the Euler equation with respect to δ_K yields the condition that

$$\frac{dC^*}{d\delta_K} < 0 \quad \text{if} \quad -H_{11}(K^*)\frac{K^*}{H_1(K^*)} \geq \frac{\delta_K}{(r + \delta_K + p)}$$

B.2. **Corollary 1.** Totally differentiating equation (11) with respect to the rate of depreciation yields

$$\frac{dK^*}{d\delta_K} = \frac{-1}{\delta_K}\frac{dC^*}{d\delta_K} - \frac{(R - C)}{\delta_K^2}$$

Under the sufficient condition given in proposition 1 the first term on the right hand side of this equation is positive. However, the second term is negative. For the effect on the steady state stock of greenhouse gas, differentiate equation (12) with respect to the rate of depreciation. This yields

$$\frac{dM^*}{d\delta_K} = \frac{\partial M^*}{\partial C^*}\frac{dC^*}{d\delta_K} + \frac{\partial M^*}{\partial \delta_K}$$

The first term on the right hand side of this equation is negative while the second term is positive.

B.3. **Proof for Proposition 2.** Differentiate equation (10) with respect to δ_M. This gives the result that $\frac{dC^*}{d\delta_M} > 0$.

B.4. **Corollary 2.** From equation (12)

$$\frac{dM^*}{d\delta_M} = \frac{\partial M^*}{\partial \delta_M} + \frac{\partial M^*}{\partial C^*}\frac{dC^*}{d\delta_M}$$

The first term on the right hand side of this equation is negative while the second term is positive.

B.5. **Proof for Proposition 3.** Differentiate equation (24) with respect to δ_M. This expression is omitted here because of its complexity. Its denominator is negative while its numerator is negative if

$$\frac{-\partial p}{\partial M}\frac{\partial M}{\partial \delta_M}\left(\frac{(r+\delta_M+p)}{(r+\delta_K+p)} + \frac{(2r+2p+\delta_M)}{(r+p)}\right) < 1$$

Appendix C. Proof on the Nature of the Catastrophe

Our theoretical model characterizes a catastrophe as an event that drives utility to zero forever once the catastrophe has occurred. Consider a different characterization where utility is not driven to zero but depends on the level of the stock of greenhouse gases and the stock of capital at the time of the catastrophe. Let $W(K, M, 0)$ and $W(K^*, M^*, a)$ denote the value functions before and after the catastrophe has occurred, respectively. K^* is the stock of capital and M^* is the stock of greenhouse gases at the time of the catastrophe. We assume that $W(K^*, M^*, a)$ is increasing in capital, decreasing in the stock of greenhouse gases and jointly concave in both capital and the stock of greenhouse gases. Furthermore, its functional form is common knowledge. We now show that our analytical results on irreversible abatement capital when the risk is exogenous still hold under this new definition of a catastrophe (a similar proof establishes the analytical results for irreversible stocks of greenhouse gases and for endogenous risk).

Following the method laid out in the paper it is easy to show that the Bellman-Hamilton-Jacobi equation under the new definition of a catastrophe is given by

$$rW(K,M,0) = \max_{C \le R+\Phi K}\Big[U(C,M) + W_1(K,M,0)\dot{K} + W_2(K,M,0)\dot{M} + W(K^*,M^*,a)p - W(K,M,0)p\Big]$$

With exogenous risk the Euler equation, along with the steady state equations for the stock of abatement capital and the stock of greenhouse gases, which together determine steady state levels of consumption, capital and the stock of greenhouse gases, are given by (assuming an interior solution)[26]

$$
\begin{aligned}
U_1 &= \frac{-U_2}{(r+\delta_M+p)}\left(g_1(1-H)+\frac{gH_1}{(r+\delta_K+p)}\right)+\frac{p}{r+\delta_K+p}W_1(K^*,M^*,a) \\
&-\frac{p}{r+\delta_M+p}W_2(K^*,M^*,a)\left(g_1(1-H)+\frac{gH_1}{r+\delta_K+p}\right) \\
K^* &= \frac{R-C^*}{\delta_K} \\
M^* &= \frac{g(1-H)}{\delta_M}
\end{aligned}
$$

When capital is durable, the following proposition characterizes the relationship between durability and steady state consumption (established by totally differentiating the previous three equations with respect to the rate of depreciation)

Proposition 4. *If* $-H_{11}(K^*)\frac{K^*}{H_1(K^*)} \geq \frac{\delta_K}{(r+\delta_K+p)}$ *and* $-W_{11}(K^*,M^*,a)\frac{K^*}{W_1(K^*,M^*,a)} \geq \frac{\delta_K}{(r+\delta_K+p)}$ *then* $\frac{dC^*}{d\delta_K} < 0$.

The first sufficient condition is the same as the sufficient condition needed to establish the relationship between durability and steady state consumption when a catastrophic event drives utility to zero forever. Under the new definition an additional sufficient condition is needed, namely that the gain in value caused by the increase in capital, in turn caused by the decrease in the rate of depreciation, be greater than the decrease caused by the decline in the marginal product of capital. If both the sufficient conditions are met then consumption increases as capital becomes more durable.

When capital irreversibility is defined in terms of the ability to convert capital into consumption (our preferred definition), then even with the new definition of the catastrophe there is no irreversibility effect. This can be established by the same logic as was used to establish this result under the older definition of catastrophe.

Appendix D. Parameters

Table 2 contains the parameters we use to generate the simulations presented in the paper. For reversible capital $\Phi = 1$ and for irreversible capital $\Phi = 0$. Similarly, for irreversible stocks of greenhouse gases $\delta_m = 0.05$ and for reversible stocks, $\delta_m = 0.1$.

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[26] Note that for simplicity of exposition we assume that the catastrophe occurs after the steady state has been attained.

Parameter	*Values*	*Parameter*	*Values*
β	0.97	δ_k	0.1
b	50	δ_m	0.1 or 0.05
Φ	0 or 1	R	20
σ	1	ρ	0.2
ω	0.0005	p	0.02

TABLE 3. Parameter Values Used for the Simulations

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