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A game theory approach to the Iranian forest industry raw material market

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Abstract
Dynamic game theory is applied to analyze the timber market in northern Iran as a duopsony. The Nash equilibrium and the dynamic properties of the system based on marginal adjustments are determined. When timber is sold, the different mills use mixed strategies to give sealed bids. It is found that the decision probability combination of the different mills follow a special form of attractor and that centers should be expected to appear in unconstrained games. Since the probabilities of different strategies are always found in the interval [0,1], the boundaries of the feasible set are sometimes binding constraints. Then, the attractor becomes a constrained probability orbit. In the studied game, the probability that the Nash equilibrium will be reached is almost zero. The dynamic properties of timber prices derived via the duopsony game model are found also in the real empirical price series from the north of Iran.

Keywords: Iranian forest industry, game theory, Nash equilibrium, constrained probability orbit.

1. Introduction:
The forest sector is important to the economy in northern Iran. A rather small number of large mills dominate the industry in the region. The analysis in this paper is made with the ambition to describe the market structure and to analyze the dynamic properties of the market. The study also focuses on the theory of duopsony games. The dynamics of such games, in particular with conditions typical in the region, will be studied and compared to real empirical data series.

Hence, dynamic game theory will be applied to analyze the timber market in northern Iran as a duopsony.

When timber is sold, the different mills use mixed strategies to give sealed bids. The Nash equilibrium and the dynamic properties of the system based on marginal adjustments will be determined.

Game theory is a branch of mathematical analysis developed to study decision making in situations of conflict (and sometimes cooperation). Such situations exist when two or more decision makers (player) have different objectives, act on the same system or share the same resources. Game theory provides a mathematical process for selecting an optimal strategy (that is, an optimum decision or a sequence of decisions) in the face of an opponent who has a strategy of his own. In game theory, these assumptions are usually made:

Each player has two or more strategies or specific choices.

Different possible combinations of strategies available give different payoffs to the different players.

In some games the players have perfect information about the game. This is not always the case. In some games, the information is not perfect and symmetrical.

Game theory has applications in a variety of fields, including, operation research, economics, political sciences, military strategy, psychology and biology. It has close links with economics in that it seeks to find rational strategies in situations where the outcome

depends not only on one's own strategy and "market conditions", but upon the strategies chosen by other players with possibly different or overlapping goals.
Some typical market situations to be handled within this framework in economics are oligopolies and oligopsonies, in particular duopolies and duopsonies.

A game can be classified on the basis of several criteria. Depending on the number of players, we may have two-person, three –person or n-person games.
Depending on the payoff situation a game can be classified as either constant-sum or nonconstant-sum. A constant sum game can be classified as a zero-sum or non-zero-sum games.

In a zero-sum game the sum of payoffs at the end is zero since the amounts won or lost are equal. In such games, each player knows exactly how the other player is affected by different decision combinations as long as he knows how he is affected by the combinations himself. As an economic example of this, we may consider two firms in a duopolistic market that are striving to increase the number of customers. If the total number of customers is constrained, the number of customers won by one firm must be identical to the number of customers lost by the other.

In many real games, the information is incomplete. Player A does not know exactly how player B is affected by different decision combinations without a lot of special information concerning the (economic or maybe physical or biological) environment of player B. Most economic situations are non-zero-sum.
In many cases, it is necessary to calculate the optimal behaviour of each player for each possible position in the physical state space and speed vector and for each possible position, speed vector and decision of the other players. The problem is then solved recursively in the spirit of dynamic programming for every player conditional on the behaviour of all other players. In fact, in a two person difference game, if the decisions of player B or their probability distribution are known by player A and the decisions made by player B are not affected by the decisions made by player A, then player A may regard his optimization problem in the difference game as a common dynamic programming problem. This however, is a very special case where we do not really investigate game anymore. We then have a "game against Nature". The dimensionality problem in dynamic programming is well known. In difference games, the dimensionality problem is much worse.

Then, what can be done?

If we accept low resolution in the state and time space and a low number of possible decisions (controls), then the difference games can often rapidly be solved. Furthermore we usually have to assume that the game is deterministic: Each player selects a pure position dependent strategy. If we let the players use randomized strategies, make different decisions with different probabilities in different situations, the computation time grows very rapidly.

One observation concerning the deterministic differential or difference game is that the outcome is known when the initial conditions are known.

In a deterministic differential game, each player knows exactly what to do and what the other players will do in every possible situation. There is really no need to play a game. For this reason we may say that we know the outcome of a game.

Of course in reality, the players do not know much enough or have time enough to calculate the optimal decisions in all possible positions. In real world conflicts, the technical properties of the equipment and the exact positions of the army units may not be known by the opponent. In other kinds of conflicts in a complicated society, the options available to the opponent are frequently very difficult to estimate.

In many real world games in economics, the physical and economic environment of the game problems changes rapidly and often unpredictably. One player may own a factory which produces a particular product. If the price of the product is high, this player may be very interested to buy a unit of some input factor. This input factor transaction may be a game in which the factory owner participates among other potential buyers. In this case, the factory owner highly valuates a decision combination which means that he can buy the input factor. One month later, the price of the product decreases dramatically. Again, the factory participates in a similar transaction game. This time he does not valuate a decision combination which makes him buy the input factor as highly as before. Since the economic environment unpredictably changes in this game, we can not expect that the players will select the same strategy for ever. Hence, we can not be sure that a player who estimates the probabilities of the other player's decisions via the frequencies in the complete historical decision observation series, and optimizes his strategy accordingly, will optimize his expected result in the changing environment.

In this paper, dynamic game theory is applied to Iranian forest industry. There are two sawmills firms actively involved in the timber market area of the game. A large number of forest companies and privately planted forests sell timber to these sawmills.

In each transaction, each sawmill (player) gives a sealed bid: A high or a low bid. Here the situation is a noncooperative game. Our first aim is to determine the optimal strategy and Nash equilibrium for each player.

2. Literature review:

Cournot (1838) presents a revolutionary contribution to the theory of non cooperative equilibria in oligopoly situations. von Stackelberg (1934 and 1938) is one of the persons who has contributed to game theory before the concept was established. In particular, he was interested in dynamic duopoly theory.

