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THE ANALYTICAL RISK OF A TWO STAGE PRETEST ESTIMATOR
IN THE CASE OF POSSIBLE HETEROSCEDASTICITY

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## ABSTRACT

In this paper we evaluate under a squared error loss measure the risk of a two stage pretest estimator (2SPE) for the two sample problem when there is uncertainty concerning both the equality of the location vectors and the scale parameters. Analytical proofs are used to compare the risk performance of the 2SPE with other traditional estimators.



Key Words: Squared Error Loss, Preliminary Test Estimator, Aitken Estimator, Gauss Markov Estimator, risk function, Wald Test Statistic.

# THE ANALYTICAL RISK OF A TWO STAGE PRETEST ESTIMATOR IN THE CASE OF POSSIBLE HETEROSCEDASTICITY*

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Within the context of a two sample linear statistical model, in this paper we consider the problem of testing linear hypotheses between vectors of location parameters when there is uncertainty relative to the equality of the scale parameters. Exact risk properties are derived for the two stage pretest estimator (2SPE) that combines the least squares estimator (OLSE), the two stage Aitken estimator (2SAE) and the Gauss Markov estimator (GME). The risk surface of 2SPE is developed and it is shown analytically that this procedure is superior to the GME estimator for all possible combinations of the variance ratio and location parameter specification errors. Consequently, if one does testing with an eye toward estimation when using squared error loss as a measure of estimator performance, we recommend a two stage testing and estimation procedure, since it is uniformly risk superior to the GME estimator that estimates each location vector directly from each sample of data without testing.

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1. Introduction

Consider the following normal linear statistical model:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & \\ & X_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = X\alpha + e$$

$$e \sim \left[ \begin{matrix} 0_1, \\ 0_2, \end{matrix} \begin{bmatrix} \sigma_{11} I_{n_1} & \\ & \sigma_{22} I_{n_2} \end{bmatrix} \right] \tag{1}$$

where $y_i$ is $(n_i x1)$, $X_i$ is $(n_i xp)$, $\alpha_i$ is $(px1)$, $e_i$ is $(n_i x1)$, $\alpha = (\alpha_1' \ \alpha_2')$, for expository purposes we assume $X_i'X_i = I_{n_i}$.

The estimation problem for the linear two sample heteroscedastic model has been examined before by many authors where it was assumed that the location parameters were unchanged from one sample to the other. For example, Othani and Toyoda (1980) examined, under a mean squared error measure, a pretest estimator after a test for heteroscedasticity. Greenberg (1980) numerically evaluated the sample moments of the same estimator with nonorthonormal regressors. The small sample properties of the two stage Aitken estimator (2SAE) are given by Taylor (1977, 1978) for the same model. All these authors made the assumption of equal location parameters.

Frequently, two samples of economic data may be consistent with different scale parameters and location vectors. Consequently, in this paper, in considering the two sample problem, we relax the assumption of equal location vectors. A familiar test for equality of location vectors is the Chow (1960) test. Toyoda (1974) investigated the accuracy of the Chow test under heteroscedasticity, and found that even with moderate heteroscedasticity the nominal size the test may be quite different than the true size if both samples are small. Schmidt (1977) redid Toyoda's calculations using the exact distribution of the Chow test under varying degrees of inequality of scale parameters. Hence, both authors indicated the lack of robustness of the Chow test under heteroscedasticity. Othani (1987) considered the bias and power of a two-stage test involving both the location and scale parameters.

Given uncertainty about the magnitude of both the location and scale parameters, we specify and evaluate the sampling performance of a two stage pretest estimator (2SPE) that combines the least squares estimator (OLSE), the two step Aitken estimator (2SAE), and the Gauss Markov (GME) estimator.

The plan of the paper is as follows: in Section 2 we define the 2SPE and discuss the corresponding estimators and present the risk characteristics of each. In Section 3, the risk of the 2SPE is explored conditional on the estimates of the sample variances. Section 4 contains the unconditional risk of the 2SPE, and in Section 5, the evaluation of the risk performance of the 2SPE is presented and contrasted to the other estimators. A summary and the important conclusions are presented in Section 6. The derivations of the theorems presented in Sections 3 and 4 are given in the Appendices A through E.

