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ALTERNATIVE FUNCTIONAL FORMS FOR PRODUCTION, COST AND RETURNS TO SCALE FUNCTIONS

by

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1. Introduction

In this paper, we relax two commonly used assumptions in formulating the functional form of a production function namely, the constant returns to scale and the constant elasticity of substitution assumptions. In general, the scale elasticity, which we equivalently call the returns to scale (RTS) function, may be a function of output and the input mix. Fare, Jansson, and Lovell (1985) introduced ray-homothetic production functions which permit scale economies to vary with the rate of output and the input mix. Zellner and Revankar (1969) linked the definition of a RTS function with Euler's theorem and McElroy (1969) generalized Euler's theorem to derive classes of production functions for which the scale elasticity depends either on output alone or on factor proportions alone. However, for simplicity of exposition, we utilize a homothetic assumption in this paper so that the RTS function is a function of output and the substitution elasticity a function of input combinations as has been done in Revankar (1971), Shephard (1973), and Zellner and Revankar (1969). Based on the generalized production function approach, Avishur (1994) considers the efficiency effect of the privatization of British telecom and Kumbhakar et al. (1991) estimated determinants of inefficiency in U.S. dairy farms.

Frisch (1965) considered a production function with a decreasing scale elasticity and several other researchers have introduced various functional forms for the scale elasticity. For given RTS functions, Zellner and Revankar [ZR (1969)] indicated that a differential equation can be solved to obtain generalized production functions that have prespecified RTS properties. Nerlove (1963) and Ringstad (1967), NR, introduced a linear function of the logarithm of output for the reciprocal of the scale elasticity. Ringstad (1974) suggested using a translog function or a combination of one of ZR's functions and NR's function denoted by, RG. Since the translog function is not a quasi-concave function, it does not satisfy the neoclassical properties of a production function unless further side conditions are imposed In this paper, we introduce a Box-Cox type RTS function that is very flexib. and rich in representing the RTS function. In this approach, we can derive various RTS functions as special cases.

If technical change is incorporated in the production function, certain a priori hypotheses have to be made if technical change is to be distinguished from the returns to scale effect. Sato (1980) has introduced the concept of a "holothetic technology" for this purpose. The separate estimation of technical progress and returns to scale will be possible if the production structure is not holothetic; see Avishur (1994) for an example. Furthermore, Calem (1990) indicated how misspecification of the underlying technology can result in inappropriate, though seemingly reliable, estimates of technical progress and a returns to scale function using one of the ZR RTS functions and a quadratic time trend.

For the substitution elasticity, we show how to estimate RTS functions when we use a unit elasticity of substitution function (Cobb-Douglas), a constant elasticity of substitution function (CES), and a variable elasticity of substitution (VES) function suggested by Revankar (1971). As a generalization of this approach, we introduce a polar coordinate representation for a homogeneous function so that it can be separated into a radial part and an angular part. As will be seen, with the proper choice of a function defining the isoquants' slope, we can represent CD, CES, VES, or more general functional forms.

As in ZR (1969), we consider output, denoted by y to be given by $y = g(f)$ where g is a monotonically increasing function of f and $f = f(K,L)$ is a homogeneous function of degree μ with capital (K) and labor (L) inputs and thus $g(f)$ is homothetic. Our objectives are threefold: (1) to establish a globally quasi-concave production function with convex isoquants; (2) for this production function to have the RTS function be a decreasing function of output and to have an increase in all inputs not decrease output; and (3) to obtain an AC function that is U-shaped (or L-shaped). We shall take up these issues in what follows.

In constructing a production function, the homotheticity restriction on $g(f)$ is a useful simplifying assumption but is not a required property. Christensen and Greene (1976) derived a RTS function based upon a translog cost function and compared the effect of imposing homotheticity, homogeneity, and unitary elasticities of substitution restrictions. The interesting question of testing the validity of the homotheticity assumption will not be discussed in this paper. Alternatively, under the homotheticity assumption, we discuss properties of various production functions, compare the parametric functional forms with the flexible functional forms, and interpret some empirical results. Other interesting issues such as models with dynamic effects will be considered in future work.

An overview of the paper is as follows. In Section 2, we review how parametric production functions can be derived from given RTS functions. Various fixed functional form choices of the RTS and the homogeneous parametric functions, denoted by $f(K,L)$ above, are considered. A polar coordinate representation is introduced for the homogeneous function, f. In Section 3, semiparametric approaches are introduced for both the RTS function and the homogeneous function, $f(K,L)$. In Section 4, maximum likelihood and Bayesian estimation procedures for this class of models are presented. In Sections 5 and 6, we apply our methods using data for the U.S. transportation equipment industry. Estimates of alternative returns to scale functions are presented in Section 5 and estimates of input substitution effects are presented in Section 6. A summary and some concluding remarks are presented in Section 7.

2. Mathematical Description of Alternative Models

In this section, after reviewing the properties of a production function, we construct various generalized production functions, using a homotheticity assumption, that is $y = g(f)$ with $g(\cdot)$ a monotonic function and $f(K,L)$ a homogeneous function. We introduce specific functional forms for the returns to scale function, denoted by $\alpha(y)$, and the homogeneous function $f = f(K,L)$. In particular, we introduce a Box-Cox type RTS function because many well known RTS functions can be defined as special cases of this function. Some of these well known functions are sensible production functions with reasonable associated average cost functions (U-shaped or L-shaped). We also generalize the homogeneous function $f(K,L)$, using a polar coordinate representation. CES and VES functions can be derived with proper choices of the slopes of the isoquants.

The properties of a production function, stated in Fuss et al. (1978) for a single output y and n inputs $(x_1,...,x_n)$ are

1) *Domain.* $y = w(x_1,...,x_n)$ is *a real-valued function of* $(x_1,...,x_n)$ *defined for every non-negative input* ($x_i \geq 0$ *for all* $i = 1,2,...,n$) *and it is finite if* $(x_1,...,x_n)$ *is finite;* $w(0,...,0) = 0$.
2) *Monotonicity. An increase in inputs cannot decrease production.*
3) *Continuity.*
4) *Concavity. w is quasi-concave over every non-negative input* ($x_i \geq 0$ *for all* $i = 1,2,...,n$).

In what follows, we impose the following two additional conditions:

5) *Homotheticity. w is a homothetic function.*
6) *Decreasing RTS. The RTS is larger than one at small output levels and decreases monotonically below as the output level increases.*

Conditions 5) and 6) are introduced because condition 5) is a useful simplifying assumption and condition 6) leads to average cost (AC) functions having a unique minimum. As explained above, a homothetic production function is defined as

$$y = g(f) \tag{2.1}$$

where g is a monotone transformation function and $f = f(K,L)$ is a homogeneous function of degree μ[1]. The RTS function $\alpha(y)$, is defined as follows,

$$\alpha(y) = \mu \frac{dy/y}{df/f} \tag{2.2}$$

and is related to Euler's theorem in McElroy (1969) and ZR (1969). The relationship between properties of the RTS function and of associated AC functions is derived in Hanoch (1975) and in Sandler and Swimmer (1978). Based upon their results, we can state:

Lemma 2-1. If the RTS function is greater (smaller) than one, then the slope of AC curve is negative (positive) or equivalently if the slope of AC curve is negative (positive), then the RTS function is greater (smaller) than one. If RTS=1 at some output level, then the AC curve has a unique minimum at this point provided that the RTS function is a decreasing function of y.

In what follows, we shall review one of ZR's (1969) generalized production functions, and then consider extensions of this model.

[1]Without loss of generality, we could take $\mu = 1$; however, we shall use the standard definition of homotheticity.

2.1 Generalized Production Functions.

ZR (1969) introduced a neo-classical production function $f(K,L)$ of homogeneity μ and considered a monotonic transformation of this neo-classical production function $y = g(f)$. Defining the RTS function, $\alpha(y)$, as in (2.2), and taking a particular parametric functional form for $\alpha(y)$, they solved the differential equation in (2.2) for output as a function of f. By choosing a functional form for f, the shapes of the isoquants are determined. The function f can be a CD, CES, or any other homogeneous function. They considered several forms for $\alpha(y)$, and we shall use one of them namely,

$$\alpha(y) = \frac{\mu}{1 + \theta y} \tag{2.3}$$

with $0 < \theta < \infty$, where μ is the degree of homogeneity of f. Substituting (2.3) in (2.2) and solving the differential equation, the result is:

$$\log y + \theta y = \log f. \tag{2.4}$$

If we assume $1 < \mu$ and $0 < \theta$, then the RTS function in (2.3) decreases monotonically below one as the output level increases and the AC curve will have a unique minimum. In addition to the RTS functions in (2.3), ZR considered two additional RTS functions and their associated production functions. Other RTS functions are shown below.

2.2 Analysis of Several Returns to Scale (RTS) Functions

Several RTS functions are given below:

Zellner-Revankar (ZR): $$\alpha_1(y) = \frac{\mu}{1 + \theta_1 y} \tag{2.3}$$

Nerlove-Ringstad (NR): $$\alpha_2(y) = \frac{\mu}{1 + 2\gamma_1 \log y} \tag{2.5}$$

$$\text{Ringstad (RG):} \quad \alpha_3(y) = \frac{\mu}{1 + \theta_2 y + 2\gamma_2 \log y} \tag{2.6}$$

$$\text{Box-Cox (BC):} \quad \alpha_4(y) = \frac{\mu}{1 + \nu_1\left(\frac{y^{\lambda_1} - 1}{\lambda_1}\right)} \tag{2.7}$$

$$\text{Combined ZR and BC (CB):} \quad \alpha_5(y) = \frac{\mu}{1 + \theta_3 y + \nu_2\left(\frac{y^{\lambda_2} - 1}{\lambda_2}\right)} \tag{2.8}$$

where ZR corresponds to one of Zellner and Revankar's models, NR denotes the Nerlove (1963) and Ringstad model (1967), RG denotes that of Ringstad (1974), BC corresponds to use of the Box-Cox transformation, which we introduce in this paper, and CB is a combination of ZR and BC. We have introduced the Box-Cox transformation on y because it is very flexible. It is clear that (2.3), (2.5), (2.6), and (2.7) are special cases of (2.8). For example, if $\lambda_2 = 0$, we get the RG RTS function in (2.6).

We require the RTS be larger than one at low output levels, and decrease monotonically below one as the output level increases. For output level $1 \le y \le +\infty$, we need, $\mu > 1$. We restrict the output range be larger than or equal to one so that the denominators of (2.5)-(2.8) are positive and increase monotonically as the output level increases. In addition, we impose $\theta_1 > 0$ and $1 + \theta_1 < \mu$ for ZR, $\gamma_1 > 0$ for NR, $\theta_2 > 0$, $1 + \theta_2 < \mu$, and $\gamma_2 > 0$ for RG, $\nu_1 > 0$ and $\lambda_1 \ge 0$ for BC, and $\theta_3 > 0$, $1 + \theta_3 < \mu$, $\nu_2 > 0$, and $\lambda_2 \ge 0$ for CB. We note these conditions are sufficient conditions for the RTS function to have the above properties.[2]

Solving the differential equation (2.2) using the RTS functions in (2.3)-(2.8), we obtain:

$$\text{ZR:} \quad \log f = \log y + \theta_1 y \tag{2.9}$$

[2]Let RTS = $\mu/D(y)$. The required properties for the RTS function can be satisfied if $dD(y)/dy > 0$ and $D(y) < \mu$ at $y = 1$. These conditions are satisfied if the parameters satisfy the conditions shown in the text and $D(y) > 0$.

$$\text{NR:} \quad \log f = \log y + \gamma_1(\log y)^2 \tag{2.10}$$

$$\text{RG:} \quad \log f = \log y + \gamma_2(\log y)^2 + \theta_2 y \tag{2.11}$$

$$\text{BC:} \quad \log f = \log y + \nu_1\left[\frac{y^{\lambda_1}}{\lambda_1^2} - \frac{\log y}{\lambda_1}\right] \tag{2.12}$$

$$\text{CB:} \quad \log f = \log y + \theta_3 y + \nu_2\left[\frac{y^{\lambda_2}}{\lambda_2^2} - \frac{\log y}{\lambda_2}\right] \tag{2.13}$$

The RHSs of (2.9)-(2.13) are increasing functions of y and the transformations from y to f are unique. In (2.12) and (2.13), we assume $\lambda_1, \lambda_2 > 0$.