The mathematical theory of games was described by Neumann and Morgenstern (1944).

Nash (1950 and 1951) gave us the important concept "Nash equilibrium". In

Nash equilibrium, no player has an incentive to deviate from the strategy chosen, since no player can choose a better strategy given the choices of the other players.

The Nash equilibrium has been very useful in most developments of game theory.

Brown and von Neumann (1950) discussed how to use differential equations in the solution of games. Robinson (1951) used an iteration method where each player sequentially estimated the probability distributions of the other players decisions and adapted the own decision probabilities optimally. Brown (1951) investigated a problem similar to the problem in Robinson (1951). Bellman (1953) continued the studies of iterative algorithms and so did von Neumann (1954).

Luce and Raffia (1957) studied many important game problems with mathematics and numerical methods.

Schelling (one of the winners of the price in economic sciences in memory of Alfred Nobel 2005) gave a good survey of the field strategy of conflict in 1960. Dresher (1961) stressed the time dimension and optimal decisions over time in connection to several games of conflict. Isaacs (1965) introduced the theory and several applications of differential games.

Selten (1975), Kalai and Smorodinsky (1975) and Rasmusen (1990) present a wide spectrum of game models from economics and related fields. Aumann and Hart (1992 and 1994) wrote a useful handbook of game theory with economic applications. (Aumann was the other winner of the price in economic sciences in memory of Alfred Nobel 2005). They dynamics of Cournot games has been studied by Flåm (1990), Flåm and Moxnes (1991) and Flåm and Zaccour (1991).

Lohmander (1994) studied the dynamics and non cooperative decisions in stochastic markets with pulp industry application. Lohmander (1997) contains a general investigation of the constrained probability orbit of mixed strategy games with marginal adjustment. A general two person non-zero sum game (with zero as a special case) is analyzed. A doupsony application where two sawmills are competing in the timber market is included and the dynamic properties of the system are determined.

Game theory has found new forest sector applications in recent years. Koskela and Ollikainen (1998) described a game-theoretic model of timber prices and the capital stock for the Finnish pulp and paper industry. Carter and Newman (1998) examined the impact of reservation prices on timber revenues from federal timber sale auctions in North Carolina from a game-theoretic perspective by recognizing the effect of competition on optimal bid strategies.

3. Cooperation or conflict in the timber market: A doupsony discussion.

Two sawmills buy timber from a large number of independent forest owners in an area. Every time a unit of timber is available, the forest owner receives sell bids from the potential buyers. Clearly, this is a case where the buyers as a group may benefit from cooperation and low bids. The extra profit obtained via the low timber price may then be distributed between the buyers in some way.

In some cases, the strongest sawmill (in the sense of ability to survive high timber prices), may prefer not to cooperate and to destroy the input market of other sawmill via high bids). This way, both sawmills loose profits during some time period and the strongest sawmill has the option to use his monopsony power and to increase his profits even more than before via low timber prices. The sawmill example contains two kinds of solutions:

In the cooperation case, way may expect the sawmills to calculate the timber price which maximizes the profit of the two sawmills as a group. Then they distribute the extra profit somehow within the group. Sometimes we may expect that the sawmills decide not only the timber price but also the distribution of the timber. The forest owners may not notice this cooperation directly. They may notice that all bids are low or that only one of the saw mills gives a bid on each unit of timber, or finally, that one sawmill gives a low bid and the other sawmill gives a very low bid on each timber unit. In the latest case, the very low bid is there just to hide the cooperation from the sellers. It does not affect the plan of buyers anyway.

In timber price fight case, the timber price bids are high until one of the buyers leaves the market. Then the bids instantly fall and the low price level remains until increased competition appears.

In a third case the buyers do not cooperate because they do not believe that other buyers will keep an agreement. Maybe they are also aware that the government will discover market cooperation and punish cartels. Hence, the buyers act according to the law and sometimes deliver sealed bids. (Some countries have such laws.)

When they decide to give a bid, they first have to inform themselves about the quality of the timber and other practical details. This activity is not costless. Then, they have to decide the level of the bid.

Of course, they can give a low bid and hope that the other sawmill will not give a higher bid. In that case they will buy the timber cheaply. If they have bad luck, the other sawmill buys the timber with a higher bid and the only economics consequence of the activity is the cost of the investigation.

On the other hand, they may give a high bid and hope that the other sawmill will give a lower bid. The probability of obtaining the timber is of course higher in this case, but the price is also higher.

This last version of the game is interesting in several ways and the methodology to be used in the analysis is not obvious. Each player has in the example two different possible decisions: A high (H) or a low (L) bid. The players are denoted A and B.

If the sum of the total profit made by the two players is zero (or a constant), it is obvious that no cooperation will appear. If the players know all the economic consequences for both players of all decision combinations exactly, then we can use the two person zero sum game theory.

The optimal strategies may turn out to be pure (only one decision) or mixed for each player where a mixed strategy means that different decision should be made with different probabilities.

In the sense that one sawmill has no (or very limited) information concerning the economic consequences in the other sawmill of different decision combinations, the obvious way for player A to deal with the problem is to observe and estimate the frequencies of the different decision taken by player B.

4. Timber price (Numerical data analysis):

Numerical data were collected from two forest companies in the north of Iran (Fig. 1). These companies are called Shafarod and Neka Chub.

They buy more than 70 percent of the timber in the region. We may call this a duopsony situation.

These companies rent some forests from the government. They harvest and manage these rented forests but they also buy the timber from other sources such as privately planted forests and forest companies. These companies produce different products in their own sawmills such as sawnwood, veneer, plywood, pulpwood, firewood and charcoal.

Now, we denote sawmills Shafarod and Neka Chub, A and B, respectively.

Figure. 1. The distribution of Iranian northern forests and two sawmills.

The real timber price series from the year 1990 until 2004 were collected from the two sawmills. Appendix A and Fig. 2 show these series. The differences of the real timber prices are shown in Fig. 3.

