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ERSION TO INCOME RISK IN THE PRESENCE OF MULTIVARIATE

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Abstract

We define a risk premium that captures aversion to income risk in the presence of other random attributes in the utility function and use it to construct a new matrix measure of aversion to income risk. Unlike univariate or other multivariate measures, the new measure permits the analysis of decisions that involve a subset of all risks faced by an individual. The restrictions on preferences needed for comparative risk aversion are identified and provide new insight into the relationship between risk attitudes and ordinal preferences.

Aversion to Income Risk in the Presence of Multivariate Risk

1. Introduction

Economic agents often must make decisions under risk that concern a subset of arguments of the objective function, in the presence of unresolved risks in others. Would consumers fully insure when offered actuarially fair unemployment or reimbursement health insurance? How does uncertainty about the relative prices of goods affect the portfolio choices of investors? Does the law of supply hold under price risk, in a peasant economy where farm households consume a portion of their product? Neither the Arrow (1965) and Pratt (1964) characterization of risk aversion nor existing generalizations provides a suitable framework for such analyses. For example, since health itself is uninsurable, a consumer with a concave utility function may prefer to be less than fully insured, even when offered actuarially fair reimbursement health insurance.[1] Such problems require a characterization of aversion to one risk in the presence of others, which is the objective of this paper.

The results can thus be placed in the context of recent studies that have generalized the Arrow-Pratt theory of risk aversion. That framework is not suitable for such problems because they violate three of its fundamental assumptions—utility must be a function of only one argument, it must not depend directly on the state of nature, and insurance schemes must be complete, in that they eliminate *all* of the risks affecting utility. This paper finds the restrictions on either preferences or risks necessary to relax those assumptions.

Several papers (*e.g.* Stiglitz (1969), Kihlstrom and Mirman (1974), Karni (1979)) have examined the case of multivariate risk—randomness in several arguments of the utility function. The risk premium was defined as the willingness to pay to avoid all risks simultaneously.[2] A second body of work has examined risk aversion with preferences that are state-dependent (*e.g.* Karni (1985)). The risk premium is then defined as the amount an individual would pay to replace his random income with a position on his reference set, an optimal allocation of a given actuarial wealth across states of nature.

These definitions are most appropriate for situations of optimal or complete insurance. However, transactions costs, moral hazard, and adverse selection are reasons why the menu of insurance schemes is usually much more limited. Often, it is possible to stabilize income only, while other random attributes of the objective function, interpretable as states of nature, remain random. For instance, insurance against relative price risks is usually unavailable.

We extend the results in these two areas of the literature on risk aversion to the case where decisions affect only a subset of risks. The key is to define aversion to income risk in the presence of background risks in other attributes. Hence, the paper can also be viewed as a generalization of studies of risk aversion in the presence of background income risks (*e.g.* Ross (1981), Kihlstrom, Romer, and Williams (1981), Jewitt (1987)). This point of view was originated by Pratt (1988).

New insights are also provided into the relationship between attitudes toward multivariate risks and ordinal preferences. In most previous studies (*e.g.* Kihlstrom and Mirman (1974)), comparisons of attitudes toward multivariate risks were limited to the case of identical ordinal preferences. These studies also showed that identical ordinal preferences are sufficient for the Arrow-Pratt result that more concavity implies a higher risk premium.[3] This study shows that if interest is in attitudes toward a subset of risks, identity of ordinal preferences excludes interpersonal comparisons, in the sense that no characterization of "greater risk aversion" exists for arbitrary risks. On the other hand, utility functions with distinct ordinal preferences can have identical risk attitudes. Hence, some separability emerges between attitudes toward risks and the underlying preferences for commodities and/or attributes.

2. Multivariate Risk Premia: Definitions and Properties

The individual's objective function is assumed to be a von Neumann-Morgenstern utility function, $U(y, c)$, defined over income (or final wealth) y and a vector of uncertain attributes, c, of length N. A familiar example is the consumer's indirect utility function, where the vector c denotes the prices of goods p, but c could also include health or other attributes affecting utility. U is assumed to be twice continuously differentiable, homogeneous of degree zero in monetary arguments, and increasing in y, everywhere in its domain.