For the homogeneous function f in (2.9)-(2.13), we can use a Cobb-Douglas unit elasticity of substitution function (CD), a constant elasticity of substitution function (CES) or a variable elasticity of substitution function (VES)[3] suggested by Revankar (1971), as shown below:

$$\text{Cobb-Douglas (CD):} \quad \log f = b_0 + b_1 \log K + b_2 \log L \tag{2.14}$$

$$\text{Const. Elas. of Subst. (CES):} \quad \log f = b_0 - (\mu/\rho)\log[\delta K^{-\rho} + (1-\delta)L^{-\rho}] \tag{2.15}$$

$$\text{Var. Elas. of Subst. (VES):} \quad \log f = b_0 + b_1 \log K + b_2 \log[L + (\rho-1)K] \tag{2.16}$$

The form of an AC function's dependence on output does not depend upon the choice of the homogeneous function f but only on the degree of homogeneity, μ, and the transformations involving y in (2.9)-(2.13). The total cost function, denoted by $C(y) = yAC(y)$, is obtained as follows for a given RTS function $\alpha(y)$:[4]

[3]Revankar's VES model is defined by

$$\log f = \log\gamma_0 + \mu(1-\delta\rho)\log K + \mu\delta\rho\log[L + (\rho-1)K]$$

where the parameters can be denoted by: $b_0 = \log\gamma_0$, $b_1 = \mu(1-\delta\rho)$, $b_2 = \mu\delta\rho$.

[4]Find an extremum of the Lagrangian expression, $L = \sum_{j=1}^{J} w_j x_j + \lambda[y - g\{f(x_1,...,x_J)\}]$. At the optimum,

$$C^* = \sum_{j=1}^{J} w_j x_j^* = \lambda^* g' \sum_{j=1}^{J} f_j^* x_j^* = \mu\lambda^* g' f^*$$

Since λ^* equals marginal cost, we get (2.17).

$$\alpha(y) = \frac{AC(y)}{MC(y)} \Rightarrow \log C(y) = \text{Constant} + \int \frac{dy}{y\alpha(y)} \tag{2.17}$$

The AC curves can exhibit symmetric and asymmetric shapes each with a minimum at an output level for which the RTS function is equal to one. Shown below are the AC functions associated with use of the RTS functions in (2.3)-(2.8):

$$\text{ZR:} \quad \log AC = \log C_1 + \frac{(1-\mu)\log y + \theta_1 y}{\mu} \tag{2.3'}$$

$$\text{NR:} \quad \log AC = \log C_2 + \frac{(1-\mu)\log y + \gamma_1(\log y)^2}{\mu} \tag{2.5'}$$

$$\text{RG:} \quad \log AC = \log C_3 + \frac{(1-\mu)\log y + \gamma_2(\log y)^2 + \theta_2 y}{\mu} \tag{2.6'}$$

$$\text{BC:} \quad \log AC = \log C_4 + \frac{(1-\mu)\log y + v_1\left[\frac{y^{\lambda_1}}{\lambda_1^2} - \frac{\log y}{\lambda_1}\right]}{\mu} \tag{2.7'}$$

$$\text{CB:} \quad \log AC = \log C_5 + \frac{(1-\mu)\log y + \theta_3 y + v_2\left[\frac{y^{\lambda_2}}{\lambda_2^2} - \frac{\log y}{\lambda_2}\right]}{\mu} \tag{2.8'}$$

where $C_1, C_2, C_3, C_4,$ and C_5 are constants depending on the choice of the homogeneous function $f(K,L)$ but not upon the choice of the RTS function, and $\lambda_1, \lambda_2 > 0$. We now turn to provide generalized forms of the homogeneous function $f(K,L)$.

2.3 Generalized VES Function

In describing a homogeneous function, we are free to use any coordinate system. Recognizing this fact, we show how CD, CES, and VES functions can be represented in a polar coordinate system. The motivation for using this coordinate system comes from the simplicity of the functional forms which permits us to write a bivariate function as a product of two univariate functions. The usefulness of this system is further discussed in section 3.2 where we impose a convexity condition on the isoquants. Suppose we rewrite capital and labor inputs in polar coordinates.

$$K = r\cos\phi \quad \text{and} \quad L = r\sin\phi \quad \text{where } 0 \le \phi \le \pi/2. \tag{2.18}$$

Lemma[5] *2-2. A function is homogeneous of degree μ if and only if*

$$f = f(K,L) = r^{\mu}\Phi(\phi). \tag{2.19}$$

As Clemhout (1968), Sandler and Swimmer (1978), and others have pointed out, a class of homogeneous production functions can be derived if the slopes of isoquants are specified everywhere, $-dL/dK = (\partial f/\partial K)/(\partial f/\partial L)$. For example, a CES function is derived by Clemhout (1968) using this method. In this paper, we shall generalize Clemhout's method using the polar coordinate representation defined in (2.18). For a two input homogeneous function $f = f(K,L)$, we define the functional form of f by specifying the slope of the isoquant in terms of a polar angle $\phi = \arctan L/K$,

$$-\frac{dL}{dK} = \frac{\partial f/\partial K}{\partial f/\partial L} = h(\tan\phi).$$

If we define $f \equiv r^{\mu}\Phi(\tan\phi)$, then

$$\frac{\partial f}{\partial K} = \mu r^{\mu-1}\cos\phi\,\Phi - r^{\mu-1}\sin\phi\,\Phi_{\phi}$$

$$\frac{\partial f}{\partial L} = \mu r^{\mu-1}\sin\phi\,\Phi + r^{\mu-1}\cos\phi\,\Phi_{\phi}$$

where $\Phi_{\phi} = \partial\Phi/\partial\phi$. Then

[5] The proofs of all lemmas are available on request to authors.

$$-\frac{dL}{dK} = \frac{\partial f/\partial K}{\partial f/\partial L} = \frac{\mu\Phi - \tan\phi(1+\tan^2\phi)\Phi'}{\mu\tan\phi\Phi + (1+\tan^2\phi)\Phi'}$$

where $\Phi' = \partial\Phi/\partial\tan\phi$.

Since the elasticity of substitution is

$$\sigma = \frac{d\log(L/K)}{d\log(f_K/f_L)} = \frac{d\log(\tan\phi)}{d\log h(\tan\phi)} = \frac{1}{\frac{d\log h(\tan\phi)}{d\log(\tan\phi)}}, \tag{2.20}$$

a unit elasticity of substitution can be obtained by choosing the slope of the isoquant as $h_1(\tan\phi) = c\tan\phi$. A constant elasticity of substitution different from one can be obtained by choosing the slope of the isoquant as $h_2(\tan\phi) = d(\tan\phi)^{1+\rho}$, and a variable elasticity of substitution can be obtained by choosing the slope of the isoquant as $h_3(\tan\phi) = A + B\tan\phi$. This VES model has elasticity of substitution $\sigma = 1 + (A/B\tan\phi)$ which is equivalent to (2.2) of Revankar (1971). To derive the homogeneous function from the slope function of the isoquants, $h(x)$, both Revankar (1967) and Clemhout (1968) solved differential equations. Revankar used a linear functional form for $h(x)$, $h(x) = A + Bx$, while Clemhout considered more general forms for $h(x)$.

As an application of Clemhout's method, let us consider a combination of CES and VES by introducing the following general form for the slope of the isoquant,

$$h_4(x) = d(A+Bx)^{1+\rho} + [A+(B-1)x] \quad \text{where} \quad x \equiv L/K = \tan\phi.$$

Following Clemhout's procedure,[6] define

$$\psi(x) \equiv \frac{1}{x+h} - \frac{1}{d(A+Bx)^{1+\rho} + (A+Bx)} \tag{2.21}$$

and then by integration, the result is:

[6]Clemhout considered a homogeneous function of degree one, whereas we are considering a homogeneous function of degree μ.

$$f(K,L) = K^{\mu}\exp\left[\mu\int\psi(x)dx\right] = K^{\mu}\left[\frac{(A+Bx)^{\rho}}{d(A+Bx)^{\rho}+1}\right]^{(\mu/\rho B)} e^{C} \tag{2.22}$$

where e^C is a constant of integration, or

$$\log f(K,L) = \mu\log K - \frac{\mu}{\rho B}\log\left[d+(A+Bx)^{-\rho}\right] + C. \tag{2.23}$$

A CES function or a VES function can be obtained from (2.23) by choosing A, B and ρ appropriately. That is, we get a VES function for $\rho=-1$, $(d+A) = (\rho'-1)/\delta\rho', B=1/\delta\rho'$ and a CES function for $A=0$, $B=1$, $d=\delta/(1-\delta)$. Here ρ and ρ' are two different parameters.

3. Use of Semiparametric Functions

3.1 Semiparametric Homogeneous Functions

To generalize the fixed functional forms used in the previous section for the homogeneous functionf, we use a semiparametric approach.[7] However, we need the convexity of isoquants to have the resultant functions be production functions.

Clemhout (1968) also suggested using a semiparametric expression for the homogeneous function, but it was not clear how to impose the convexity restriction on the isoquants with respect to the origin. In the following, we shall establish a necessary and sufficient condition for this restriction to hold.

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[7]We note that the RTS function $\alpha(y)$ can also be represented semiparametrically, e.g. $\alpha(y) = u \bigg/ \left(A + \sum_{n=1}^{N} n a_n y^n\right)$, which when inserted in (3.2) leads to $A\log y + \sum_{n=0}^{N} a_n y^n = \log f$.

Theorem 3-1. The determinant of a Hessian matrix for a homogeneous function of degree one is zero. That is, for a two input homogeneous function, f(K, L),

$$\det \begin{vmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{vmatrix} = 0. \tag{3.1}$$

To impose concavity on a two input homogeneous function of degree one, we need to check $|H_1| = f_{KK} < 0$ *because* $|H_2| = f_{KK}f_{LL} - f_{KL}^2 = 0$. *If* $f(K,L) = r\Phi(\phi)$, *then a necessary and sufficient condition for the concavity restriction on f is*

$$f_{KK} = \frac{\sin^2\phi}{r}(\Phi + \Phi_{\phi\phi}) < 0. \tag{3.2}$$

For the semiparametric representation of the homogeneous function, Clemhout (1968) suggested a functional form in which the logarithm of output is a function of the input combination ratio, tan ϕ,

$$\log f = \log L + \textstyle\sum_{n=0}^{N} c_n(\tan\phi)^n. \tag{3.3}$$

Alternatively, we may use a functional form based upon the homogeneity of degree μ technology. For these functions, output can be decomposed into a scale effect (r) and an input characteristic part, $\Phi(\phi)$,

$$f = r^{\mu}\Phi^{\mu}(\phi) = r^{\mu}\exp[\Psi(\phi)] = r^{\mu}\exp\left[\textstyle\sum_{n=0}^{N} d_n(\tan\phi)^n\right] \tag{3.4}$$

and similarly,

$$f = r^{\mu}\Phi^{\mu}(\phi) = r^{\mu}\exp[\Psi(\phi)] = r^{\mu}\exp\left[\textstyle\sum_{n=0}^{N} e_n(\phi)^n\right] \tag{3.5}$$

where d_n and e_n, n = 0,1,2,... are parameters. To impose concavity on (3.4) and (3.5), we use theorem

3-1 and require that $\mu + \Psi_\phi^2/\mu + \Psi_{\phi\phi} < 0$ for $\Psi = \sum_{n=0}^{N} d_n(\tan\phi)^n$ or $\Psi = \sum_{n=0}^{N} e_n(\phi)^n$.