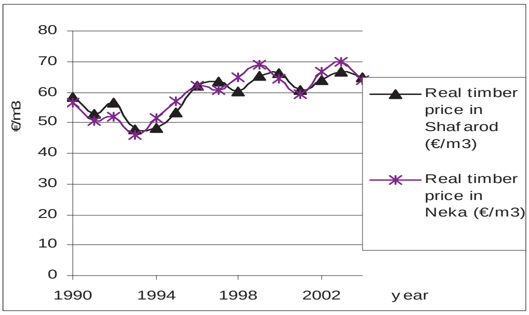

Figure. 2. Real timber prices in two sawmills in the north of Iran.

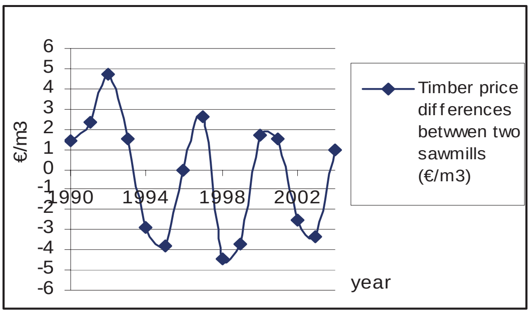

Figure. 3. The differences between real timber prices between two sawmills in the north of Iran.

As a start, we investigate the prices using autoregressive (AR) time series analysis.
Timber prices are treated as stochastic and assumed to follow a first order Markov process. A Markov price expectation structure refers to any stochastic model in which price is conditional on previous prices. Current prices are known, but the future prices are uncertain.
Fifteen years of timber price data were used to estimate the following model:

$P_{t+1} = \alpha + \beta P_t + \varepsilon_t$, where P_{t+1} is the expected price in period t+1, P_t is price in the current period and ε_t is the error term. ε_t is assumed to be independent identical distribution and Gaussian, with expected value 0 and standard deviation σ_{ε_t}.
The estimated parameters α, β are found below: (t statistics in parentheses).

Sawmill A:

$$P_{t+1} = 23.394 + 0.678 P_t + \varepsilon_t \qquad (1)$$

$$(1.727) \quad (3.496) \qquad \sigma_{\varepsilon_t} = 8.880.$$

Sawmill B:

$$P_{t+1} = 23.915 + 0.667 P_t + \varepsilon_t \qquad (2)$$

$$(1.808) \quad (3.518) \qquad \sigma_{\varepsilon_t} = 8.422.$$

The parameter estimates of the two first order AR price processes above indicate that prices are stationary. Nonstationary Martingale prices, on the other hand, have the property $P_{t+1} = P_t + \varepsilon_t$.
Detailed inspections of the results however show that the t-values of the constants are too low to give a statistically significant indication of stationarity at the 95% level.

Furthermore, the estimated first order AR processes do not give any information concerning possible dependence between the prices in the two sawmills. Hence, some alternative models would be interesting. As a start, we investigate the price difference.

The first order AR model for the timber price differences between the two sawmills, $\bar{P}_t = P_{t,A} - P_{t,B}$ is:

$$\overline{P}_{t+1} = 0.368 + 0.084\overline{P} + \varepsilon_t \qquad (3)$$

(0.325) (0.273) $\qquad \sigma_{\varepsilon_t} = 4.231.$

Also the second order AR process for timber price differences was estimated:

$$\overline{P}_t = \alpha + \beta\overline{P}_{t-1} + \delta\overline{P}_{t-2} + \varepsilon_t \qquad (4)$$

$$\overline{P}_t = -0.22369 - 0.01548\overline{P}_{t-1} - 0.30816\overline{P}_{t-2} + \varepsilon_t$$

(-0.1759) (-0.04678) (-0.93313) $\qquad \sigma_{\varepsilon_t} = 4.386796.$

We observe that the first and second order AR models of the price differences give very low t-values. Such models do not seem to capture the properties and possible dependencies of the prices very well. The price difference equilibrium of the price differences according the second order AR model can be calculated:

$$\overline{P}_{eq} = \alpha + \beta\overline{P}_{eq} + \delta\overline{P}_{eq} \text{ or } (1-\beta-\delta)\overline{P}_{eq} = \alpha$$

and
$$\overline{P}_{eq} = \frac{\alpha}{(1-\beta-\delta)}. \qquad (5)$$

Using the estimated parameter values, we get the equilibrium price $\overline{P}_{eq} = -0.1690€.$
So, if we use the second order process, even if it gives low t-values, it indicates that the expected long run difference between the prices in the two mills is very low. This is what we find also if we investigate the price differences shown in Fig.3.

Maybe we could get some interesting results if we estimate the prices of the two mills as a function of the earlier prices in both mills?

$$P_{A,t+1} = \alpha_A + \beta_A P_{A,t} + \beta_B P_{B,t} + \varepsilon_t \qquad (6)$$

where

$$P_{A,t+1} = 16.148 - 0.177 P_{A,t} - 0.952 P_{B,t}$$

(1.249) (-0.283) (1.501) $\qquad \sigma_{\varepsilon_t} = 8.562.$

and

$$P_{B,t+1} = \alpha_B + \beta_B P_{B,t} + \beta_A P_{A,t} + \varepsilon_t \qquad (7)$$

Where

$$P_{B,t+1} = 19.774 - 0.280\, P_{B,t} + 0.997 P_{A,t} + \varepsilon_t$$

(1.470) (-0.431) (1.511) $\qquad \sigma_{\varepsilon_t} = 8.907.$

Again, we observe that the models give very low t-values. Some other approach is needed.

We may also run the following regressions:

$$dP_A = \alpha_1 + \alpha_2 P_A + \alpha_3 P_B + \varepsilon_{t,A} \tag{8}$$

Where dP_A is defined as $P_{A,t+1} - P_{A,t}$.

$$dP_B = \alpha_4 + \alpha_5 P_A + \alpha_6 P_B + \varepsilon_{t,B} \tag{9}$$

dP_B is $P_{B,t+1} - P_{B,t}$.

Table 1 shows the results of these regressions.

Table 1. Parameters based on the timber price data.

	α_1	α_2	α_3	α_4	α_4	α_6	$\varepsilon_{t,A}$	$\varepsilon_{t,B}$
Parameter value	16.148	0.952	-1.177	19.774	-0.003	-0.280		
Standard deviation	12.933	0.634	0.625	13.454	0.660	0.650	8.562	8.907
t- statistics	1.249	1.501	-1.883	1.470	-0.005	-0.431		

We conclude this section by the following observations:
The AR process estimations of different types gave low t-values. The two price processes seem to be stationary but no definite results were obtained this way. Hence, we will move over to a dynamic game theory approach and investigate if we can interpret the empirical findings that way.