Risk enters U through an $N+1$-dimensional random vector z, with symmetric covariance matrix Ω and i^{th} element, z_i, defined as the deviation of the i^{th} argument of U from its mean. z is partitioned to distinguish income risk from other risks as $z=(z_y, z_c)'$. The probability distributions of z, z_y, and z_c are denoted by $\tilde{z}$, $\tilde{z}_y$, and $\tilde{z}_c$, respectively.[4]

To avoid the difficulty that arises with comparisons of individuals with distinct ordinal preferences, we follow Karni (1979) to restrict comparisons to a particular direction in commodity space, defining the risk premium in terms of the cheapest of all bundles of commodities providing the consumer with a given level of utility. The cost of the bundle is evaluated at the mean of the price vector.[5]

The premium for insuring against all risks, $\pi(y, c, \tilde{z})$, was defined by Karni (1979), for the case where $c \equiv p$, as

$$EU(y+z_y, c_1+z_1, \ldots, c_N+z_N) = U(y-\pi, c_1, \ldots, c_N).$$

π is the maximum amount that an individual is willing to pay to avoid the joint income and attribute risk, $\tilde{z}$, stabilizing utility at $U(y-\pi, c)$. The properties of U, together with the assumption that all of the expectations are finite, are sufficient to ensure both existence and uniqueness of π and the other risk premia defined below, on any finite closed interval. To develop measures of aversion to any particular subset of risks, it is useful to decompose π into several risk premia, each measuring the willingness to pay for partial insurance.

Pratt's (1964) risk premium, $\pi^{y|c}$, measures the maximum amount that an individual would pay to avoid income risk when nothing else is random. It is defined by

$$EU(y+z_y, c) = U(y-\pi^{y|c}, c).$$

A multivariate income risk premium, π^y, may be defined analogously, as the maximum amount that the individual would pay to avoid income risk when attributes c remain random:

$$EU(y+z_y, c+z_c) = EU(y-\pi^y, c+z_c).$$

In a similar manner, we denote by $\pi^{c|y}$ the maximum amount an individual would pay to stabilize the attributes in c with income certain:

$$EU(y, c + z_c) = U(y - \pi^{c|y}, c).$$

Finally, we define π^{ind} by the following equation:

$$E_{\tilde{z}}[U(y + z_y, c + z_c)] = E_{\tilde{z}_y}\{E_{\tilde{z}_c}[U(y + z_y - \pi^{ind}, c + z_c)]\}.$$

The expectation on the left hand side is taken with respect to the joint distribution of income and attributes, while on the right hand side it is taken with respect to the product of their marginal distributions. π^{ind} is thus interpreted as the maximum amount that the individual would be willing to pay to eliminate the stochastic dependence between z_y and z_c.

Intuition to accompany these definitions is provided by the approximate expressions in the following theorem, which does not hold for large risks without modification.

Theorem 1: Assuming small risks and using the above definitions,

$$(i)\ \ \pi = \pi^y + \pi^{c|y}$$

$$(ii)\ \ \pi^y = \pi^{ind} + \pi^{y|c}$$

Proof: Ignoring terms that are $o[tr(\Omega)]$, a second-order Taylor series expansion at the point (y, c) of each side of the equation that defines π yields

$$\pi = -\frac{1}{2}\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\sigma_{ij}\frac{U_{ij}}{U_1}$$

$$= -\frac{1}{2}\sigma_{11}\frac{U_{11}}{U_1} - \sum_{i=2}^{N+1}\sigma_{i1}\frac{U_{i1}}{U_1} - \frac{1}{2}\sum_{i=2}^{N+1}\sum_{j=2}^{N+1}\sigma_{ij}\frac{U_{ij}}{U_1}$$

where subscripts on U denote partial derivatives. By similar approximations, $\pi^{y|c}$ equals the leftmost term in the right hand side of the above equation, π^{ind} equals the middle term, and $\pi^{c|y}$ equals the rightmost term. Finally, π^y equals the sum of the first two terms.[6] □

When individuals must make decisions concerning income while risks in other attributes are unavoidable, only the attitude toward income risk captured by π^y governs behavior.

Neither π nor the usual Pratt (1964) risk premium, $\pi^{y|c}$, are informative. By Theorem 1, π^y consists of $\pi^{y|c}$ plus the premium for eliminating the stochastic interaction of attributes and income. If the individual gains enough from the covariation of income and attributes, π^y and π^{ind} could both be negative, even if U is concave in y for fixed c (*i.e.* $\pi^{y|c} \geq 0$). Similarly, π consists of π^y plus the premium for eliminating attributes risk when income is fixed. The latter is of no relevance for choices that affect the income risk alone. Furthermore, concavity of U implies concavity of U in c for any value of y, and hence implies that both π and $\pi^{c|y}$ will be non-negative. However, since $\pi^{c|y}$ may be larger than π, concavity of U is neither necessary nor sufficient for $\pi^y \geq 0$. This motivates a characterization of aversion to income risk corresponding to π^y.