There is a tradeoff in establishing a general functional form. By introducing generality in the homogeneous function as in (3.4) and (3.5), we can impose concavity with ease but the substitution elasticity will be in a complicated form. Alternatively, by introducing a general form for the substitution elasticity, the derived homogeneous function is in a complicated form and imposition of the concavity condition is difficult.

There is an alternative way, introduced by Barnett et al. (1988) to derive a concavity-restricted homogeneous function. Based upon Clemhout's method stated in (2.22),

$$f(K,L) = K^{\mu}\exp\left[\mu\int\psi(x)dx\right] \tag{3.6}$$

where $\psi(x) = 1/(x+h)$ and $h = -dL/dK$. Therefore if we choose $\psi(x) = \xi'(x)/\xi(x)$ where $\xi(x)$ is a concave function, then an explicit form can be obtained, namely

$$f(K,L) = K^{\mu}\xi^{\mu}(x). \tag{3.7}$$

The required convexity condition of an isoquant with respect to the origin is satisfied because $h = -dL/dK = 1/\psi - x = \xi/\xi' - x$ is a monotone increasing function when $\partial h/\partial x = -\xi\xi''/(\xi')^2 > 0$ for a concave ξ. Since the sum of two concave functions is another concave function, introduce a sequence of concave functions, $g_1, \ldots, g_N$ and let $\xi = g_1 + \ldots + g_N$. Then we get $f = K^{\mu}[g_1(x) + \ldots + g_N(x)]^{\mu}$. This technique of adding several concave functions to establish a flexible concave function can be applied using, e.g., the Müntz-Szätz series expansion and others. However, imposition of the global constraints severely reduces the flexibility of the chosen complete set expansion. For example, if we restrict the expansion coefficients of the Müntz-Szätz approximation to be positive to approximate a concave function, then such imposition of sufficient conditions for the concavity restriction reduces the flexibility of the series expansion. The critical issue is whether a semiparametric estimated form without concavity restrictions imposed is or is not significantly different from the restricted function, an empirical proposition that can be tested.

In this subsection, we have established a relationship between the homogeneous function and the slope of an isoquant. In particular, we have shown how to derive a homogeneous function of degree one from the given slope of the isoquant using Clemhout's (1968) method. In the next section, we shall estimate Clemhout functions of a polynomial of degree 2 as shown in (4.16)-(4.20) in Table 1.

4. Estimation of Models

Below we shall consider maximum likelihood (ML) and Bayesian (B) estimation of the production functions described above and shown in Table 1. With respect to B estimation, we have just as yet estimated a subset of the functions in Table 1 using a finite sample B approach. Large sample B results are available for all the functions in Table 1.

4.1 Maximum Likelihood Estimation

In this section, we use the maximum likelihood method to estimate production functions' parameters. Various combinations of RTS functions and homogeneous functions are considered and listed in Table 1.

For a general representation of a model and maximum likelihood parameter estimation, we follow the procedures of Zellner and Revankar (1969) and Greene (1991). The models listed in (4.1)-(4.20) of Table 1 are conveniently written as

$$z(y_i, \xi) = h(K_i, L_i, \Theta) + u_i \tag{4.21}$$

where ξ and Θ denote parameters associated with the RTS functions and homogeneous functions, respectively and K_i and L_i are regarded as exogenous input variables, as in Zellner, Drèze and Kmenta

Table 1: Combination of Various Returns to Scale Functions and Homogeneous Functions

Zellner-Revankar's (ZR) RTS with CD:	$\log y + \theta_1 y = b_0 + b_1 \log K + b_2 \log L + u_1$	(4.1)
Nerlove-Ringstad's (NR) RTS with CD:	$\log y + \gamma_1 (\log y)^2 = b_0 + b_1 \log K + b_2 \log L + u_2$	(4.2)
Ringstad's (RG) RTS with CD:	$\log y + \gamma_2 (\log y)^2 + \theta_2 y = b_0 + b_1 \log K + b_2 \log L + u_3$	(4.3)
Box-Cox (BC) RTS with CD:	$\log y + \nu_1 \left[y^{\lambda_1}/\lambda_1^2 - \log y/\lambda_1 \right] = b_0 + b_1 \log K + b_2 \log L + u_4$	(4.4)
Comb. ZR and BC (CB) RTS with CD:	$\log y + \theta_3 y + \nu_2 \left[y^{\lambda_2}/\lambda_2^2 - \log y/\lambda_2 \right] = b_0 + b_1 \log K + b_2 \log L + u_5$	(4.5)
ZR's RTS with CES:	$\log y + \theta_1 y = b_0 - (\mu/\rho) \log\left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right] + u_6$	(4.6)
NR's RTS with CES:	$\log y + \gamma_1 (\log y)^2 = b_0 - (\mu/\rho) \log\left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right] + u_7$	(4.7)
RG's RTS with CES:	$\log y + \gamma_2 (\log y)^2 + \theta_2 y = b_0 - (\mu/\rho) \log\left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right] + u_8$	(4.8)
BC RTS with CES:	$\log y + \nu_1 \left[y^{\lambda_1}/\lambda_1^2 - \log y/\lambda_1 \right] = b_0 - (\mu/\rho) \log\left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right] + u_9$	(4.9)
CB RTS with CES:	$\log y + \theta_3 y + \nu_2 \left[y^{\lambda_2}/\lambda_2^2 - \log y/\lambda_2 \right] = b_0 - (\mu/\rho) \log\left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right] + u_{10}$	(4.10)
ZR's RTS with VES:	$\log y + \theta_1 y = b_0 + b_1 \log K + b_2 \log[L + (\rho - 1)K] + u_{11}$	(4.11)
NR's RTS with VES:	$\log y + \gamma_1 (\log y)^2 = b_0 + b_1 \log K + b_2 \log[L + (\rho - 1)K] + u_{12}$	(4.12)
RG's RTS with VES:	$\log y + \gamma_2 (\log y)^2 + \theta_2 y = b_0 + b_1 \log K + b_2 \log[L + (\rho - 1)K] + u_{13}$	(4.13)
BC RTS with VES:	$\log y + \nu_1 \left[y^{\lambda_1}/\lambda_1^2 - \log y/\lambda_1 \right] = b_0 + b_1 \log K + b_2 \log[L + (\rho - 1)K] + u_{14}$	(4.14)
CB RTS with VES:	$\log y + \theta_3 y + \nu_2 \left[y^{\lambda_2}/\lambda_2^2 - \log y/\lambda_2 \right] = b_0 + b_1 \log K + b_2 \log[L + (\rho - 1)K] + u_{15}$	(4.15)
ZR's RTS with Clemhout:	$\log y + \theta_1 y = b_0 + b_1 \log r + b_2 \phi + b_3 \phi^2 + u_{16}$	(4.16)
NR's RTS with Clemhout:	$\log y + \gamma_1 (\log y)^2 = b_0 + b_1 \log r + b_2 \phi + b_3 \phi^2 + u_{17}$	(4.17)
RG's RTS with Clemhout:	$\log y + \gamma_2 (\log y)^2 + \theta_2 y = b_0 + b_1 \log r + b_2 \phi + b_3 \phi^2 + u_{18}$	(4.18)
BC RTS with Clemhout:	$\log y + \nu_1 \left[y^{\lambda_1}/\lambda_1^2 - \log y/\lambda_1 \right] = b_0 + b_1 \log r + b_2 \phi + b_3 \phi^2 + u_{19}$	(4.19)
CB RTS with Clemhout:	$\log y + \theta_3 y + \nu_2 \left[y^{\lambda_2}/\lambda_2^2 - \log y/\lambda_2 \right] = b_0 + b_1 \log r + b_2 \phi + b_3 \phi^2 + u_{20}$	(4.20)

(1966).[8] The subscript i denotes variables pertaining to the ith unit ($i = 1,2,...,T$) and the u_i's are random error terms which are assumed normally and independently distributed each with mean zero and common variance σ^2. The logarithm of the likelihood function is given by

$$\log L = -\frac{T}{2}\log 2\pi - \frac{T}{2}\log\sigma^2 + \log J - \frac{1}{2\sigma^2}\sum_{i=1}^{T}\left\{z(y_i,\xi) - h(K_i,L_i,\Theta)\right\}^2 \tag{4.22}$$

where J is the Jacobian of the transformation from the u_i's to the y_i's

$$J = \prod_{i=1}^{T}\left|\frac{\partial u_i}{\partial y_i}\right| = \prod_{i=1}^{T} J_i \tag{4.23}$$

Differentiating (4.22) partially with respect to σ^2 and setting the derivative equal to zero, we obtain $\hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^{T}\left\{z_i(y_i,\xi) - h(K_i,L_i,\Theta)\right\}^2$. Substituting this conditional maximizing value for σ^2 in (4.22), we obtain the "concentrated" likelihood function,

$$\log L^* = \text{constant} + \log J - \frac{T}{2}\log\left[\sum_{i=1}^{T}\left\{z_i(y_i,\xi) - h(K_i,L_i,\Theta)\right\}^2\right] \tag{4.24}$$

Since unconditional maximization of the concentrated likelihood function is not easy, we apply the conditional maximization method employed in ZR (1969) and Zellner (1971, p.177ff.). A subset of the parameters, e.g. b_0, b_1 and b_2 in (4.4) in Table 1, are estimated by least squares conditional on various given values of ν_1 and λ_1 and for each pair the conditionally maximized $\log L^*$ is evaluated. The values of the parameters for which $\log L^*$ is maximized are the ML estimates. Alternatively, $\log L^*$ can be maximized by use of nonlinear numerical optimization routines. In addition, $\log L^*$ can be maximized subject to inequality constraints mentioned in connection with (2.3)-(2.8).

An estimate of the asymptotic covariance matrix of the maximum likelihood estimators can be obtained by inverting the estimated information matrix. However, the Berndt et al. (1974) estimate

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[8] The problems associated with the possible endogeneity of input variables and measurement errors in the input variables will not be treated herein. See, e.g. Tybout (1992) for consideration of these issues and references to the literature.

is easier to compute, as noted by Greene (1991). To estimate the variances of the estimated RTS functions, we apply the delta method which is explained in Goldberger (1991) and other textbooks; see also, Anderson and Thursby (1986). Of course, these asymptotic methods may not produce reliable results in small samples.

4.2 Bayesian Estimation of Parameters

When the sample size is large, the posterior density of a model's parameters is asymptotically normal centered at the ML estimate–see, e.g. Jeffreys (1967), Hartigan (1983), Heyde and Johnson (1979), and Chen (1985). However, since the exact finite sample posterior density can be computed, there is in general no need to rely on asymptotic approximate results in the Bayesian approach. If a sample is not large, computed posterior densities will not generally be normally shaped. In fact such a finding is an indication that asymptotic results are not appropriate. Below, examples will be provided to illustrate this point.

For Zellner and Revankar's RTS function and the Cobb-Douglas homogeneous function given in (4.1) of Table 1, Zellner (1971) pursued a Bayesian analysis to estimate the model's parameters. Following Box and Cox's (1964) argument, the prior pdf for the parameters of (4.1) was assumed to be $\pi(b_0, b_1, b_2, \sigma, \theta) \propto 1/J^{3/T}\sigma$ where J is defined in (4.23), and he derived the marginal posterior pdfs. We employed similar methods for the CD and CL models in (4.1)-(4.20). An alternative approach is to obtain a prior density that provides maximal prior average data information relative to the information in the prior distribution, the maximal data information prior (MDIP) approach. In establishing the prior density, we restricted the parameter space in order to obtain U- or L-shaped average cost curves. For example, the ZR RTS function requires $\theta_1 > 0$ as previously indicated in section 2.2. For other returns to scale functions, the restrictions on the parameters are discussed in section 2.2. See Zellner (1977, 1991, 1993, 1995), Kass and Wasserman (1995) and Soofi (1996) for discussion and applications of the MDIP approach.