5. Expected pay off and Nash equilibrium

The profit in one of the mills may be calculated by the following function:

$$\pi = -F + P_S V_S + P_P V_P - P_T V_T \tag{10}$$

Where π is the net profit, F is the fix production cost, P_T is the timber price, P_S is the net sawnwood price, P_P is the net pulpwood price V_S is the volume of sawnwood production, V_P is the volume of pulpwood production and V_T is the purchased timber volume.
Below, we ignored the fix cost, because it has the same effect in two sawmills.
We assume that from 1.2 m^3 timber it possible to produce 1 m^3 sawnwood and pulpwood (0.7 m^3 sawnwood and 0.3 m^3 pulpwood).

We may rewrite equation 10 like:

$$\pi = 0.7VP_S + 0.3VP_P - P_T 1.2V \tag{11}$$

Where V is the sum of sawnwood and pulpwood $(V = V_S + V_P)$.

Sawmill A has higher capacity than sawmill B. They are both located close to the forest, about 500 km away from each other. The independent forest harvesters and privately planted forests sell their timber to these two sawmills.
Here the situation is a non cooperative game. Each sawmill uses a mixed strategy and gives a high or a low bid. Compare Table 2.

We determine the elements of the profit (pay off) matrix this way:

In case the timber price is high:

Ps = 110 (€/m^3), Pp = 20 (€/m^3), P_T = 65(€/m^3), V=1 m^3
If we substitute these values into equation 11, the profit is 5 €/ m3

In case the timber price is low:

Ps = 110 (€/m^3), Pp = 20 (€/m^3), P_T =55 (€/m^3), V=1 m^3
By substituting these values into equation 11, the profit is 17 €/ m3.

Table. 2. The payoffs matrix for two sawmills.

	Low (Y)	High (1-Y)
Low (X)	V_A = 126 (1.) V_B = 108 P_A = 55 P_B = 55 π_A = 1785 (2.) π_B = 1530	V_A = 120 V_B = 336 P_A = 55 P_B = 65 π_A = 1700 π_B = 1400
High (1-X)	V_A = 456 V_B = 84 P_A = 65 P_B = 55 π_A = 1900 π_B = 1190	V_A = 360 V_B = 300 P_A = 65 P_B = 65 π_A = 1500 π_B = 1250

V is the timber volume (1000 m^3).
π is the net profit (1000 €).

Let us determine the Nash equilibrium:

The expected payoff of mill A is:

$$E_A = 17.85XY + 17X(1-Y) + 19Y(1-X) + 15(1-X)(1-Y) \quad (12)$$

$$E_A = 15 + 2X + 4Y - 3.15XY \quad (13)$$

$$\delta E_A / \delta X = 2 - 3.15Y = 0 \quad (14)$$

From this, we conclude that firm A has no reason to change X if Y = 0.634

The expected payoff of mill B is:

$$E_B = 15.3XY + 14X(1-Y) + 11.9Y(1-X) + 12.5(1-X)(1-Y) \quad (15)$$

$$E_B = 12.5 + 1.5X - 0.6Y + 1.9XY \quad (16)$$

$$\delta E_B / \delta Y = -0.6 + 1.9X = 0 \quad (17)$$

Hence, firm B has no reason to change Y if X = 0.316

The mixed Nash equilibrium is $(N_X, N_Y) = (0.316, 0.634)$

We may with the mixed Nash equilibrium values of N_X and N_Y, determine the expected payoffs of mills A and B: E_A = 1753708 € and E_B = 1313382 €

Hence, we realize that both mills expect to get these payoffs if both buy the timber according to the mixed Nash Equilibrium.

6. The dynamics of the mixed strategy game

As we mentioned, it is not likely that the managers of the two mills have complete information concerning the properties of the other mills. The costs and revenues of the competitor are not perfectly known. The mixed strategy frequencies are however observed. Now, we introduce the dynamic rules of the game:

Each mill continuously observes the frequencies of the other mills action.

The expected marginal profits, $\delta E_A / \delta X$ and $\delta E_B / \delta Y$ are calculated based on this information. In case the marginal profit of mill A is strictly positive (zero or strictly negative), mill A increases (leaves unchanged, decreases) X. In case the marginal profit of mill B is strictly positive (zero or strictly negative), mill B increases (leaves unchanged, decreases) Y. We assume that the speed of adjustment (of X and Y) is proportional to the expected marginal profits and that both mills A and B have the same relation between speed of adjustment and expected marginal profit.

We assume that $W_1$and W_2 are the speed of adjustment for mills A and B, respectively and W1=W2.

We may rewrite the Eq. (14) like:

$$\dot{X} = W_1(\delta E_A / \delta X) \quad (18)$$

or $\dot{X} = W_1(2 - 3.15Y)$ (19)

We can rewrite the Eq. (17) like,

$$\dot{Y} = W_2(\delta E_B / \delta Y) \quad (20)$$

or $\dot{Y} = W_2(-0.6 + 1.9X)$ (21)

The resulting mixed strategy trajectories are found in Fig. 4.

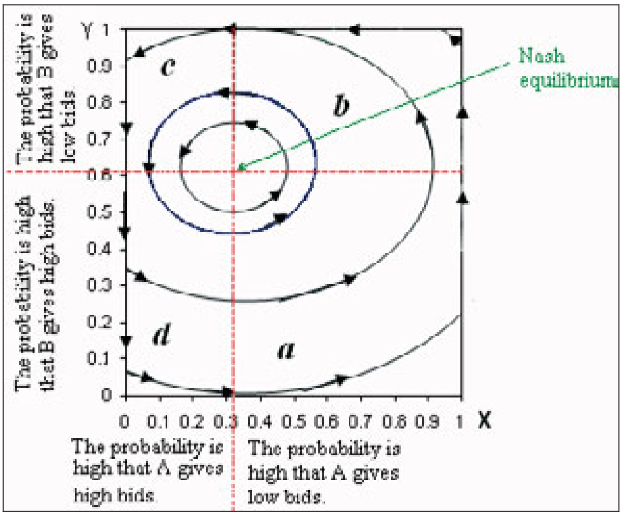

Figure. 4. The dynamics of the mixed strategy probabilities of the timber game.