A measure of risk aversion corresponding to π was defined by Karni (1979) as the $N+1$ by $N+1$ matrix R with typical element $-U_{ij}/U_1$. The approximate expression for π^y suggests an alternative measure of local aversion to income risk, which we denote H. H is a symmetric $N+1$ by $N+1$ matrix with $i,j^{\underline{th}}$ element given by

$$h^{ij} = \begin{cases} -U_{ij}(y,c)/U_1(y,c) & \text{where } i=1 \text{ and/or } j=1 \\ 0 & \text{where } i \neq 1 \text{ and } j \neq 1 \end{cases}$$

Thus, H is a zero matrix except for elements in row 1 and in column 1. The first diagonal element, h^{11}, is proportional to the risk premium per unit of income variance (σ_{11}), while off-diagonal elements of the type h^{i1} are proportional to the risk premium per unit of covariance between the risk in the $i^{\underline{th}}$ attribute and the income risk, thereby capturing local aversion to the stochastic interaction between income and attribute risks. It is worth noting that $\pi^y = \pi$ and $H = R$ for all risks if and only if the consumer is indifferent between stable and unstable attributes, implying that U is linear in the elements of c.

3. Comparative Aversion to Income Risk

The measure H provides a partial ordering of utility functions according to aversion to income risk, as captured by π^y. Theorem 2 establishes that, for small risks, the relationship between the risk premia of two individuals is summarized by the sign-definiteness of the difference between their respective measures of local aversion to income risk. Theorem 3 demonstrates that the analogous characterization for large risks is available, but implies rather severe limitations on preferences. The conditions for a non-negative risk premium and a risk premium that decreases with wealth then follow as immediate corollaries.

3.1. Comparisons Between a Pair of Utility Functions

Definition: Let U^1 and U^2 be utility functions with a common domain, D. U^1 is said to be at least as averse to income risk as U^2 if and only if $\pi^y{}_1(y,c,\tilde{z}) \geq \pi^y{}_2(y,c,\tilde{z})$ for all $(y+z_y, c+z_c)$ in D and all $\tilde{z}$.

Theorem 2: Let $\pi^y{}_1$, $\pi^y{}_2$, H_1, and H_2 be the income risk premia and the risk aversion measures corresponding to utility functions U^1 and U^2 with a common domain D. $\pi^y{}_1(y^0,c^0,\tilde{z}) \geq \pi^y{}_2(y^0,c^0,\tilde{z})$ for all small risks in the neighborhood of (y^0,c^0) if and only if $H_1(y^0,c^0) - H_2(y^0,c^0)$ is positive semi-definite (PSD).[7]

Proof: The difference between two risk premia can be written as

$$\pi^y{}_1(y^0,c^0,\tilde{z}) - \pi^y{}_2(y^0,c^0,\tilde{z}) \;=\; \frac{1}{2}\, tr\,\{\Omega\,[H_1(y^0,c^0) - H_2(y^0,c^0)]\}.$$

Sufficiency follows from the fact that Ω is PSD. To prove necessity, choose the joint distribution of income and attributes such that the absolute values of all the correlation coefficients equal 1. Under this assumption

$$\pi^y{}_1 - \pi^y{}_2 \geq 0 \Rightarrow tr\,[\Omega(H_1 - H_2)] = tr\,[vv'(H_1 - H_2)] = tr\,[v'(H_1 - H_2)v] = v'(H_1 - H_2)v \geq 0,$$

where v is the vector of standard deviations, premultiplied by either 1 or -1. Since elements in v can take any values, the above inequality establishes that $H_1 - H_2$ must be PSD. □

If one agent is locally at least as averse to income risk as another at every point of the domain, then by Theorem 3 he is also globally at least as averse to income risk.

Theorem 3: Let $\pi^y{}_1$, $\pi^y{}_2$, H_1, and H_2 be the income risk premia and the risk aversion measures corresponding to utility functions U^1 and U^2 with a common domain D. The following statements are equivalent:

(i) $\pi^y{}_1(y,c,\tilde{z}) \geq \pi^y{}_2(y,c,\tilde{z})$ for all $(y+z_y, c+z_c)$ in D and all $\tilde{z}$;

(ii) $H_1(y,c) - H_2(y,c)$ is PSD for all (y,c) in D;

(iii) $U^1(y,c) = a^1(c) + b(c)e^1(y)$ and $U^2(y,c) = a^2(c) + b(c)e^2(y)$,

where $e^1 = h(e^2)$ and h is concave.