5. Application of Models and Methods

We utilize 1957 U.S. annual survey of manufactures state data for the transportation equipment industry employed in calculations of ZR (1969). For each of 25 states ($T = 25$), we have observations of aggregate value added measured in millions of dollars, aggregate capital service flow measured in millions of dollars, and aggregate man-hours worked measured in millions of man-hours. By dividing these observations by the number of establishments, we put variables on a per establishment basis. Across the states, the value added per establishment ranged from 0.193 to 7.18 with mean value 2.87 while the ratios of man-hours worked to capital ranged from 1.27 to 8.32 with mean 3.26. Such large differences in value added and ratios of man-hours worked to capital are useful in estimating the RTS functions and the homogeneous functions.

In Table 2, we report the results of maximum likelihood estimation of production functions' parameters with asymptotic standard errors in parentheses. For ZR-CD model, $\hat{\theta} = 0.107$ and an approximate 95% confidence interval for θ extends from -0.0474 to +0.261 which is rather broad and the usual approximate, asymptotic test of $\theta = 0$, implying a non-U shaped AC, is not very powerful. The estimated parameters $\hat{b}_1 = 0.350$, $\hat{b}_2 = 1.09$ and $\hat{b}_1 + \hat{b}_2 = \hat{\mu} = 1.44$ are reasonable and indicate increasing returns to scale for low output per establishment. In Figure 1a, the RTS function is evaluated using $\hat{\theta} = 0.107$. We have 1.41 for Florida and 0.816 for Michigan. The RTS for Kentucky is estimated to be 1.01. In Figure 2, we report estimates of average cost, 1.68 for Florida, 0.930 for Michigan, and 0.879 for Kentucky. The minimum point of the AC curve coincides with RTS = 1 as shown by the RTS and AC estimates for Kentucky.

In Figure 3, we report the posterior probability density of the ZR-CD parameter θ, $f(\theta|D)$ where D stands for data and prior information, based on a diffuse MDIP density, $\pi(\theta,\beta,\sigma) \propto 1/\sigma$. The shape (solid line) is non-normal, skewed to the right with a long fat tail giving evidence that the sample size is not large enough to produce a normal posterior distribution which would be the shape of this distribution if the sample size were large. The approximate asymptotic normal density, given by the dotted curve in Figure 3, has mean value $\hat{\theta} = 0.107$ and standard deviation 0.0788. From the finite sample posterior density, $\Pr(\theta > 0|D) = 0.998$ is evidence that θ is probably positive. In Figure 5, we report the restricted posterior probability density of θ, $f(\theta|D, \theta>0)$. There is not much difference between Figures 3 and 5. When the sample size is large, the MLE estimated RTS and AC curves can be considered as approximately equal to the posterior mean RTS and AC curves. For the RTS function, $\eta = \mu/(1+\theta y_0)$, the Bayesian result is approximately $f(\eta|D) \approx N[\hat{\mu}_{MLE}/(1+\hat{\theta}_{MLE}\, y_0), \hat{\sigma}^2_\eta]$ where $\hat{\sigma}^2_\eta$ is the approximate posterior variance. However, if desired, the exact posterior pdf for η can be computed. The same holds for the posterior density of the AC function. We interpret results for other models in a similar way.

For the models (4.2)-(4.5) in Table 1, as shown in Table 2, the parameters of the transformations are not precisely estimated due to the small sample of observations ($T = 25$) and/or the large number of parameters. For all cases, we have $\hat{b}_1 + \hat{b}_2 > 1$. For NR-CD, the asymptotic confidence interval for γ is broad. However, the finite sample posterior pdf for γ, shown in Figure 4, yields $\Pr[\gamma > 0|D] = 0.992$. The estimated RTS functions, plotted in Figure 1, are not much different for models (4.1)-(4.5). They have minima in the range of output 4.03 to 5.22. The estimated AC curves, shown in Figure 2, are more or less the same for the models (4.1)-(4.5). For (4.5), the estimate $\hat{b}_0 = 7.90\times10^{10}$ is rather large which is due to $\lim_{\lambda_2 \to 0} y^{\lambda_2}/\lambda_2^2 \to \infty$ in (4.5). We have used an estimated value, $\hat{\lambda}_2 = 0.000001$ to estimate $\hat{b}_0$. As λ_2 approaches zero, the CB RTS function in (2.8) reduces to the RG RTS function in (2.6) and the associated production function (4.3) in Table 1, the RG-CD. As seen from Table 2, the estimates of the RG-CD parameters γ_2 and θ_2 are very imprecise. This implies that it is difficult to distinguish among the RG-CD, NR-CD and the ZR-CD models with our present sample of data.

Table 2: MLE of Parameters for Various RTS Functions with CD Forms for Homogeneous Function (asymptotic standard errors in parentheses)

Equations	$\hat{\theta}_1$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\theta}_2$	$\hat{\nu}_1$	$\hat{\lambda}_1$	$\hat{\theta}_3$	$\hat{\nu}_2$	$\hat{\lambda}_2$	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{\sigma}^2$
(4.1) ZR-CD	0.107	-	-	-	-	-	-	-	-	2.92	0.350	1.09	0.0428
	(0.0788)	-	-	-	-	-	-	-	-	(0.450)	(0.100)	(0.161)	(0.0151)
(4.2) NR-CD	-	0.0807	-	-	-	-	-	-	-	2.49	0.312	0.949	0.0319
	-	(0.0560)	-	-	-	-	-	-	-	(0.171)	(0.0841)	(0.107)	(0.00883)
(4.3) RG-CD	-	-	0.0372	0.0603	-	-	-	-	-	2.74	0.335	1.03	0.0379
	-	-	(0.441)	(0.593)	-	-	-	-	-	(2.31)	(0.223)	(0.822)	(0.0514)
(4.4) BC-CD	-	-	-	-	0.123	0.666	-	-	-	2.81	0.317	0.979	0.0343
	-	-	-	-	(0.366)	(5.17)	-	-	-	(4.99)	(0.0839)	(0.226)	(0.0108)
(4.5) CB-CD[a]	-	-	-	-	-	-	0.0575	0.0790	0.0000	7.90×10^{10}	0.333	1.03	0.0376

[a]Standard errors for the CB-CD model are not presented since they are not defined for $\hat{\lambda}_2 = 0$.

Table 3: MLE of Parameters for Various RTS Functions with CES Forms for Homogeneous Function (asymptotic standard errors in parentheses)

Equations	$\hat{\theta}_1$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\theta}_2$	$\hat{\nu}_1$	$\hat{\lambda}_1$	$\hat{\theta}_3$	$\hat{\nu}_2$	$\hat{\lambda}_2$	$\hat{b}_0$	$\hat{\mu}$	$\hat{\rho}$	$\hat{\delta}$	$\hat{\sigma}^2$
(4.6) ZR-CES	0.0933	-	-	-	-	-	-	-	-	2.77	1.43	0.650	0.127	0.0399
	(0.0766)	-	-	-	-	-	-	-	-	(0.870)	(0.454)	(0.143)	(0.0850)	(0.0243)
(4.7) NR-CES	-	0.0723	-	-	-	-	-	-	-	2.41	1.28	0.809	0.109	0.0305
	-	(0.0544)	-	-	-	-	-	-	-	(0.616)	(0.341)	(0.174)	(0.0829)	(0.0156)
(4.8) RG-CES	-	-	0.0578	0.0200	-	-	-	-	-	2.48	1.31	0.787	0.111	0.0324
	-	-	(0.402)	(0.533)	-	-	-	-	-	(2.11)	(0.992)	(0.171)	(0.0858)	(0.0438)
(4.9) BC-CES	-	-	-	-	0.145	0.0000	-	-	-	8.58×10^{8}	1.28	0.802	0.111	0.0305
(4.10) CB-CES[a]	-	-	-	-	-	-	0.0207	0.115	0.0000	2.88×10^{8}	1.31	0.765	0.114	0.0325

[a]Standard errors for the BC-CES and CB-CES models are not presented

For the models (4.6)-(4.10) in Table 1, we used a CES homogeneous function rather than a CD function, and the estimated parameter values are reported in Table 3. The estimated RTS functions for ZR-CD and ZR-CES are similar. However, sample standard errors are large indicating that estimation is not precise. The location of RTS = 1 is in the range of output 4.5 to 6.5. For (4.9) and (4.10) models, the $\hat{b}_0$ values are quite large due to small estimated values of λ_1 and λ_2.

For models (4.11)-(4.15) of Table 1, we introduced a VES homogeneous function rather than a CD function, and estimated parameter values are reported in Table 3. The point estimates of b_0, b_1, and b_2 look reasonable, but again estimation is imprecise.

For the models (4.16)-(4.20), we introduced a CL homogeneous function rather than a CD function, and the estimated parameter values are reported in Table 5. The numerical algorithm produced the results for (4.11)-(4.15) and (4.16)-(4.20) shown in Tables 4 and 5. Note that the asymptotic standard errors are very large, probably because of the small sample size, and thus it's difficult to reach definitive conclusions. Unfortunately, other estimates in Tables 3 and 4 are subject to the same reservations. More data are needed to reach firmer conclusions.

In Figure 6, two marginal posterior pdfs for θ are presented, one based on the Box-Cox prior density and the other on a diffuse MDIP prior density function. Following Box-Cox's argument, Zellner (1971) derived the marginal posterior pdf for θ, $f(\theta|D) \propto J^{v/T}/(s^2)^{v/2}$. In comparison, Pericchi (1981) introduced a slightly different prior $\pi(\beta,\sigma,\theta) \propto 1/\sigma^4$ while our diffuse prior is $\pi(\beta,\sigma,\theta) \propto 1/\sigma$. The posterior pdf based on our diffuse prior pdf has a lower peak and a fatter right tail compared to the posterior pdf based on the Box-Cox prior as shown in Figure 6.

In Figures 3-6, Bayesian parameter estimation was performed for a CD homogeneous function. Extension of the Bayesian approach to apply to the nonlinear homogeneous functions such as CES and VES has not as yet been done but results of Bayesian estimation for the CL homogeneous function are reported

Table 4: MLE of Parameters for Various RTS Functions with VES Forms for Homogeneous Function
(asymptotic standard errors in parentheses)

	Parameters													
Equations	$\hat{\theta}_1$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\theta}_2$	$\hat{\nu}_1$	$\hat{\lambda}_1$	$\hat{\theta}_3$	$\hat{\nu}_2$	$\hat{\lambda}_2$	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{\rho}$	$\hat{\sigma}^2$
(4.11) ZR-VES	0.0884	-	-	-	-	-	-	-	-	3.16	0.503	0.919	0.592	0.0389
	(0.0742)	-	-	-	-	-	-	-	-	(0.515)	(0.219)	(0.264)	(0.528)	(0.0139)
(4.12) NR-VES	-	0.0686	-	-	-	-	-	-	-	2.86	0.491	0.785	0.501	0.0299
	-	(0.0515)	-	-	-	-	-	-	-	(0.361)	(0.190)	(0.183)	(0.480)	(0.00820)
(4.13) RG-VES	-	-	0.0686	0.0000	-	-	-	-	-	2.86	0.491	0.785	0.501	0.0299
	-	-	(0.383)	(0.508)	-	-	-	-	-	(2.21)	(0.335)	(0.629)	(0.487)	(0.0372)
(4.14) BC-VES[a]	-	-	-	-	0.140	0.0000	-	-	-	1.40×10^{11}	0.490	0.787	0.504	0.0300
(4.15) CB-VES	-	-	-	-	-	-	0.000290	0.140	0.0000	1.40×10^{11}	0.489	0.789	0.508	0.0301

[a]Standard errors for the BC-VES and CB-VES models are not presented.