We can make the following observations in Fig. 4.
The trajectories found in Fig.4 show possible time paths of the strategy combination (X, Y).

Region a:
$X>0.316$, $Y<0.634$. Sawmill A often gives a low bid, and sawmill B often gives a high bids. Since A frequently gives a low bid, B finds it profitable to increase the frequency of low, so he decides to give low bids more often and the system moves upwards and to the right and soon reaches the region b.
Region b:
$X>0.316$, $Y>0.634$.Both mills often give low bids.
A realizes that it profitable if he increases the frequency of high bids, so he gives high bids more often and the system moves upwards and to the left, reaching region c.

Region c:
$X<0.316$, $Y>0.634$. Sawmill A often gives high bids, and sawmill B often gives low bids.
B finds that it profitable to give high bid more often and the system moves down reaching region d.

Region d:
$X<0.316$, $Y<0.634$. B prefers frequently give high bids.

A finds that it profitable if he more often gives low bids. He decides to increase the frequency of low bids and the system is moved to the right reaching region a again.

7. Formal analysis of the dynamics

The aim is to show that the mixed strategy probabilities follow the trajectories in Fig. 4. The formal analysis of the differential equation system is found in the Appendix B.

$$\dot{X} = \alpha_1 + \beta_1 Y \qquad (22)$$

$$\dot{Y} = \alpha_2 + \beta_2 X \qquad (23)$$

The following assumptions are satisfied:

$(\beta_1\beta_2 < 0)$, $(\alpha_1\beta_1 < 0)$, $(\alpha_2\beta_2 < 0)$

The solution is:

$$X(t) = A_1 \cos(\theta t) + A_2 \sin(\theta t) + N_X \qquad (24)$$

$$Y(t) = A_3 \cos(\theta t) + A_4 \sin(\theta t) + N_Y \qquad (25)$$

(N_X, N_Y) is the Nash Equilibrium and $N_X = -\frac{\alpha_2}{\beta_2}$, $N_Y = -\frac{\alpha_1}{\beta_1}$.

$X(0) = X_0$

$Y(0) = Y_0$

$A_1 = X_0 + \frac{\alpha_2}{\beta_2}$, $A_2 = \frac{\beta_1 A_3}{\theta}$, $A_3 = Y_0 + \frac{\alpha_1}{\beta_1}$, $A_4 = \frac{\beta_2 A_1}{\theta}$, $\theta = \sqrt{-\beta_1\beta_2}$.

The trajectories X(t) and Y(t) are shown in Fig. 5 and 6.

(X(t), Y(t)) will follow an orbit around the Nash equilibrium (N_X, N_Y). This is called a center in the theory of dynamical systems.

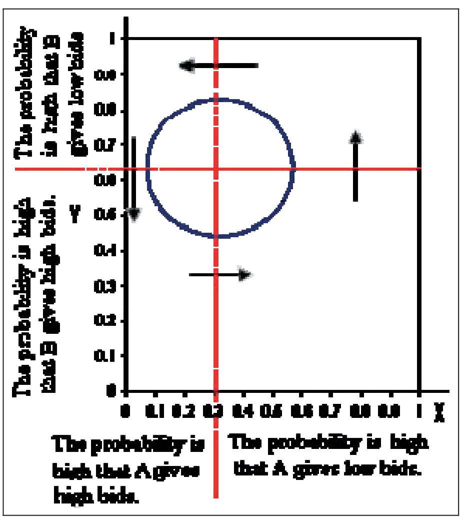

Figure. 5. The dynamics of the mixed strategy probabilities of the timber game for two players A and B.

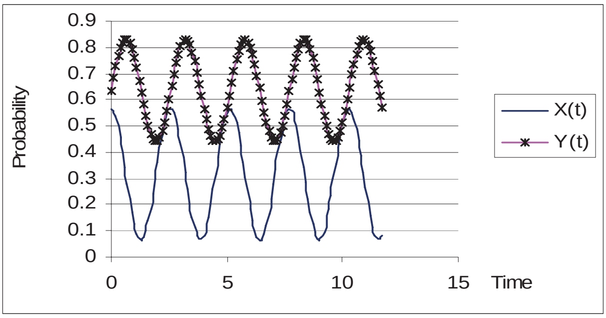

Figure. 6. The probability path of the mixed strategy timber game.

Now, we may determine the expected price, the expected profits and marginal profit for the two players. A simulation model programmed in Lingo found in Appendix C was used to determine these values. The results show the dynamics of the expected prices, the expected profits and the expected marginal profits for each player.

Fig. 7 shows how the expected price difference changes for two players when the high and low price offers are 55 and 65 €/m^3 respectively and W_1=W_2=1 for both players.

Now, it is time to recall the price differences in the real world, found in Fig. 3.

To obtain a price differences path similar to the empirical data found in Fig. 3, we consider a price difference of 15 €/m^3 between high and low bids and W_1=W_2=1 for both players. We assume that the Nash equilibrium is still the same as in the case with high and low prices of 65 €/m^3 and 55 €/m^3, respectively.

Now, however, we assume that, for different reasons, there are differences between the two areas where the two mills A and B are located. A high price is 4 €/m^3 higher in the area of mill B than in the area of mill A. this is quite reasonable since there may be all kinds of local reasons why the conditions are different. We do not have documented reasons for such possible differences in cost and revenue background data, however.

Now, we determine θ such that the period of the system fits the empirical data.

The period is 4 years according to the data found in Fig. 3. That means that $\frac{2\Pi}{4} = \theta$, which gives $\theta = 1.57$

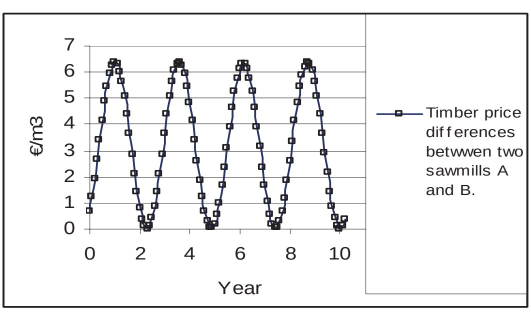

Figure. 7. The expected price difference path with the first game model version.