## 2. Estimators and tests

A traditional way of estimating the location parameters is by using the OLSE:

$$\underset{\sim}{\alpha}^*(1) = \begin{bmatrix} (X_1'y_1 + X_2'y_2)/2 \\ (X_1'y_1 + X_2'y_2)/2 \end{bmatrix} \qquad (2)$$

This estimator, $\underset{\sim}{\alpha}^*(1)$ is biased (unless $\underset{\sim}{\alpha}_1 = \underset{\sim}{\alpha}_2$), and has a risk (or expected squared error loss)

$$E||(\alpha^*(1) - \alpha)||^2 = R(\alpha^*(1),\alpha) = (p/2)(1+\tau)\sigma_{22} + \pi'\pi/2 \qquad (3)$$

where $\tau$ is the variance ratio, $\sigma_{11}/\sigma_{22}$, and $\pi = \alpha_1 - \alpha_2$, is a (px1) vector of specification errors.

Alternatively, following Taylor (1977, 1978) the 2SAE,

$$\alpha^*(2) = \begin{bmatrix} \Theta X_1'y_1 + (1-\Theta)X_2'y_2 \\ \Theta X_1'y_1 + (1-\Theta)X_2'y_2 \end{bmatrix} \qquad (4)$$

may be used, where $\Theta = s_{22}/(s_{11}+s_{22})$, and $s_{ii} = SSE_i/(n_i-p)$ is an unbiased estimator of $\sigma_{ii}(i=1,2)$. Taylor found this procedure efficient relative to OLSE and GME. The risk properties of this estimator is different under our version of the model, since in case $\alpha_1 \neq \alpha_2$, the 2SAE is biased and has an unbounded risk as the specification error, $\pi$ goes to infinity. Therefore, in our case the risk of the 2SAE is

$$E||\alpha^*(2)-\alpha||^2 = \sigma_{22}^2 p\{\theta^2\tau + (1-\theta)^2\} + (2\theta^2 + 1 - 2\theta)\pi'\pi \qquad (5)$$

The derivation in (5) is given in Ozcam (1987). The distribution of $\theta$, for the nonorthonormal case is derived by Taylor (1978). If the risk of (5) is numerically integrated with respect to the density of $\theta$, it reduces to the risk of 2SAE when $\pi = 0$.

In view of the possible inequality of the location and scale parameters, a third candidate, the GME,

$$\alpha^*(3) = \begin{bmatrix} X_1'y_1 \\ X_2'y_2 \end{bmatrix} \qquad (6)$$

is minimax under squared error loss, and has a risk

$$E||\alpha^*(3),\alpha||^2 = p(1+\tau)\sigma_{22} \qquad (7)$$

In formulating the 2SPE, we make use of the following test statistics: Goldfeld and Quandt (GQ)= $SSE_1/SSE_2$, Wald (W) = $(X_1'y_1 - X_2'y_2)'(X_1'y_1 - X'_2y_2)/(s_{11}+s_{22})$, and Chow (CH) = $(X_1'y_1 - X_2'y_2)'(X_1'y_1-X_2'y_2)\ \frac{(n-2p/2p)}{(SSE_1+SSE_2)}$, where $n=n_1+n_2$.

## 3. The Conditional Risk of 2SPE

Within the context of section 2, the two stage pretest estimator (2SPE) is defined as follows:

(i) Complete separate regressions on each of the two samples, and test $H_{o1}$: $\sigma_{11} = \sigma_{22}$ by using the Goldfeld and Quandt (GQ), (1965) test statistic which under the null is distributed as F(p, n-2p).

(ii) If in step (i) we conclude $\sigma_{11} = \sigma_{22}$ test $H_{o2}$: $\underset{\sim}{\alpha}_1 = \underset{\sim}{\alpha}_2$ (versus $\underset{\sim}{\alpha}_1 \neq \underset{\sim}{\alpha}_2$) by using the Chow (CH) (1960), test statistic, or test $H_{o2}$ by using the Wald (W) test statistic if in step (i) we reject the null hypothesis of equality of variances. The reason for not using the CH test statistic in (ii) is the well known non-robustness of CH when the scale parameters are different (Toyoda (1974) and Schmidt (1977)). Othani and Toyoda (1985) using Monte Carlo sampling experiments has examined the small sample properties of the Wald, the Lagrange Multiplier and the Likelihood Ratio tests. They find that the Wald and the likelihood ratio tests have an upward bias in the size, while the Lagrange Multiplier test tends to have a downward bias.