Proof: By Theorem 2, (i) implies (ii). To prove that (ii) implies (iii), note that

$$x'(H_1 - H_2)x \ = \ x_1 \, [x_1 (h^{11}{}_1 - h^{11}{}_2) + 2 \sum_{i=2}^{N+1} x_i (h^{1i}{}_1 - h^{1i}{}_2)].$$

Hence, for each x_i and $(h^{1i}{}_1 - h^{1i}{}_2)$, $i = 2,\ldots,N+1$, $x'(H_1 - H_2)x = 0$ is a quadratic equation in x_1 that has two distinct real solutions:

$$x_1 = \frac{-2 \sum_{i=2}^{N+1} x_i \, (h^{1i}{}_1 - h^{1i}{}_2)}{(h^{11}{}_1 - h^{11}{}_2)} \qquad \text{or} \qquad x_1 = 0.$$

Thus, a suitable choice of x_1 ensures that $x'(H_1 - H_2)x$ could take positive, negative, or zero values, implying that $H_1 - H_2$ is an indefinite matrix. Indefiniteness is ruled out only when $x_1 = 0$ is a unique solution to $x'(H_1 - H_2)x = 0$. Thus, $H_1 - H_2$ is globally PSD if and only if

$$(1) \quad h^{11}{}_1 \geq h^{11}{}_2 \qquad \text{and} \qquad (2) \quad h^{1i}{}_1 = h^{1i}{}_2 \quad \text{for } i = 2,\ldots,N+1.$$

Integrating (2) with respect to c_j yields

$$log\,[U^1{}_y(y,c)] = log\,[U^2{}_y(y,c)] + log\,[\,\Phi(y, c_{i \neq j})] \quad \text{for each } j = 2,\ldots,N+1,$$

where $c_{i \neq j}$ denotes the vector c with c_j excluded and Φ denotes an arbitrary function. These N equations are equivalent to

$$U^1{}_y(y,c) = U^2{}_y(y,c)\,\Phi(y, c_{i \neq j}) \quad \text{for } j = 2,\ldots,N+1.$$

Since the ratio of marginal utilities of income does not depend on c_j (for every j), the utility functions must be of the following forms:[8]

$$U^1(y,c) = a^1(c) + b(c)e^1(y) \ \text{ and } \ U^2(y,c) = a^2(c) + b(c)e^2(y).$$

From (1), it follows that $e^1=h(e^2)$ is a concave transformation. Finally, for each U^k,

$$E_{\tilde{z}_{y,b}}[(b+z_b)e^k(y+z_y)] = E_{\tilde{z}_b}[(b+z_b)e^k(y-\pi^y)] = \bar{b}\cdot e^k(y-\pi^y{}_k) \qquad k=1,2,$$

where z_b is the deviation of $b(c+z_c)$ from its mean, $\bar{b}$, $\tilde{z}_{y,b}$ is the joint probability distribution of z_b and z_y, induced by $\tilde{z}$, and $\tilde{z}_b$ is the marginal distribution of z_b. Theorem 3 of Karni (1979) can be applied to $b\,e^1(y)$ and $b\,e^2(y)$ to show that $\pi^y{}_1 \geq \pi^y{}_2$ for all risks and all (y,c) in D. □

When the two individuals have the same ordinal preferences, $U^1=T(U^2)$, where T is a monotonically increasing transformation. If T is linear, the two individuals also have the same risk preferences. If it is nonlinear, condition (iii) of Theorem 3 is not met. Thus we can conclude that, if two individuals have identical indifference maps, no restrictions exist on a pair of utility functions such that one is more averse to arbitrary income risks than the other. U^1 will pay a larger premium for some risks and a smaller one for others.[9] On the other hand, it follows from the proof of Theorem 3 that two individuals may display the same aversion to risk—*i.e.* have equal risk premia for all risks—and still have distinct ordinal preferences. Thus, focusing only on income risks allows some separation between preferences for attributes of utility and attitudes toward risks.

Kihlstrom and Mirman (1974) limited comparisons between individuals' attitudes toward multivariate risk to cases of identical ordinal preferences. Karni (1979) showed, however, that individuals with different indifference maps can be compared, provided that the risk premium is measured in a certain direction in commodity space. A conclusion emerging from both studies is that, given identical ordinal preferences, the degree of concavity of the utility function provides both necessary and sufficient conditions for comparative risk aversion.