Table 5: MLE of Parameters for Various RTS Functions with Clemhout (CL) Forms for Homogeneous Function
(asymptotic standard errors in parentheses)

	Parameters													
Equations	$\hat{\theta}_1$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\theta}_2$	$\hat{\nu}_1$	$\hat{\lambda}_1$	$\hat{\theta}_3$	$\hat{\nu}_2$	$\hat{\lambda}_2$	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\hat{\sigma}^2$
(4.16) ZR-CL	0.0844	-	-	-	-	-	-	-	-	-1.60	1.41	7.34	-3.32	0.0381
	(0.0728)	-	-	-	-	-	-	-	-	(2.30)	(0.149)	(3.79)	(1.59)	(0.0136)
(4.17) NR-CL	-	0.0670	-	-	-	-	-	-	-	-1.88	1.27	7.30	-3.29	0.0300
	-	(0.0502)	-	-	-	-	-	-	-	(2.13)	(0.0652)	(3.58)	(1.50)	(0.00825)
(4.18) RG-CL	-	-	0.0670	0.0000	-	-	-	-	-	-1.88	1.27	7.30	-3.29	0.0300
	-	-	(0.354)	(0.477)	-	-	-	-	-	(2.16)	(0.828)	(5.17)	(2.26)	(0.0352)
(4.19) BC-CL[a]	-	-	-	-	0.134	0.0000	-	-	-	3.35×10^{8}	1.27	7.29	-3.29	0.0300
(4.20) CB-CL	-	-	-	-	-	-	0.0000	0.134	0.0000	3.35×10^{8}	1.27	7.29	-3.29	0.0300

[a]Standard errors for the BC-CL and CB-CL models are not presented.

in Figures 7-10. When we chose the CL homogeneous function, Bayesian estimated peak points of θ and γ moved slightly to the left side compared to the CD homogeneous function cases. Besides this, there seems to be no apparent difference between the Figures 3-6 and 7-10, and that the choice of homogeneous function has a minor effect on parameter estimation of the RTS functions.

In Table 6, we report the RTS values for Florida, New Jersey, and Michigan for the 20 models we have chosen. These states are chosen because of their extreme and central output levels. When the functional form of the homogeneous function is general, the choice of RTS function mattered little. For the CL homogeneous function, the RTS functions for NRCL, RGCL, BCCL, and CBCL are more or less the same at all output levels. For the VES homogeneous function, we have results similar to those associated with use of the CL homogeneous function. For the CD and CES homogeneous functions, not enough flexibility is included in the homogeneous functions and different choices of the RTS function produced different RTS values. In particular, the ZR RTS function produced slightly smaller RTS values at all output levels. Since the RGCL and CBCL RTS functions include ZRCL as a special case, and the RTS values of NRCL, RGCL, BCCL, and CBCL were more or less similar at all output levels, we believe that the ZRCL form may lack some flexibility.

The RTS estimates for New Jersey are quite similar across all forms ranging from 1.09 for the NR-CD to 1.15 for the ZR-VES. Florida's RTS varies from 1.40 in the ZR-VES to 1.71 in the NR-CD. The asymptotic standard errors of the estimated RTS, shown in Table 6 are rather large in a number of cases, probably a reflection of our small sample size. However, the point estimates of the RTS for Florida range from 1.39 to 1.72, for New Jersey from 1.09 to 1.15 and for Michigan from .814 to 1.02 indicating a general decline in RTS as output increases.

Table 6: Comparison of RTS and Substitution Elasticity Functions for Various Models

MODELS	Estimated RTS			Estimated Elasticity of Substitution, $\hat{\sigma}$		
	y = 0.193 (Florida)	y = 2.70 (New Jersey)	y = 7.18 (Michigan)	L/K = 1.13 (Michigan)	L/K = 3.29 (Maine)	L/K = 8.32 (Kansas)
ZRCD	1.41 (0.150)	1.11 (0.0809)	0.814 (0.175)	1.0	1.0	1.0
NRCD	1.71 (0.489)	1.09 (0.0947)	0.957 (0.143)	1.0	1.0	1.0
RGCD	1.54 (1.58)[a]	1.10 (0.194)	0.864 (0.752)	1.0	1.0	1.0
BCCD	1.48 (0.769)	1.10 (0.176)	0.863 (0.388)	1.0	1.0	1.0
CBCD	1.55 (***)[b]	1.10 (***)	0.869 (***)	1.0	1.0	1.0
ZRCES	1.40 (0.433)	1.14 (0.283)	0.856 (0.237)	0.606 (0.0525)	0.606 (0.0525)	0.606 (0.0525)
NRCES	1.68 (0.717)	1.12 (0.270)	0.996 (0.247)	0.553 (0.0532)	0.553 (0.0532)	0.553 (0.0532)
RGCES	1.61 (1.85)	1.12 (0.321)	0.955 (0.941)	0.560 (0.0535)	0.560 (0.0535)	0.560 (0.0535)
BCCES	1.72 (***)	1.15 (***)	1.02 (***)	0.555 (***)	0.555 (***)	0.555 (***)
CBCES	1.61 (***)	1.12 (***)	0.952 (***)	0.567 (***)	0.567 (***)	0.567 (***)
ZRVES	1.40 (0.138)	1.15 (0.0885)	0.870 (0.200)	-0.0234 (0.460)	0.649 (0.158)	0.861 (0.0624)
NRVES	1.65 (0.401)	1.12 (0.0983)	1.00 (0.151)	-0.151 (0.424)	0.606 (0.145)	0.844 (0.0575)
RGVES	1.65 (1.78)	1.12 (0.189)	1.00 (1.02)	-0.151 (0.458)	0.606 (0.157)	0.844 (0.0621)
BCVES	1.66 (***)	1.12 (***)	1.00 (***)	-0.147 (***)	0.607 (***)	0.845 (***)
CBVES	1.66 (***)	1.12 (***)	1.00 (***)	-0.141 (***)	0.609 (***)	0.845 (***)
ZRCL	1.39 (0.129)	1.15 (0.0906)	0.878 (0.206)	-0.361 (0.256)	0.897 (0.113)	11.1 (0.196)
NRCL	1.63 (0.377)	1.12 (0.0998)	1.00 (0.152)	-0.523 (0.152)	0.823 (0.206)	11.8 (0.147)
RGCL	1.63 (1.59)	1.12 (0.174)	1.00 (0.984)	-0.523 (0.984)	0.823 (0.582)	11.8 (0.808)
BCCL	1.63 (***)	1.12 (***)	1.00 (***)	-0.508 (***)	0.829 (***)	12.2 (***)
CBCL	1.63 (***)	1.12 (***)	1.00 (***)	-0.508 (***)	0.829 (***)	12.2 (***)

[a] Standard errors of RGCD, RGCES, RGVES, and RGCL are large because standard errors of γ and θ are large for these models as shown in Tables 2-5.

[b] Standard errors for certain models are not presented since they are not defined for $\gamma = 0$. Estimation of asymptotic covariance requires calculation of $\partial z_i(y_i, \xi)/\partial\lambda$. For CBCD, we have a term $\partial[(y^{\lambda_2})/\lambda_2^2 - \log y/\lambda_2]/\partial\lambda_2$ which goes to infinity as $\lambda_2 \to 0$.

To summarize the empirical results, the methods we introduced are quite operational. There is evidence that RTS vary with output and AC functions are U-shaped, but it is hard to differentiate among different models with our small sample. In Figures 3-6, we have provided exact finite sample posterior densities which are quite non-normal, an indication that large sample, approximate, normal results, Bayesian or non-Bayesian are probably inaccurate. From the small estimated values of the λ parameter in the CB RTS function, there is an indication that the NR, RG or ZR RTS functions may be adequate. Also, the results seem to indicate that an elasticity of substitution different from one, the CD case, is required. Finally, our estimated AC curves are L-shaped with a distinct minimum in every case.

6. Application of Models and Methods: Elasticity of Substitution Function

In this section, we estimate the elasticity of substitution at various labor-capital ratios for the 20 models tabulated in Table 1. We have used a CD function for the models (4.1)-(4.5) and a CES homogeneous function for (4.6)-(4.10). The estimated parameter values are reported in Table 3. For the ZR-CES function, the point estimate of ρ is $\hat{\rho} = 0.650$, with a 95% approximate confidence interval, 0.370 to 0.930, indicates that we probably have a CES function not a CD function. The same conclusion holds for the models listed in (4.7)-(4.10).

For the models (4.11)-(4.15), we introduced a VES homogeneous function rather than a CD function, and estimated parameter values are reported in Table 4. We have $\hat{\rho} = 0.592$ for ZR-VES and $\hat{\rho}$ for other models with such large standard errors that it's difficult to reach a conclusion with respect to VES vs. Cobb-Douglas forms. The point estimates of b_0, b_1, and b_2 look reasonable, but again estimation is imprecise. For the models (4.16)-(4.20), with a CL homogeneous function rather than a CD function, estimated parameter values are reported in Table 5. Again, the asymptotic standard errors are very large probably because of the small sample size. Unfortunately other estimates in Tables 3 and 4 are subject to the same reservations. More data are needed to reach firmer conclusions.

We now consider input substitution for the models listed in (4.1)-(4.20) in Table 1. The elasticity of substitution functions are:

1) CD: $\sigma = 1$.

2) CES: $\sigma = 1/(1+\rho)$.

3) VES: Revankar (1971) showed[9] that:

$$\sigma = 1 + (\rho - 1)\left(1 + \frac{b_2}{b_1}\right)\frac{K}{L} = 1 + B\frac{K}{L}.$$

Since B is a constant either positive or negative and L/K is positive, σ will be below one or above one depending on the sign of B.

4) CL:

$$\sigma \equiv \frac{d\log(L/K)}{d\log(f_K/f_L)} = \frac{d\log[\tan\phi]}{d\log[H(\phi)]}$$

where we define

$$\tan\phi = L/K \text{ and } H(\phi) \equiv f_K/f_L \equiv \frac{A}{B}$$

with $f = r^{\mu}\Psi(\phi)$,

$$A = \mu\cos\phi\Psi - \sin\phi\Psi'$$

$$B = \mu\sin\phi\Psi + \cos\phi\Psi'$$

[9]Rewrite Revankar's expression,

$$\sigma = 1 + \frac{\rho - 1}{1 - \delta\rho}\cdot\frac{K}{L}$$

to match our notation of eq. (4.11) by defining $b_0 = \log\gamma$, $b_1 = \mu(1-\delta\rho)$ and $b_2 = \mu\delta\rho$, so that $1 - \delta\rho = b_1/(b_1 + b_2)$.

where $\Psi' = d\Psi/d\phi$. Hence,

$$\sigma = \frac{H}{\cos\phi\sin\phi} \cdot \frac{1}{dH/d\phi}$$

where

$$\frac{dH}{d\phi} = \frac{d}{dH}\left(\frac{A}{B}\right) = \frac{A'B - AB'}{B^2}$$

and

$$A' = -\mu(\sin\phi)\Psi + \mu\cos\phi\Psi' - \cos\phi\Psi' - \sin\phi\Psi''$$

$$B' = \mu(\cos\phi)\Psi + \mu\sin\phi\Psi' - \sin\phi\Psi' + \cos\phi\Psi''$$

For the CL homogeneous functions listed in (4.16)-(4.20), using a logarithmic function, we have $\log\Psi = b_0 + b_2\phi + b_3\phi^2$, $\Psi' = (b_2 + 2b_3)\Psi$, and $\Psi'' = 2b_3\Psi + (b_2 + 2b_2\phi)\Psi'$.