Under this specification the two stage pretest estimator (2SPE) is

$$\alpha^*(2SPE) = \underset{\sim}{\alpha}^*(1) \text{ if } \begin{cases} 0 \leqq GQ \leqq c_1 \\ 0 \leqq CH \leqq c_2 \end{cases}$$

$$= \alpha^*(3) \text{ if } \begin{cases} 0 \leqq GQ \leqq c_1 \\ c_2 < CH < \infty \end{cases}$$

$$= \alpha^*(2) \text{ if } \begin{cases} c_1 < GQ < \infty \\ 0 \leqq W \leqq c_3 \end{cases}$$

$$= \alpha^*(3) \text{ if } \begin{cases} c_1 < GQ < \infty \\ c_3 < W < \infty \end{cases}$$

or

$$\underset{\sim}{\alpha}*(2SPE) = I^{(GQ)}[0,c_1]I^{(CH)}[0,c_2]\underset{\sim}{\alpha}*(1) + I^{(GQ)}[0,c_1]I^{(CH)}(c_2,\infty)\underset{\sim}{\alpha}*(3) + I^{(GQ)}(c_1,\infty)I^{(W)}[0,c_3]\underset{\sim}{\alpha}*(2) + I^{(GQ)}(c_1,\infty)I^{(W)}(c_3,\infty)\underset{\sim}{\alpha}*(3) \quad (8)$$

where $c_1$, $c_2$ and $c_3$ are critical values of the GQ, CH and W test statistics, respectively, and $I^{(\cdot)}$ is a zero-one indicator function. In other words, this means that $\underset{\sim}{\alpha}*(2SPE)$is comprised of the GME If we reject $H_{o2}$ (whatever the outcome of the first test), or it is the OLSE if we accept both $H_{o1}$ and $H_{o2}$, or it is the 2SAE if we reject $H_{o1}$, but accept $H_{o2}$.

All cross products vanish since $I_{[o,a]}I_{(a,\infty)} = 0$ for all $a\varepsilon R^+$. The 2SPE has risk

$$E||(\alpha*(2SPE) - \underset{\sim}{\alpha})||^2 = E(I^{(GQ)}[0,c_1]I^{(CH)}[0,c_2]||(\alpha*(1) - \underset{\sim}{\alpha})\ ||^2) + E(I^{(GQ)}[0,c_1]I^{(CH)}(c_2,\infty)||(\alpha*(3) - \underset{\sim}{\alpha})\ ||^2) + E(I^{(GQ)}(c_1,\infty]I^{(W)}(0,c_3)||(\alpha*(2) - \underset{\sim}{\alpha})\ ||^2) + E(I^{(GQ)}(c_1,\infty]I^{(W)}(c_3,\infty)||(\alpha*(3) - \underset{\sim}{\alpha})\ ||^2 \quad (9)$$

In order to evaluate (9) we need the following Theorem.

<u>Theorem A</u>: For the linear statistical model specified in (1), the conditional risk of 2SPE (conditioned on $s_{11}$, $s_{22}$), given in (8) is

$$R(\alpha*(2SPE), \alpha|s_{11}, s_{22}) = \sigma p - E(I^{(GQ)}[0, c_1]\ \{\sigma/2\ p\ pr(\chi^2{}_{(p+2,\delta)} < c_2^*) + \pi'\pi/2\ pr(\chi^2{}_{(p+4,\delta)} < c_2^*) - \pi'\pi\ pr(\chi^2{}_{(p+2\delta)} < c_2^*)\} + E(I^{(GQ)}(c_1,\infty)(-2((1-\theta)\sigma_{11}+\theta\sigma_{22})+\sigma(1+2\theta^2-2\theta)) \{p\ pr(\chi^2_{(p+2,\delta)}<c_3*)+\pi'\pi/\sigma\ pr(\chi^2{}_{(p+4,\delta)}<c_3*)\}) + E(I(GQ)(c_1,\infty)2\pi'\pi/\sigma((1-\theta)\sigma_{11} + \theta\sigma_{22}) pr(\chi^2{}_{(p+2,\delta)} < c_3*)). \quad (10)$$

For the convenience of the reader, the derivation of 10 is shown in Appendix A, where $c_2^*$, $c_3^*$ are some stochastic critical values and $\chi^2{}_{(p,\delta)}$ is a noncentral chi square random variable with p degrees of freedom, and a noncentrality parameter of $\delta$. We now turn to the unconditional risk of the 2SPE. In the proofs we use the procedure of conditioning on the estimates of the sample variances.