The apparent conflict between the results of these studies and those from Theorem 3 is reconciled by the following observations. When the two individuals' ordinal preferences coincide, the multivariate risk is reducible to a univariate risk, where the random variable is the order of the equivalence class.[10] The Kihlstrom and Mirman (1974) or Karni (1979) results exploit this fact. After the stabilization of *every* argument of the utility function, the distribution over the equivalence classes becomes a degenerate one, so the Arrow-Pratt characterization of more risk averse follows.

However, stabilization of the income risk alone merely implies a new distribution over the equivalence classes, not a degenerate distribution. Depending on both the interaction between the income risk and the attributes risk and attitudes towards them, the new distribution over the equivalence classes may be more or less risky. Hence, a larger degree of concavity does not imply a larger risk premium. Only if the two utility functions have identical attitudes towards the interaction between risks in income and other attributes will the utility function that is more concave in income pay a higher premium to stabilize the income risk. This is exactly condition (iii) of Theorem 3.

3.2. A Positive Risk Premium

By the definitions of π^y and H, if $U(y,c)$ is linear in y and additively separable in y and c, both π^y and H are identically zero. Using these observations and Theorems 2 and 3 facilitates local and global characterizations of utility functions with a positive risk premium. In the interest of brevity we consider only large risks; the small risk results are analogous.

Corollary 1: The following statements are equivalent:

(i) $\pi^y(y,c,\tilde{z}) \geq 0$ for all $(y+z_y, c+z_c)$ in D and all $\tilde{z}$;

(ii) $H(y,c)$ is PSD for all (y,c) in the domain of U;

(iii) $U(y,c) = a(c) + b(y)$, where $b' > 0$ and $b'' \leq 0$.

For the case $c \equiv p$, these conditions are also equivalent to

(iv) $U(y,p) = [log(y) - log(G(p))]$,

where $G(p)$ is linearly homogeneous, and to

(v) $r = \eta_i = 1$ for each i,

where r is the Arrow-Pratt measure of relative risk aversion and η_i is the income elasticity of the demand for good i.

Proof: The equivalence of (i), (ii), and (iii) is established from Theorem 3 by comparing U to the utility function corresponding to $\pi^y \equiv 0$. For the case $c \equiv p$, Roy's Identity implies that (iii), (iv), and (v) are equivalent. □

The implications should be emphasized—in the presence of attribute risk, an economic agent is averse to all income risks if and only if his marginal utility of income is invariant to changes in the levels of other attributes and he is risk averse in the univariate sense.[11]

3.3. Decreasing Aversion to Income Risk

As U^1 and U^2 can represent the utility functions of a single individual at two different levels of wealth, Theorem 3 is also useful for characterizing the utility functions of individuals whose risk premium is nonincreasing with wealth.

Corollary 2: The following statements are equivalent:

(i) $\dfrac{\partial \pi^y(y,c,\tilde{z})}{\partial y} \geq 0$ for all $(y+z_y, c+z_c)$ in D and all $\tilde{z}$;

(ii) $\dfrac{\partial H(y,c)}{\partial y}$ is PSD for all (y,c) in the domain of U;

(iii) $U(y,c) = a(c) + b(c)e(y)$, where $-e''(y)/e'(y)$ is nonincreasing in y.

For the case $c \equiv p$, these conditions are also equivalent to

$$(iv) \quad U(y,p) = \begin{cases} \dfrac{1}{1-r}\left[\dfrac{y}{G(p)}\right]^{1-r} - K(p) & r \neq 1 \\ \log\left[\dfrac{y}{G(p)}\right] & r = 1 \end{cases}$$

where $G(p) > 0$ linearly homogeneous and $K(p)$ is homogeneous of degree zero.

Proof: Upon the substitution $U^1(y,c) \equiv U(y,c)$ and $U^2(y,c) \equiv U(y+w,c)$, where $w \geq 0$, (i)-(iii) are equivalent by Theorem 3. To see the equivalence of (iii) and (iv), note that $U(y,p) = a(p) + b(p)e(y)$ if and only if r does not depend on p. Homogeneity of U implies that r is homogeneous of degree zero in y and p; hence it must also be constant with respect to y. Stiglitz (1969) and Hanoch (1977) showed that r is constant if and only if (iv) holds. □

Thus, individuals display decreasing aversion to income risk in the presence of attribute risk if and only if their attitudes toward the covariation of income and attribute risks are constant with respect to the level of wealth *and*, conditioned on any c, their aversion to income risk is decreasing in wealth. These properties are equivalent to an r that is constant with respect to c, which, when $c \equiv p$, implies that r is also constant with respect to y. Thus, when facing relative price risk, an individual displays decreasing aversion to income risk if and only if his Arrow-Pratt measure of relative risk aversion is constant.