In Table 6, we report the elasticities of substitution for the states with extreme and mean values of the labor-capital ratio, namely Michigan, Maine, and Kansas. When we use the CD homogeneous function, the elasticities are one by definition. For the CES homogenous function, the estimated elasticities ranged between 0.553 for NRCES to 0.606 for ZRCES. For the VES homogeneous function, the elasticities should be either above or below one as indicated above. Therefore the elasticities ranged between -0.151 to 0.844 for both NRVES and RGVES. In Table 4, we noted that RGVES becomes identical to NRVES when $\theta = 0$. However, contrary to standard economic theory, the substitution elasticity for Michigan is negative. This may be due to either the small sample of observations or the homotheticity assumption that we made. For the CL homogeneous function, the elasticities varied from -0.508 to 12.2 for CBCL.

In Figures 11-13, we show the estimated input substitution elasticities when we choose various functional forms, combinations of the ZR, NR, and BC RTS functions and CD, CES, VES, and CL functions. We found that use of the CL homogeneous function produces large differences in the input substitution elasticities, but the CD, CES, and VES forms have very limited variability. Also we note that the choice of a RTS function appears to be not very important for estimation of the input substitution parameter.

7. Summary and Concluding Remarks

We have estimated returns to scale functions and average cost curves using the generalized production function approach without assuming fixed scale elasticities or fixed substitution elasticities. The functional forms can be either parametric forms or flexible semiparametric forms. For the parametric form approach, to generalize the returns to scale function, we introduced a Box-Cox type function and derived Zellner and Revankar, Nerlove and Ringstad, and Ringstad's returns to scale functions as special cases of the Box-Cox type returns to scale function. To generalize the homogeneous function, we introduced a polar coordinate representation, defined the substitution elasticity as a function of an angular variable, and derived Cobb-Douglas, CES, and variable elasticity of substitution functions. For a flexible functional form approach, we introduced a sequence of polynomials for the reciprocal of the returns to scale function and also for the angular part of the homogeneous function. We called the polynomial expansion of the homogeneous function a Clemhout type function and showed how to impose convexity on the isoquants with respect to the origin. We recognize that various model selection procedures can be employed to help choose among alternative models and/or combine them.

Extension to many input generalized production functions is planned in future work. The choice of a returns to scale function does not cause any additional complications but imposition of the concavity restriction on the isoquants of the homogeneous function will be difficult. We have introduced a polar coordinate representation to separate a homogeneous function into a radial part and an angular part, and explained that the determinant of Hessian matrix of a homogeneous function of degree one is zero and thus the burden of imposing the concavity restriction is lessened in this formulation. We expect that the decrease in dimension of the homogeneous function will provide some simple rules for imposing the concavity restriction in a many input homogeneous function.

References

Anderson, R. G. and J. G. Thursby (1986), "Confidence Intervals for Elasticity Estimators in Translog Models," *Review of Economics and Statistics*, November, 647-656.

Avishur, A. (1994), "The Effect of the Privatization of British Telecom on Efficiency: A Generalized Production Function Approach," mimeo.

Barnett, W., J. Geweke, and P. Yue (1989), "Seminonparametric Bayesian Estimation of the Asymptotically Ideal Model: The AIM Consumer Demand System," in William Barnett, James Powell, and George Tauchen (eds.), *Nonparametric and Semiparametric Methods in Econometrics and Statistics, Proceedings of the Fifth International Symposium in Economic Theory and Econometrics* (Cambridge University Press, Cambridge).

Berndt, E., B. Hall, R. Hall, and J. Hausman (1974), "Estimation and Inference in Nonlinear Structural Models," *Annals of Economic and Social Measurement*, 3/4, 653-665.

Calem, P. (1990), "On Estimating Technical Progress and Returns to Scale," in R. Sato (ed.), *Conservation Laws and Symmetry* (Kluwer Academic Publishers, Boston, MA).

Chen, C. (1985), "On Asymptotic Normality of Limiting Density Functions with Bayesian Implications," *Journal of the Royal Statistical Society*, B, 47, 540-546.

Christensen, L. and W. Greene (1976), "Economics of Scale in U.S. Electric Power Generation," *Journal of Political Economy*, 84, 655-676.

Clemhout, S. (1968), "The Class of Homothetic Isoquant Production Functions," *Review of Economic Studies*, 35, 91-104.

Färe, R., L. Jansson, and C. Lovell (1985), "Modeling Scale Economies with Ray-Homothetic Production Functions," *The Review of Economics and Statistics*, 624-629.

Frisch, R. (1965), *Theory of Production* (Reidel, Dordrecht, Holland).

Fuss, M., D. McFadden, and Y. Mundlak (1978), *A Survey of Functional Forms in the Economic Analysis of Production* (North-Holland Publishing Co., Amsterdam).

Goldberger, A. (1991), *A Course in Econometrics* (Harvard University Press, Cambridge, MA).

Greene, W. (1991), *Econometric Analysis* (Macmillan, New York, NY).

Hanoch, G. (1975), "The Elasticity of Scale and the Shape of Average Costs," *American Economic Review*, 65, 492-497.

Hartigan, J. (1983), *Bayes Theory* (Springer-Verlag, New York, NY).

Heyde, C. and I. Johnstone (1979), "On the Asymptotic Posterior Normality for Stochastic Processes," *Journal of the Royal Statistical Society*, B, 41, 184-189.

Jeffreys, H. (1967), *Theory of Probability*, third ed. (Oxford University Press, London).

Kass, R.E. and L. Wasserman (1995), "The Selection of Prior Distributions by Formal Rules," Tech. Report #583, Dept. of Statistics, Carnegie Mellon University, published in *Journal of the American Statistical Association*, 91 (1996), 1343-1370..

Kumbhakar, S., S. Ghosh, and J. McGuckin (1991), "A Generalized Production Frontier Approach for Estimating Determinants of Inefficiency in U.S. Dairy Farms," *Journal of Business and Economic Statistics*, 9, 279-286.

McElroy, R. (1969), "Returns to Scale, Euler's Theorem and the Form of Production Functions," *Econometrica*, 37, 275-279.

Nerlove, M. (1963), "Returns to Scale in Electricity Supply," in C. Christ, et al., *Measurement in Econometrics: Studies in Mathematical Economics and Econometrics in Memory of Yehuda Grundfeld* (Stanford University Press, Stanford).

Pericchi, L. (1981), "A Bayesian Approach to Transformations to Normality," *Biometrika*, 68, 35-43.

Revankar, N. (1967), "Production Functions with Variable Elasticity of Substitution and Variable Returns to Scale," unpublished PhD dissertation (The University of Wisconsin, Madison, WI).

Revankar, N. (1971), "A Class of Variable Elasticity of Substitution Production Functions," *Econometrica*, 39, 61-71.

Ringstad, V. (1967), "Econometric Analyses Based on a Production Function with Neutrally Variable Scale-Elasticity," *Swedish Journal of Economics*, 115-133.

Ringstad, V. (1974), "Some Empirical Evidence on the Decreasing Scale," *Econometrica*, 42, 87-101.

Sato, R. (1980), "The Impact of Technical Change on the Holotheticity of Production Functions," *Review of Economic Studies*, 47, 767-776.

Sandler, T. and A. Swimmer (1978), "The Properties of Generation of Homothetic Production Functions: A Synthesis," *Journal of Economic Theory*, 18, 349-361.

Shephard, R. (1973), "Semi-Homogeneous Production Functions and Scaling of Production," in M. Beckmann (ed.), *Lecture Notes in Economics and Mathematical Systems*, Vol. 99 (Springer-Verlag, Berlin).

Soofi, E.S. (1994), "Capturing the Intangible Concept of Information," *Journal of the American Statistical Association*, 89, No. 428, 1243-1254.

Soofi, E. S. (1996), "Information Theory and Bayesian Statistics," in D. B. Berry, K. M. Chaloner and J. F. Geweke (eds.), *Bayesian Analysis in Statistics and Econometrics: Essays in Honor of Arnold Zellner* (John Wiley & Sons, Inc., New York), 179-189.

Tybout, J. (1992), "Making Noisy Data Sing: Estimating Production Technologies in Developing Countries," *Journal of Econometrics*, 53, 25-44.

Zellner, A. (1971), *An Introduction to Bayesian Inference in Econometrics* (John Wiley & Sons, New York, NY), reprinted in Wiley Classics Library, 1996.

Zellner, A. (1977), "Maximal Data Information Prior Distributions," in A. Aykac and C. Brumat (eds.), *New Developments in the Applications of Bayesian Methods* (North-Holland Publishing Co., Amsterdam), 211-232 and reprinted in A. Zellner (1984).

Zellner, A. (1984), *Basic Issues in Econometrics* (University of Chicago Press, Chicago).

Zellner, A. (1991), "Bayesian Methods and Entropy in Economics and Econometrics," in W. Grandy, Jr. and L. Schick (eds.), *Maximum Entropy and Bayesian Methods* (Kluwer Academic Publishers, Boston, MA).

Zellner, A. (1993), "Models, Prior Information and Bayesian Analysis," published in *Journal of Econometrics Annals* issue, *Bayes, Bernoullis and Basel: Proceedings of the First Multi-National Conference on Bayesian Econometrics and Statistics*, edited by L. Bauwens, W. Polasek and H.K. van Dijk, 75 (1996), 51-68.

Zellner, A. (1995), "Past and Recent Results on Maximal Data Information Priors," ms., H.G.B. Alexander Research Foundation, University of Chicago, published in Zellner (1997).

Zellner, A., J. Drèze, and J. Kmenta (1966), "Specification and Estimation of Cobb-Douglas Production Models," *Econometrica*, 34, 784-795.

Zellner, A. and N. Revankar (1969), "Generalized Production Functions," *The Review of Economic Studies*, 36, 241-250.

Zellner, A. (1997), Bayesian Analysis in Econometrics and Statistics: The Zellner View and Papers (Edward Elgar Publ. Co., Cheltenham, U.K.,

Fig. 1a : RTS for ZRCD[a]

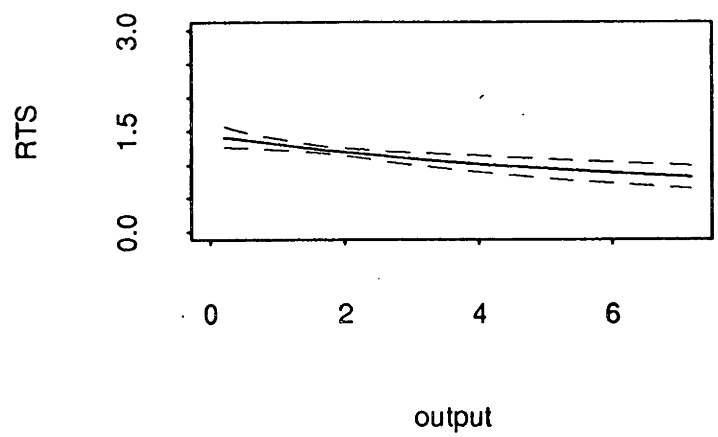


Fig. 1b : RTS for NRCD[a]

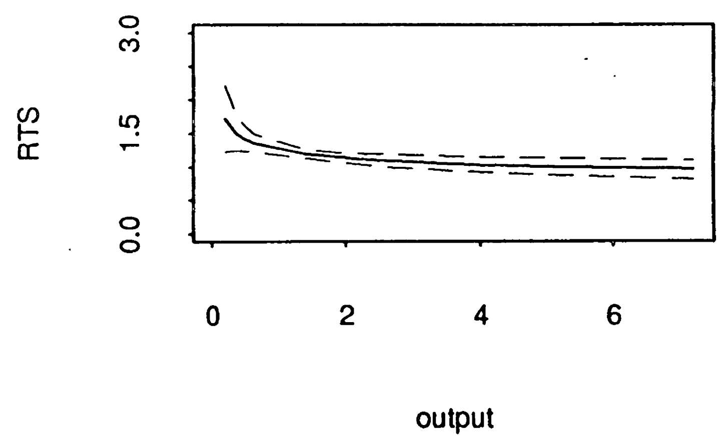


Fig. 1c : RTS for RGCD[a]