## 4. The Unconditional Risk of 2SPE.

The unconditional risk of 2SPE is given in the next theorem, and its derivation is shown in Appendices B, C, D and E.

Theorem B. For the linear statistical model of (1), the unconditional risk of 2SPE is

$$
\begin{aligned}
R(\alpha^*(2SPE), \alpha) = {} & \sigma_{22}(1+\tau)p + (\pi'\pi-\sigma_{22}(1+\tau)p/2)H1 - \pi'\pi/2\, H1' \\
& - 2p(\sigma_{11} H2 - \sigma_{11}H3 + \sigma_{22}H3) - 2\,\pi'\pi/\sigma_{22}(1+\tau) \\
& (\sigma_{11} H2' - \sigma_{11}H3' + \sigma_{22}H3') + \sigma_{22}(1+\tau)p(H2 \\
& + 2H4 - 2H3) + \pi'\pi(H2' + 2H4' - 2H3') \\
& + 2\pi'\pi/\sigma_{22}(1+\tau)(\sigma_{11}H2 - \sigma_{11}H3 + \sigma_{22} H3) \qquad (11)
\end{aligned}
$$

where

$$
\begin{aligned}
H1 &= E[I^{(GQ)}_{[0,c_1]}\, pr(\chi^2_{(h,\delta)} < c_2^*)] \\
H2 &= E[I^{(GQ)}_{[0,c_1]}\, pr(\chi^2_{(h,\delta)} < c_3^*)] \\
H3 &= E[I^{(GQ)}_{(c_1,\infty)}\, \theta\, pr(\chi^2_{(h,\delta)} < c_3^*)] \\
H4 &= E[I^{(GQ)}_{(c_1,\infty)}\, \theta^2\, pr(\chi^2_{(h,\delta)} < c_3^*)]
\end{aligned}
$$

and $H_i'$ (i=1,2,3,4) are the same expressions where the degrees of freedoms are p+4. These four expectations are derived in Appendices B, C, D and E, respectively. The risk in (11) depends on the specification error, on the variance ratio and the critical values used for testing the equality of the scale and location parameters. As the specification error $\pi$ grows larger the noncentrality parameter of the chi square random variable grows, and all eight expectations $H_i$, $H_i'$ (i=1,2,3,4) go to zero, since the probabilities inside the expectations go to zero.

Consequently, as $\delta$ goes to infinity, all the terms, except the first one, go to zero. The remaining term $\sigma_{22}(1+\tau)p$, is the risk of the GME.

5. Risk Performance of 2SPE

This section contains graphs of the risk of 2SPE and a comparison of its risk to that of the GME, the 2SAE and the OLSE. The risk function in Theorem B depends on both $\tau$ (variance ratio) and $\pi$ (location vector specification error). To present the risk characteristics of the 2SPE with respect to the other estimators, we cut the risk surface in two planes. Figure 1 shows the risk functions at the origin where, $\pi=0$, and Figure 2 displays them along the specification error axis for a fixed value of the variance ratio ($\log \tau = 2$).

In Figure 1, we have drawn the relative risks of the estimators, in logarithm form, with respect to the risk of the generalized least squares estimator (GLS)