4. Independent Risks

As the results thus far imply rather strong restrictions on preferences, it is also worth considering restrictions on the class of risks. If attention is restricted to distributions such that y and c are independent, the class of utility functions that are averse to income risk is significantly broadened, and is implied by the usual Arrow-Pratt characterization—concavity of U in y conditioned on any c. For small risks, this is an immediate result of Theorem 1, since independence implies $\pi^{ind} = 0$. Theorem 4 shows that this also holds for large risks.

Theorem 4: If z_y and z_c are independent, $\pi^y(y, c\ \tilde{z}) \geq 0$ for all $\tilde{z}$ and all (y, c) in the domain of U if and only if $\pi^{y|c}(y, c\ \tilde{z}_y) \geq 0$ for all $\tilde{z}_y$ and all (y, c) in the domain of U.[12]

Proof: Under independence, the equation that defines π^y is

$$E_{\tilde{z}}[U(y + z_y, c + z_c)] = E_{\tilde{z}_c}[U(y - \pi^y, c + z_c)].$$

Thus, $\pi^y \geq 0$ for all $\tilde{z}$ and all (y, c) in the domain of U if and only if $E_{\tilde{z}_c}[U(y, c + z_c)]$ is concave in y. But as pointed out by Pratt (1988), this expectation is a mixture of univariate functions and, hence, it is concave in y for all risks if and only if $U(y, c)$ is concave in y for any given c, *i.e.* $\pi^{y|c} \geq 0$ for all $\tilde{z}_y$ and all (y, c) in the domain of U. □

Independence thus implies that the sign of π^y is the same as the sign of the corresponding univariate risk premium. However, it can be shown that its magnitude is not, except for the preferences in Theorem 3. Hence, without this restriction, even independent risks in c will affect decisions about income risks.

It turns out that, in the case of independence, the Arrow-Pratt condition for decreasing risk aversion also suffices for the multivariate case. Formally, Pratt (1988) showed that if X and W are two independent random variables and $U(X,W)$ displays decreasing aversion to X, conditioned on any level of W, then the Arrow-Pratt measure of risk aversion of the derived utility function $u = E_W U(X,W)$ is also decreasing in wealth. In Theorem 5, this result is restated for completeness in the current context.

Theorem 5: Let z_c and z_y be independent. $\partial \pi^y / \partial y \leq 0$ if and only if $\partial \pi^{y|c} / \partial y \leq 0$.

Proof: Let $u(y,c,\tilde{z}_c) = E_{\tilde{z}_c} U(y,c)$. From Pratt (1988), $-u_{yy}(y,c,\tilde{z}_c)/u_y(y,c,\tilde{z}_c)$ is decreasing in y for all $\tilde{z}_c$ and all (y,c) in the domain of U if and only if $-U_{yy}(y,c)/U_y(y,c)$ is decreasing in y for any c. Theorem 2 of Pratt (1964) implies that these are equivalent to $\partial \pi^y / \partial y \leq 0$ and $\partial \pi^{y|c} / \partial y \leq 0$, respectively. □

5. Implications for Individuals' Choices

For decision problems that concern a subset of all risks, familiar propositions about the correspondence between the degree of risk aversion and individual choices no longer hold for the Arrow-Pratt measures of risk aversion. We pointed out earlier that multivariate generalizations also do not facilitate such descriptions, since they depend on attitudes toward risks that are not affected, and hence, should not affect the decision. In summary, concavity of the utility function in all arguments or in income alone is neither necessary nor sufficient for aversion to an income risk in the presence of other risks.

However, the alternative characterization of aversion to income risk developed in this paper can be used to describe behavior. This is best illustrated by generalizing a result from Diamond and Stiglitz (1974; Theorem 4, p. 349) to incorporate attribute risk. Suppose that H, the measure of aversion to income risk, is increasing with some parameter ρ, in the sense that $\partial H / \partial \rho$ is PSD. Let $U[y(z_y,\alpha), c(z_c),\rho]$ be a family of utility functions indexed by ρ, where α is a control variable affecting income y.