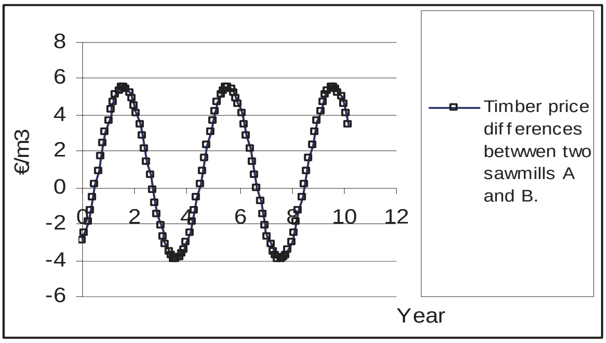

Figure. 8. The expected price difference path when the parameters have been adapted to fit the empirical price difference data.

8. Dynamic sensitivity analysis of the timber market game

Now, we will partially modify the initial Nash equilibrium to investigate the behavior of each sawmill under these new assumptions.

Case 1.
According to the duopsony game formulated above.
Equilibrium: $(N_X, N_Y) = (0.316, 0.634)$. Illustration: Fig. 4 and 5.
Case 2.
We assume that the equilibrium is $(N_X, N_Y) = (0.5, 0.5)$. In this situation, A and B will have equal probability to participate in the game with high or low bids.
Illustration: Fig. 9.

Case 3.
We assume that the equilibrium is $(N_X, N_Y) = (0.7, 0.3)$.
Compared to case 1, the probability that A gives high bids decreased and low bid increased. In this case the probability that B gives high bids increased and low bids decreased. Illustration: Fig. 10

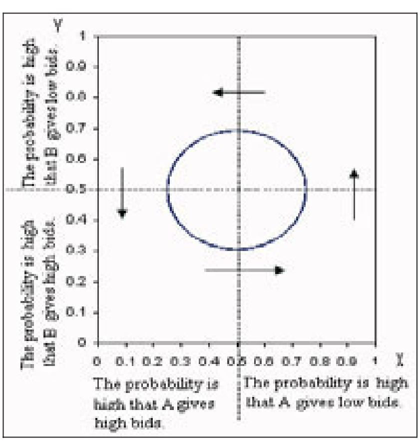

Figure. 9. The mixed strategies of timber game when the Nash equilibrium is $(N_X, N_Y) = (0.5, 0.5)$.

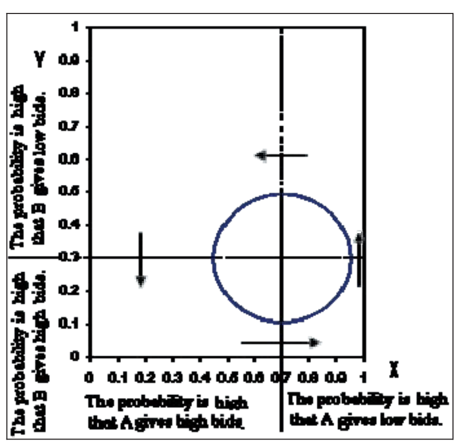

Fig. 10. The mixed strategies of timber game when the Nash equilibrium is $(N_X, N_Y) = (0.7, 0.3)$.

According to our investigation we may write the following observations:

Each player optimizes his expected payoff via a mixed strategy conditionally on the decision frequencies of the other player. In the mixed strategies, every decision should have a strictly positive probability.

The differential equation system governing the simultaneous optimal adjustments of the decision frequencies of the two players give cyclical solutions, sine and cosine functions.

The Nash equilibrium solution, $(N_X, N_Y) = (0.316, 0.634)$, will never be reached unless that happens to be the initial state of the system.

If the system follows a trajectory, an orbit or a center that passes through the four different regions without touching the boundary of the feasible area, then the system will follow this orbit for ever.

If the system follows a trajectory that somewhere touches the boundary of the feasible area, then the system will follow the boundary for some time. Finally the system will start to follow an attractor, a center, for ever. This attractor will be the largest center that can be constructed around the equilibrium, without touching the boundaries, which is consistent with the unconstrained differential equations. Note that most of the small circles in Fig. 4 have been trapped for ever in the respective attractors.

Conclusion

In this paper, a dynamic two person, non - zero sum game was applied in a duopsony situation in the timber market in the northern part of Iran where most of the industrial forests are located. The trajectories of the decision probability combination were investigated. It was found that a large number of initial conditions make the decision probability combination follow a special form of attractor and that centers can be expected to appear in typical games. The probability that the Nash equilibrium will be reached is almost zero.

Real world games are complicated. Hopefully, the reader has found the analysis in this paper to be a step in the right direction. When we find a game in reality where the players use mixed strategies and change the frequencies over time, we have an indication that the present theory is relevant. The properties of the empirical observations, found in Fig. 3, should be expected if our game model is relevant.

In Fig.7, the corresponding model results are shown. Our interpretation is that the game model results closely match the real world data. Since we have not found any other model that gives more realistic results, we conclude that our game approach may be the best choice.

Appendices:

Appendix A. Real timber purchase price in two sawmills in north of Iran during 1990 to 2004.

year	Timber price in Shafarod (€/m3)	Timber price in Neka Chub (€/m3)	Consumer Price index in Iran	Real timber price in Shafarod (€/m3)	Real timber price in Neka Chub (€/m3)	Timber price differences between two sawmills (€/m3)
1990	4.2	4	18.6	59.61	56.77	2.84
1991	4.5	4.3	22.4	53.03	50.67	2.36
1992	6	5.5	27.9	56.77	52.041	4.73
1993	6.6	7	34.3	50.79	53.87	-3.08
1994	9	10	46.3	51.31	57.01	-5.70
1995	14	15	69.2	53.41	57.22	-3.82

1996	27	26	85.2	83.66	80.56	3.10
1997	29	27	100	76.56	71.28	5.28
1998	34	36	118.1	76.00	80.47	-4.47
1999	42	40	141.8	78.19	74.47	3.72
2000	48	50	159.7	79.348	82.65	-3.31
2001	56	58	177.9	83.10	86.07	-2.97
2002	61	60	206	78.17	76.89	1.28
2003	65	63	238.2	72.04	69.82	2.22
2004	70	64	264	70	64	6.00

Appendix B:

Formal analysis of the dynamics:

$\dot{X} = \alpha_1 + \beta_1 Y$ (1)

$\dot{Y} = \alpha_2 + \beta_2 X$ (2)

We assume $(\beta_1\beta_2 < 0)$, $(\alpha_1\beta_1 < 0)$, $(\alpha_2\beta_2 < 0)$.