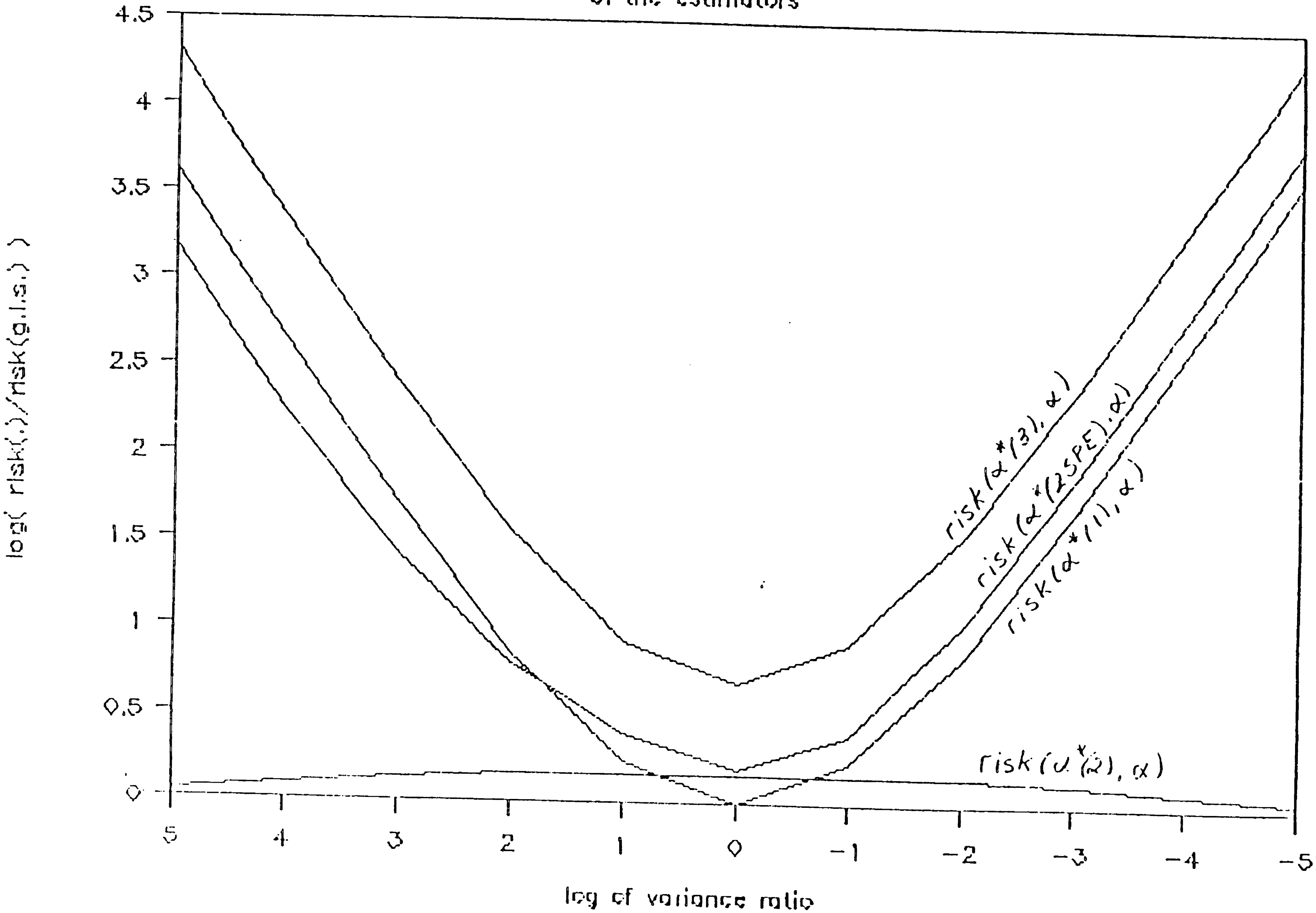
$$\frac{\sigma_{22}}{\sigma_{11} \quad \sigma_{22}} X_1'y_1 + \frac{\sigma_{11}}{\sigma_{11} \quad \sigma_{22}} X_2'y_2$$

when the scale parameters are assumed known. The GLS estimator is more efficient at the origin than the other four estimators because it uses the unknown population variances. However, it is biased since it combines both samples and consequently has unbounded risk as the specification error $\pi$ grows.