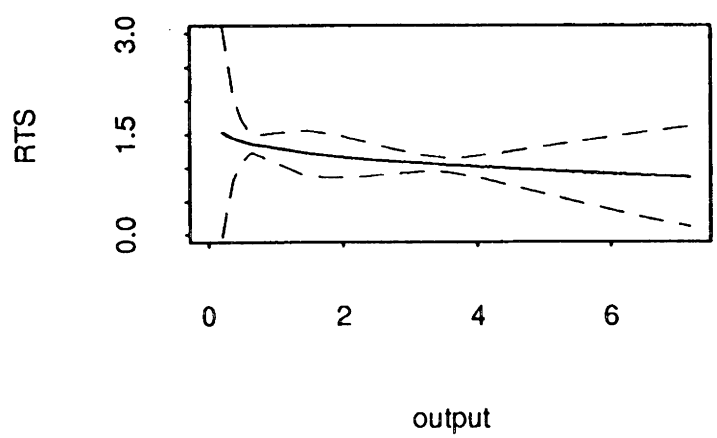
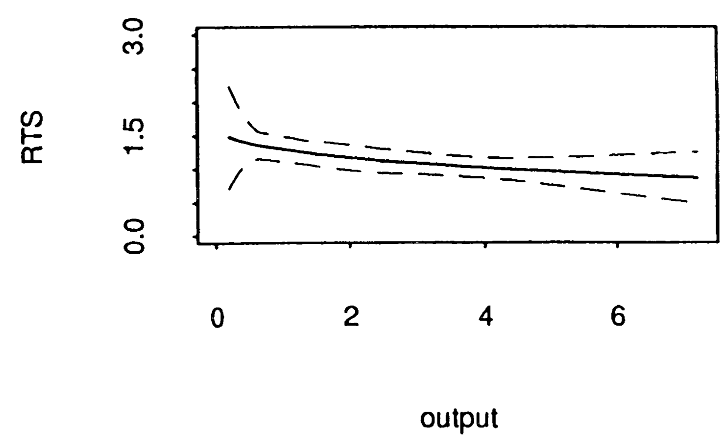


Fig. 1d : RTS for BCCD[a]



a. Solid line for RTS and dashed lines for ± one standard error.

Fig. 1e Returns to Scale Functions[a]

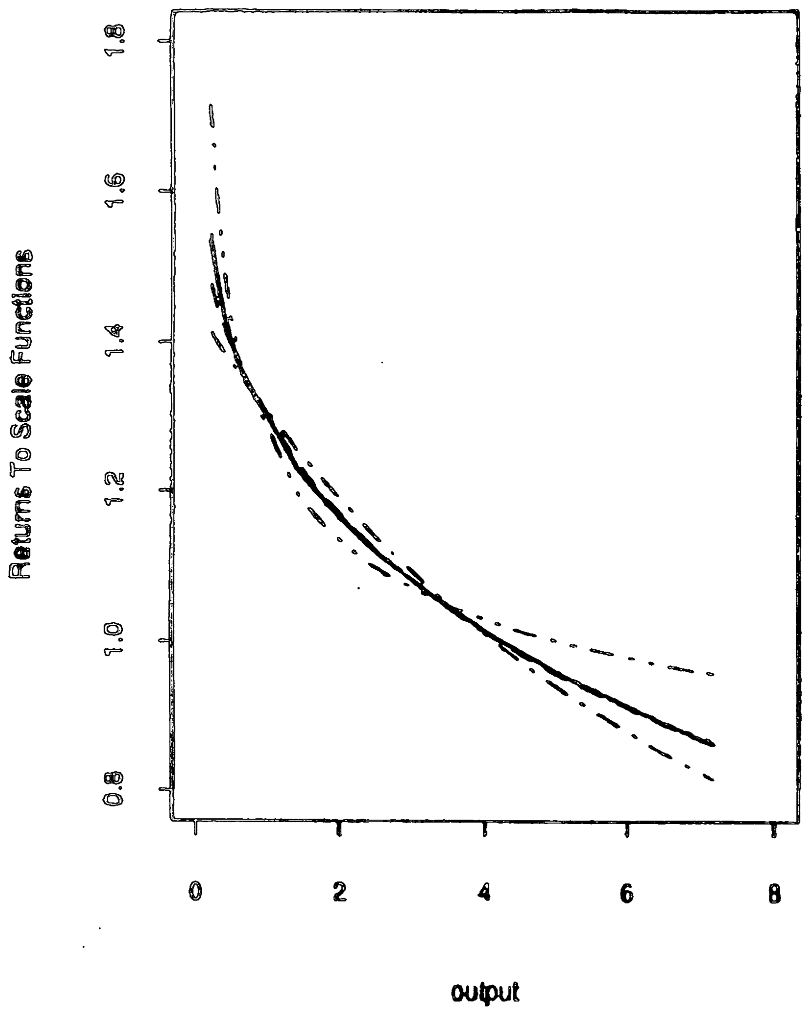
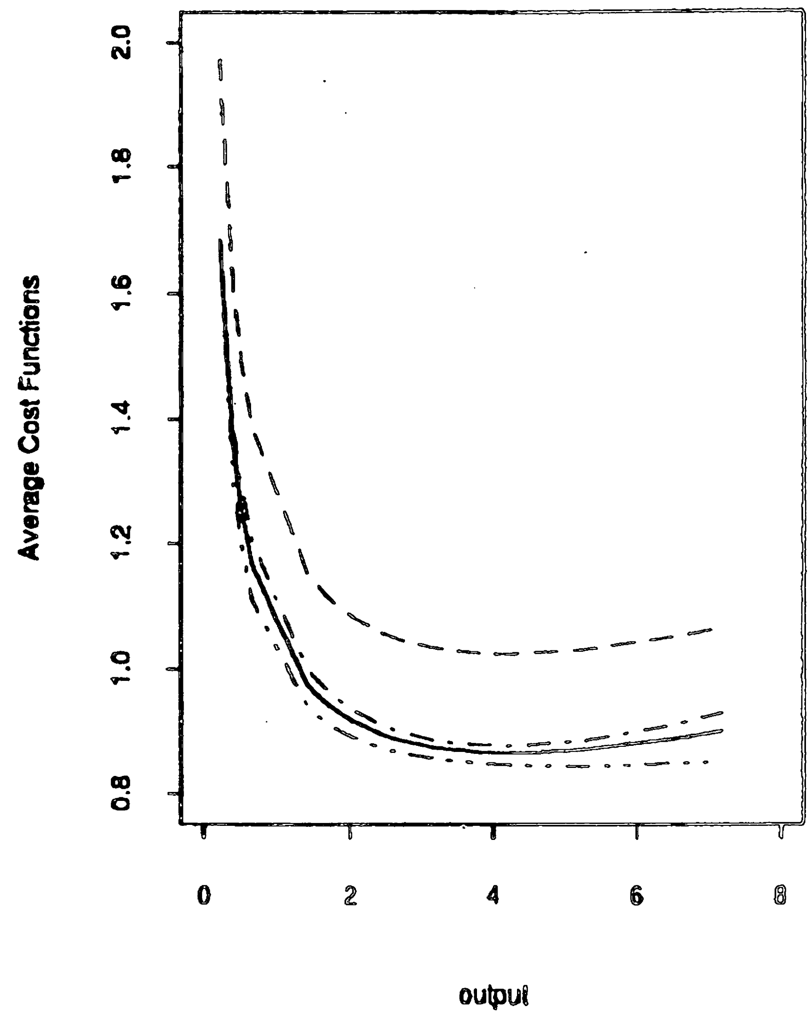


Fig. 2 Average Cost Functions



[a] Functions estimated are listed in Table 1. Plotting symbols are — · — · — for ZRCD, — ··· — ··· — for NRCD, —— for RGCD, – – – – for BCCD, and ······ for CBCD models. The RTS functions for RGCD (——) and for CBCD (······) are so closely located that they are hardly distinguishable. In Fig. 2, we use the same plotting symbols for the 5 models.

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Fig. 3 Posterior Densities of theta for ZR-CD[a]

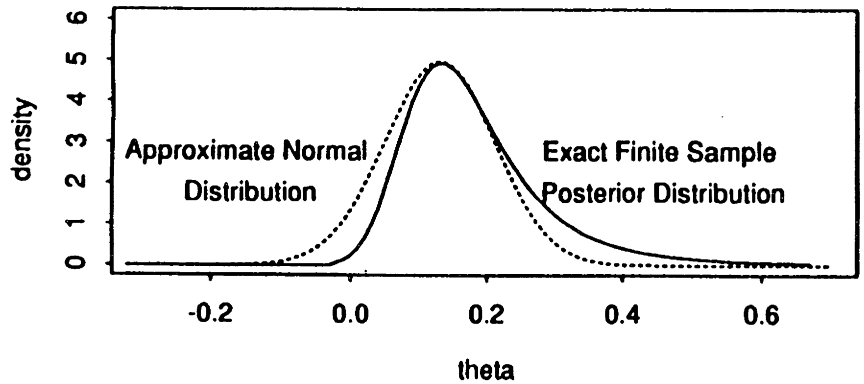


Fig. 4 Posterior Density of gamma for NR-CD[b]

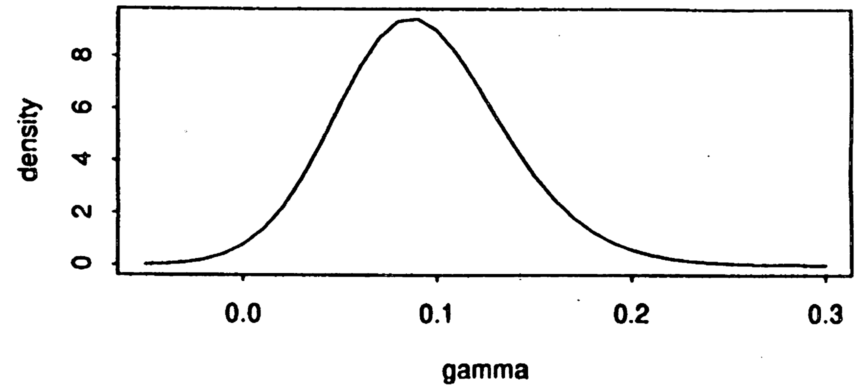


Fig. 5 Posterior Density of theta for ZR-CD
(inequality constrained pdf)

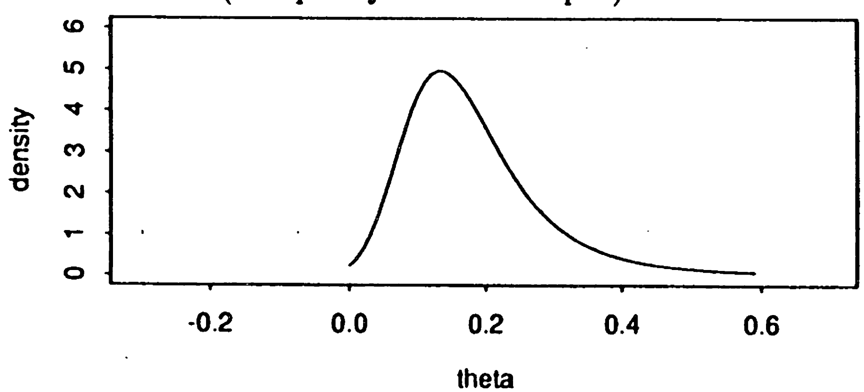
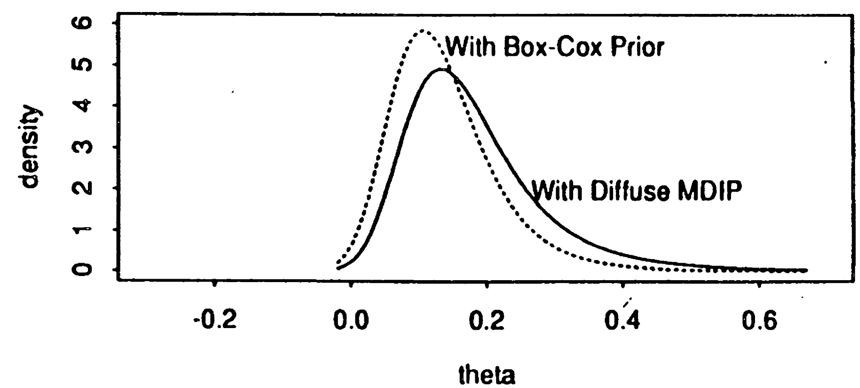


Fig. 6 Posterior Densities of theta in ZR-CD Based on Box-Cox Prior and Diffuse MDIP[c]



[a] The exact posterior density is based on a diffuse maximal data information prior (MDIP).
[b] The functional forms for ZR-CD and NR-CD models are as follows. See section 4.1.