$\ddot{X} = \beta_1 \dot{Y}$

$\ddot{X} = \beta_1(\alpha_2 + \beta_2 X)$

$\ddot{X} = \beta_1\alpha_2 + \beta_1\beta_2 X$ and $\ddot{X} - \beta_1\beta_2 X = \beta_1\alpha_2$ (3)

In general form we have $\ddot{X}$ + aX-b = 0 where a = -$\beta_1\beta_2$ and b=$\beta_1\alpha_2$

Homogenous solution of equation (3):

$\dot{X} + aX = 0$ (4)

Let X(t)=Ae^{Lt}

$\dot{X}$ =LAe^{Lt}

and

$\ddot{X}$ = L^{2}Ae^{Lt}

Ae^{Lt} (L^2 + a)=0

L=$\pm\sqrt{\beta_1\beta_2}$; i=$\sqrt{-1}$

then L=$\pm\sqrt{-\beta_1\beta_2}$ i (5)

Particular solution of equation (3):

X(t)=m+nt.

$\dot{X}$ =n and $\ddot{X}$ =0 By using this results in equation (4), we get:

0+a (m+nt)=b

n=0 then am=b and m= $\frac{b}{a}$ so we get m= $\frac{+\beta_1\alpha_2}{-\beta_1\beta_2}$ or m= $-\frac{\alpha_2}{\beta_2}$

As a consequence, we have X(t)= $Ae^{\pm\sqrt{-\beta_1\beta_2}\,it} + \left(-\frac{\alpha_2}{\beta_2}\right)$

Hence,

$$X(t) = e^{0t}(A_1 \cos(\sqrt{-\beta_1\beta_2}\ t) + A_2 \sin(\sqrt{-\beta_1\beta_2}\ t) - \frac{\alpha_2}{\beta_2}$$

or $X(t) = A_1 \cos(\sqrt{-\beta_1\beta_2}\ t) + A_2 \sin(\sqrt{-\beta_1\beta_2}\ t) - \frac{\alpha_2}{\beta_2}$ (6)

$\ddot{Y} = \beta_2 \dot{X}$ (7)

By substituting equation (2) in equation (7) we get:

$\ddot{Y} = \beta_2(\alpha_1 + \beta_1 Y)$

$\ddot{Y} = \beta_2\alpha_1 + \beta_1\beta_2 Y$

$\ddot{Y} - \beta_1\beta_2 Y = \beta_2\alpha_1$

Finally we get this solution:

$Y(t) = A_3 \cos(\sqrt{-\beta_1\beta_2}\ t) + A_4 \sin(\sqrt{-\beta_1\beta_2}\ t) - \frac{\alpha_1}{\beta 1}$ (8)

We define θ as $\sqrt{-\beta_1\beta_2}$.

We rewrite equations (6) and (8) like this:

$X(t) = A_1 \cos(\theta\ t) + A_2 \sin(\theta\ t) - \frac{\alpha_2}{\beta_2}$ (10)

$Y(t) = A_3 \cos(\theta\ t) + A_4 \sin(\theta\ t) - \frac{\alpha_1}{\beta 1}$ (11)

The first order derivatives of these equations are:

$\dot{X}$ = $-A_1\theta \sin(\theta t) + A_2\theta \cos(\theta t)$ (12)

$\dot{Y}$ = $- A_3\theta \sin(\theta t) + A_4\theta \cos(\theta t)$ (13)

If we substitute equations (10) and (11) into equations (1) and (2), we have:

$\dot{X} = \alpha_1 + \beta_1(A_3 \cos(\theta\ t) + A_4 \sin(\theta\ t) - \frac{\alpha_1}{\beta 1})$ (14)

$\dot{Y} = \alpha_2 + \beta_2(A_1 \cos(\theta\ t) + A_2 \sin(\theta\ t) - \frac{\alpha_2}{\beta_2})$ (15)

After simplifying, we get:

$\dot{X} = \beta_1 A_3\cos(\theta t) + \beta_1 A_4\sin(\theta t)$ (16)

$\dot{Y} = \beta_2 A_1\cos(\theta t) + \beta_2 A_2\sin(\theta t)$ (17)

From equations (12, 13) and (16, 17) we get the following equalities:

$$\begin{cases}\begin{cases}-A_1\theta = \beta_1 A_4 \\ A_2\theta = \beta_1 A_3\end{cases} \\ \\ \begin{cases}-A_3\theta = \beta_2 A_2 \\ A_4\,\theta = \beta_2 A_1\end{cases}\end{cases} \tag{18}$$

From equation (18), we get:

$\frac{A_1}{A_2} = -\frac{A_4}{A_3}$, $\frac{A_3}{A_4} = -\frac{A_2}{A_1}$, $A_2 = \frac{\beta_1 A_3}{\theta}$, $A_3 = \frac{A_2\theta}{\beta_1}$, $A_4 = \frac{\beta_2 A_1}{\theta}$ (19) Consequently, the following equations can be written:

$$\begin{cases} X(t) = A_1 \cos(\theta t) + A_2 \sin(\theta t) - \frac{\alpha_2}{\beta_2} \\ Y(t) = (-\frac{\beta_2 A_2}{\theta})\cos(\theta t) + (\frac{\beta_2 A_1}{\theta})\sin(\theta t) - \frac{\alpha_1}{\beta_1} \end{cases} \tag{20}$$

or

$$\begin{cases} X(t) = A_1 \cos(\theta t) + (\frac{\beta_1 A_3}{\theta})\sin(\theta t) - \frac{\alpha_2}{\beta_2} \\ Y(t) = A_3 \cos(\theta t) + (\frac{\beta_2 A_1}{\theta})\sin(\theta t) - \frac{\alpha_1}{\beta_1} \end{cases} \tag{21}$$

Then:

$X(0) = A_1 - \frac{\alpha_2}{\beta_2} \Rightarrow A_1 = X(0) + \frac{\alpha_2}{\beta_2}$

$Y(0) = A_3 - \frac{\alpha_1}{\beta_1} \Rightarrow A_3 = Y(0) + \frac{\alpha_1}{\beta_1}$.