Theorem 6: Define $\alpha^*(\rho)$ by

$$\alpha^*(\rho) = \underset{\alpha}{argmax}\ E\{U[y(z_y,\alpha), c(z_c), \rho]\}.$$

and let $\hat{z}_y$ denote a realization of z_y, the income risk. Finally, denote $\partial U/\partial\alpha$ by U_α. If aversion to income risk is increasing in ρ in the above sense, then α^* is increasing (decreasing) in ρ if there exists a $\hat{z}_y$ such that $U_\alpha \geq (\leq)\ 0$ for $z_y \leq \hat{z}_y$ and $U_\alpha \leq (\geq)\ 0$ for $z_y \geq \hat{z}_y$.

Proof: By Theorem 3, $\partial H/\partial\rho \geq 0$ if and only if

$$U[y(z_y,\alpha), c(z_c),\rho] = a[c(z_c)] + b[c(z_c)]e[y(z_y\alpha),\rho],$$

where $-e_{yy}/e_y$, is increasing in ρ (equivalently, $\partial^2[-\log e_y]/\partial y\,\partial\rho > 0$.) The first order condition for optimality is given by

$$E\{b[c(z_c)]e_y[y(z_y\alpha),\rho]y_\alpha\} = 0,$$

where $y_\alpha = \partial y/\partial\alpha$. Total differentiation with respect to α and ρ yields

$$\frac{\partial\alpha}{\partial\rho} = \frac{-E\{b[c(z_c)]e_{y\rho}[y(z_y\alpha),\rho]\cdot y_\alpha\}}{\Delta},$$

where Δ denotes $\partial^2 E(U)/\partial\alpha^2$ which is negative by the second order condition. Hence, the sign of $\partial\alpha/\partial\rho$ is the same as that of $E[b\ e_{y\rho}y_\alpha]$. After some manipulation,

$$E[b\ e_{y\rho}y_\alpha] = E\left\{\left[\frac{e_{y\rho}[y(z_y\alpha),\rho]}{e_y[y(z_y\alpha),\rho]} - \frac{e_{y\rho}[\hat{y},\rho]}{e_y[\hat{y},\rho]}\right]be_y y_\alpha\right\}.$$

Since U is increasing in y, $sgn(b)=sgn(e_y)$. Moreover, by the assumptions of the theorem, both y_α and the expression in square brackets change sign at $\hat{y}$. Thus, the expectation above is always positive (negative) when $U_\alpha \geq (\leq)\ 0$ for $z_y \leq \hat{z}_y$ and $U_\alpha \leq (\geq)\ 0$ for $z_y \geq \hat{z}_y$. □

Some straightforward corollaries of Theorem 6 include the following. Consider individuals who face relative price risk and choices between a safe asset and a risky one. The less averse is a consumer to income risk, the larger is the amount invested in the risky asset. Similarly, when offered actuarially fair reimbursement insurance for health expenditures, a consumer's optimal amount of coinsurance is inversely related to the degree of aversion to income risk.

An additional example is provided by agricultural households that consume a portion of their farm product, typical in peasant economies. The household supplies more output in the form of marketed-surplus when expected price increases if it exhibits non-increasing aversion to income risk. So in this case, the "law of supply" holds. Moreover, the larger is aversion to income risk, the smaller is the output level, so Sandmo's (1971) proposition is preserved.

Finally, the measure H also characterizes risk-sharing agreements. Let only z_y and z_1 be random. An *ex ante* risk sharing contract, $\lambda(z_y, z_1)$, is the income transfer from individual 2 to individual 1 upon the realization (z_y, z_1). As Karni (1979) showed, λ is Pareto-efficient if

$$\frac{d\lambda}{dz_y} = \frac{{h^{11}}_1 - {h^{11}}_2}{-({h^{11}}_1 + {h^{11}}_2)} \quad \text{and} \quad \frac{d\lambda}{dz_1} = \frac{{h^{12}}_1 - {h^{12}}_2}{-({h^{11}}_1 + {h^{11}}_2)}.$$

Theorem 3 implies that if individual 1 is at least as averse to income risk as individual 2, then $d\lambda/dz_y \leq 0$ and $d\lambda/dz_1 = 0$. Thus, individual 2 stabilizes the income stream of individual 1. Moreover, two individuals would mutually benefit from the risk sharing agreement if and only if they differ in their attitudes toward income risk as captured by the measure H. If their respective measures coincide, then λ is a constant, rather than a welfare-enhancing instrument.

ENDNOTES

1. Besley (1990) contains a comparative analysis of reimbursement versus optimal health insurance schemes.

2. For a recent summary and extension of these results to generalized expected utility, see Karni (1989).

3. However, see Karni (1979) for interpersonal comparisons of risk aversion for utility functions corresponding to distinct ordinal preferences.