Zellner-Revankar's (ZR) RTS with CD : $\log y + \theta_1 y = b_0 + b_1 \log K + b_2 \log L + u_1$ (4.1)

Nerlove-Ringstad's (NR) RTS with CD : $\log y + \gamma_1 (\log y)^2 = b_0 + b_1 \log K + b_2 \log L + u_2$ (4.2)

Fig. 7 Posterior Densities of theta for ZR-CL[a]

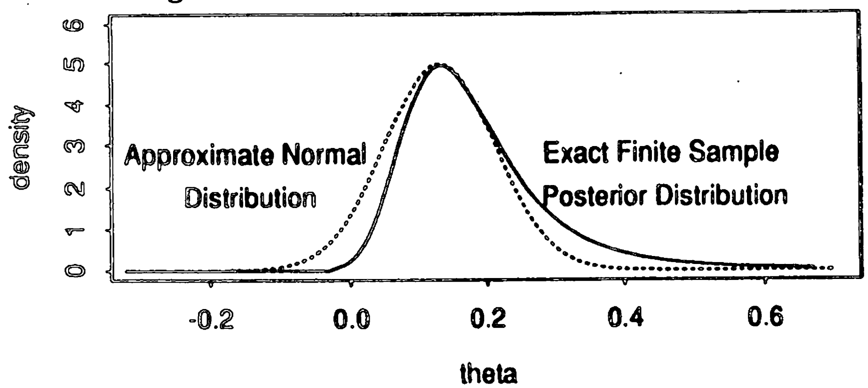


Fig. 8 Posterior Density of gamma for NR-CL[b]

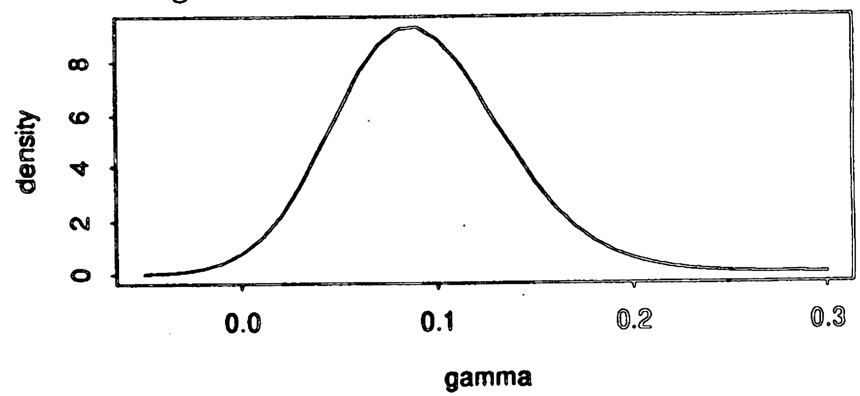


Fig. 9 Posterior Density of theta for ZR-CL
(inequality constrained pdf)

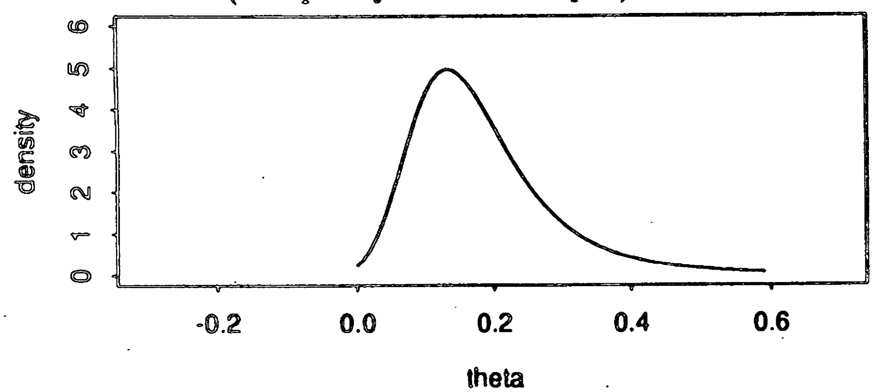
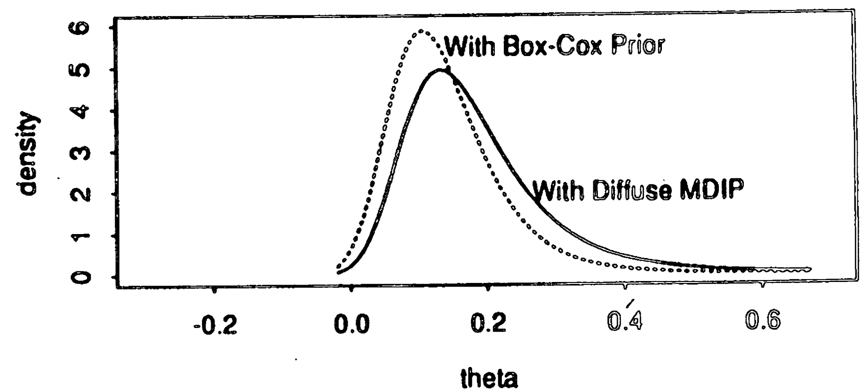


Fig. 10 Posterior Densities of theta in ZR-CL Based on Box-Cox Prior and Diffuse MDIP[c]



[a] The exact posterior density is based on a diffuse maximal data information prior (MDIP).
[b] The functional forms for ZR-CL and NR-CL models are as follows. See section 4.1.

$$\text{Zellner-Revankar's (ZR) RTS with CL} \quad : \quad \log y + \theta_1 y = b_0 + b_1 \log K + b_2 \log L + u_1 \qquad (4.1)$$

$$\text{Nerlove-Ringstad's (NR) RTS with CL} \quad : \quad \log y + \gamma_1 (\log y)^2 = b_0 + b_1 \log K + b_2 \log L + u_2 \qquad (4.2)$$

[c] See Section 5 for explicit forms for the Box-Cox and maximal data information prior (MDIP).

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Fig. 11 Sub. Elas. with ZR RTS [a]

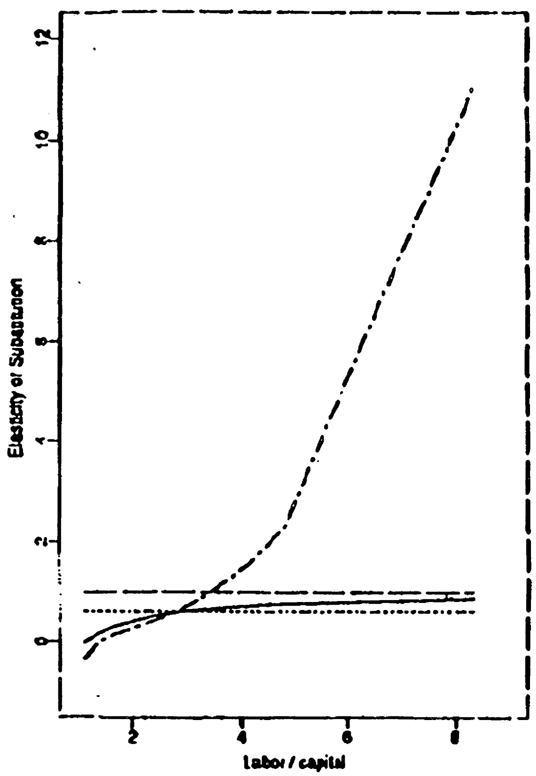


Fig. 12 Sub. Elas. with NR RTS

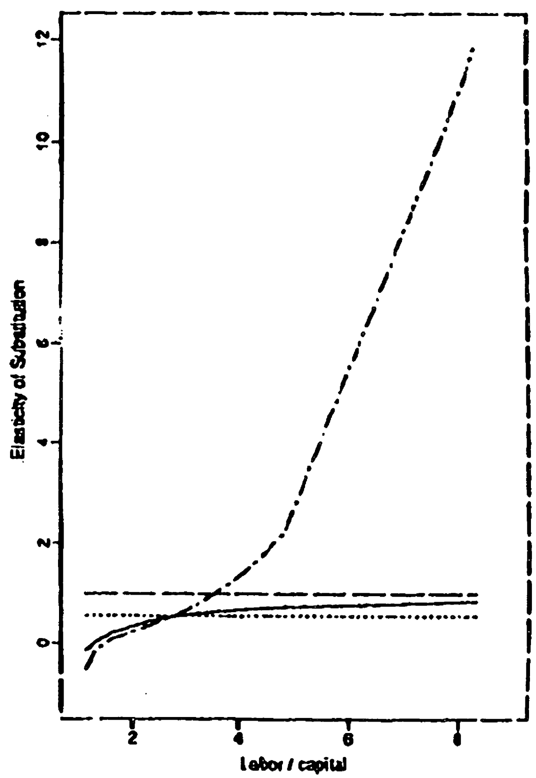
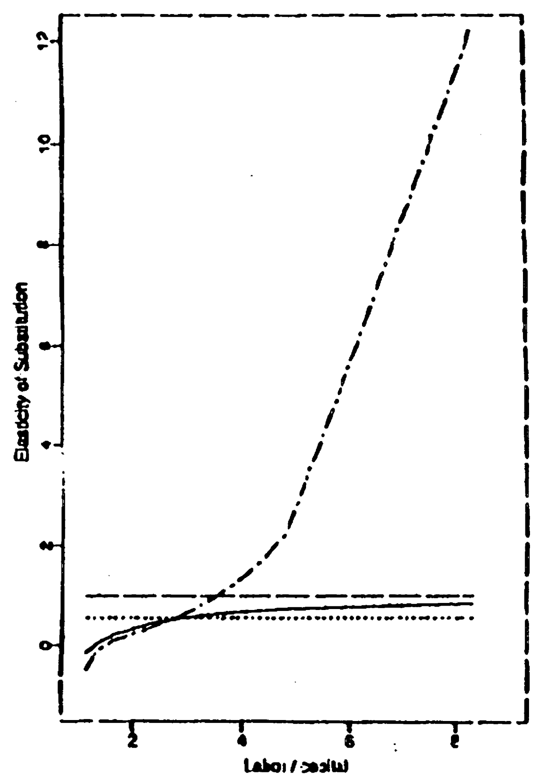


Fig. 13 Sub. Elas. with BC RTS



[a] Estimation functions are listed on P.25. Plotting symbols are — · — · — for ZRCL, —— for ZRVES, – – – – for ZRCD, and ······ for ZRCES models. In Fig. 12 and 13, we use the NR and BC RTS functions rather than ZR RTS function.