The Nash Equilibrium values for X and Y are $N_X = -\frac{\alpha_2}{\beta_2}$, $N_Y = -\frac{\alpha_1}{\beta_1}$, respectively.

Appendix C.The Lingo code is found below.

```
Model:
sets:
time/1..60/:x,y,EA, EAd, EB, EBd,MA, MB, EPA, EPB, EPDIFF;
endsets
! Speed of adjustment coefficients;
wA = 0.005;
wB = 0.005;
```

```
step = 0.1;
! Initial conditions;
x(1) = 0.35;
y(1) = 0.50;
! Parameters;
PAM = 60;
PAD = 5;
PBM = 60;
PBD = 5;
SSawnw = 0.7;
Spulpw = 1-SSawnw;
Use = 1.2;
PSawnw = 110;
PPulpw = 20;
! Calculations of profit per cubic metre finished;
ProfPm3A_LOW = SSawnw*PSawnw + SPulpw*PPulpw - Use*(PAM-PAD);
ProfPm3A_HIGH = SSawnw*PSawnw + SPulpw*PPulpw - Use*(PAM+PAD);
ProfPm3B_LOW = SSawnw*PSawnw + SPulpw*PPulpw - Use*(PBM-PBD);
ProfPm3B_HIGH = SSawnw*PSawnw + SPulpw*PPulpw - Use*(PBM+PBD);
! Volume calculations;
VolA_Alow_Blow = 105*Use;
VolB_Alow_Blow = 90*Use;
VolA_Alow_Bhigh = 100*Use;
VolB_Alow_Bhigh = 280*Use;
VolA_Ahigh_Blow = 380*Use;
VolB_Ahigh_Blow = 70*Use;
VolA_Ahigh_Bhigh = 300*Use;
VolB_Ahigh_Bhigh = 250*Use;
! Profit calculations;
ProfA_ll = ProfPm3A_LOW*VolA_Alow_Blow/Use ;
ProfA_lh = ProfPm3A_LOW*VolA_Alow_Bhigh/Use ;
ProfA_hl = ProfPm3A_HIGH*VolA_Ahigh_Blow/Use ;
ProfA_hh = ProfPm3A_HIGH*VolA_Ahigh_Bhigh/Use ;
ProfB_ll = ProfPm3B_LOW*VolB_Alow_Blow/Use ;
ProfB_lh = ProfPm3B_HIGH*VolB_Alow_Bhigh/Use ;
ProfB_hl = ProfPm3B_LOW*VolB_Ahigh_Blow/Use ;
ProfB_hh = ProfPm3B_HIGH*VolB_Ahigh_Bhigh/Use ;
! Simulation of the system;
! The expected profits per period for players A and B are denoted EA and EB;
 EA(1) = 0;
 @FOR( time(t) | t#GT#1: EA(t) = ProfA_ll*x(t-1)*y(t-1)           +
                         ProfA_lh*x(t-1)*(1-y(t-1))      +
                         ProfA_hl*(1-x(t-1))*y(t-1)      +
                         ProfA_hh*(1-x(t-1))*(1-y(t-1)) ) ;
 EB(1) = 0;
 @FOR( time(t) | t#GT#1: EB(t) = ProfB_ll*x(t-1)*y(t-1)           +
                         ProfB_lh*x(t-1)*(1-y(t-1))      +
                         ProfB_hl*(1-x(t-1))*y(t-1)      +
                         ProfB_hh*(1-x(t-1))*(1-y(t-1)) ) ;
! The expected profits per period for players A and B are changed by
```

```
 EAd and EBd if X or Y are increased by 0.001;
d = 0.001;
 EAd(1) = 0;
 @FOR( time(t) | t#GT#1: EAd(t) = ProfA_ll*(x(t-1)+d)*y(t-1)          +
                              ProfA_lh*(x(t-1)+d)*(1-y(t-1))       +
                              ProfA_hl*(1-x(t-1)-d)*y(t-1)         +
                              ProfA_hh*(1-x(t-1)-d)*(1-y(t-1)) )    ;
 EBd(1) = 0;
 @FOR( time(t) | t#GT#1: EBd(t) = ProfB_ll*x(t-1)*(y(t-1)+d)         +
                              ProfB_lh*x(t-1)*(1-y(t-1)-d)        +
                              ProfB_hl*(1-x(t-1))*(y(t-1)+d)      +
                              ProfB_hh*(1-x(t-1))*(1-y(t-1)-d)  )  ;
! The marginal expected profits per period for players A and B are
  MA and MB if X or Y are increased;
@FOR( time(t) | t#GT#1: MA(t) = (EAd(t) - EA(t))/d );
 @FOR( time(t) | t#GT#1: MB(t) = (EBd(t) - EB(t))/d );
@for(time(t): @FREE(MA(t)));
@for(time(t): @FREE(MB(t)));
! Now, X and Y are increased (or decreased) in case MA and MB are positive (negative);
 @FOR( time(t) | t#GT#1: X(t) = X(t-1) + MA(t)*wA*step  );
 @FOR( time(t) | t#GT#1: Y(t) = Y(t-1) + MB(t)*wB*step  );
! The expected prices of A and B and the expected price difference are calculated;
@FOR( time(t) | t#GT#1: EPA(t) = (PAM-PAD)*x(t) + (PAM+PAD)*(1-x(t))    );
@FOR( time(t) | t#GT#1: EPB(t) = (PBM-PBD)*y(t) + (PBM+PBD)*(1-y(t))    );
@FOR( time(t) | t#GT#1: EPDIFF(t) = EPA(t)-EPB(t)                     );
@for(time(t): @FREE(EPDIFF(t)));
END
```

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