4. The domain of U will vary with the the application. For instance, if $c \equiv p$, the domain, D, is defined by $\{y, c \mid y > 0, c > 0\}$. The discussion below is confined to distributions of z such that $Pr[(y + z_y, c + z_c) \in D] = 1$.

5. This problem does not occur under univariate income risk, since the ordinal preferences of all individuals are identical, in the sense that each prefers more income to less.

6. In the case of large risks, the various risk premia cause wealth effects which must be incorporated in the analogous decompositions. For instance, the analogue for (i) is $\pi = \pi^y(y, c, \tilde{z}) + \pi^{c|y}(y - \pi^y, c, \tilde{z}_c)$.

7. Theorem 2 is stated with a weak inequality. Alternatively, one could exclude risks with degenerate $\tilde{z}_y$, and then state the theorem with a strict inequality, which would require both that $H_1 - H_2$ is PSD and that $h^{11}{}_1 > h^{11}{}_2$, as well as that Ω be full rank.

8. Or, $U^1(y, c) = a^1(c) + b(c, y)$ and $U^2(y, c) = a^2(c) + b(c, y)$, which is equivalent to $\pi^y{}_1 = \pi^y{}_2$ and $H_1 - H_2 = 0$ for all (y, c) in D and all $\tilde{z}$, ruling out differences in attitudes toward income risk.

9. This result is consistent with what has been found in attempts to characterize individual choices under multivariate risk. For instance, Karni (1982) considered saving behavior when both future income and the rate of return to saving are uncertain, and showed that "there are no ordinal preferences that imply a monotonic relation between risk aversion

and the level of saving which is valid for all admissible risks" (p. 36).

10. The random variable could be utility, except that the two individuals need not have the same index of utility. The risk can be described in terms of what indifference curve, or equivalence class, they realize, once the risk is resolved.

11. In the intertemporal context, this is consistent with the usual assumptions that the objective function is the discounted sum of the future utilities and that the individual is instantaneously risk averse in the Arrow-Pratt sense. Moreover, it is also consistent with the functional form shown by Gilboa (1989) to represent aversion to variation in the agent's payoffs over time.

12. The theorem may be stated with a slightly weaker, but less intuitive condition than independence. It is sufficient that the conditional distribution of c given y does not depend on y.

References

Arrow, Kenneth J. *Aspects of the Theory of Risk-Bearing* (Helsinki, 1965). Reprinted in *Essays in the Theory of Risk Bearing* (Amsterdam: North-Holland, 1970).

Besley, Timothy. "Optimal Reimbursement Health Insurance: An Application of Profit Functions and Frischian Demands" *Journal of Economic Theory* 51(1990):403-422.

Epstein, Larry. "A Disaggregated Analysis of Consumer Choice Under Uncertainty" *Econometrica* 43(1975):877-92.

Gilboa, Itzhak. "Expectation and Variation in Multi-Period Decisions" *Econometrica* 57(1989):1153-69.

Hanoch, Giora. "Risk Aversion and Consumer Preferences" *Econometrica* 45(1977):413-25.

Jewitt, Ian. "Risk Aversion and the Choice Between Risky Prospects: The Preservation of Comparative Statics Results" *Review of Economics Studies* 54(1987):73-85.

Karni, Edi. "On Multivariate Risk Aversion" *Econometrica* 47(1979):1391-1401.

Karni, Edi. "Risk Aversion and Saving Behavior: Summary and Extension" *International Economic Review* 23(1982):35-42.

Karni, Edi. *Decision Making Under Uncertainty: The Case of State-Dependent Preferences* (Cambridge, Massachusetts: Harvard University Press, 1985).

Karni, Edi. "Generalized Expected Utility Analysis of Multivariate Risk Aversion" *International Economic Review* 30(1989):297-305.

Kihlstrom R. E., and L. J. Mirman. "Risk Aversion with Many Commodities" *Journal of Economic Theory* 8(1974):361-88.

Kihlstrom, R. E., D. Romer, and S. Williams. "Risk Aversion with Random Initial Wealth" *Econometrica* 49(1981):911-21.

Pratt, J. W. "Risk Aversion in the Small and in the Large" *Econometrica* 32(1964):122-36.

Pratt, J. W. "Aversion to One Risk in the Presence of Others" *Journal of Risk and Uncertainty* 1(1988):395-413.

Ross, S. A. "Some Stronger Measures of Risk Aversion in the Small and in the Large with Applications" *Econometrica* 49(1981):621-38.

Stiglitz, Joseph E. "Behavior Towards Risk with Many Commodities" *Econometrica* 37(1969):660-667.