The reason for considering the relative risks is the desire to be able to reproduce the risk of Taylor's 2SAE within our model. The risk of the GLS serves as the unit of measurement in Figure 1. Taylor works with nonorthonormal regressors and uses a transformation which renders the risk of the Gauss Markov estimator to be one. Here, the relative risk of generalized least squares (relative to itself) is also one. Also, note that the Gauss Markov estimator for Taylor's model is the

# FIGURE 1 Relative Risk functions

of the estimators

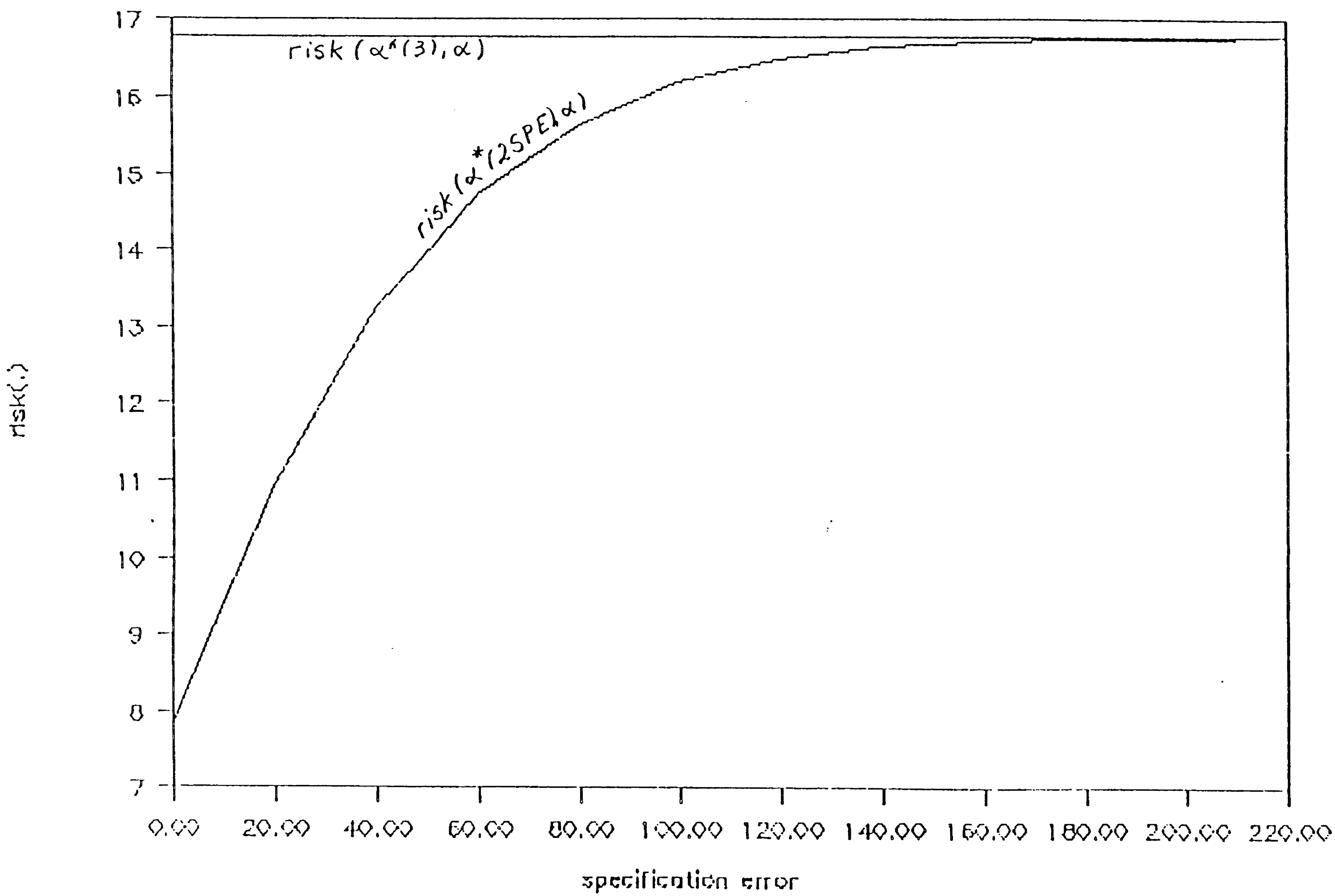


generalized least squares which is a linear combination of the OLS estimators (given above), whereas, in the case of location vectors that are possibly different, it is just the OLS estimators for each sample.

In Figure 1, the relative risk of the feasible 2SAE is highest around $\tau = 1$ (or $\log \tau = 0$), i.e., when $\sigma_{11} = \sigma_{22}$. As found by Taylor the risk converges to zero as the variances depart from equality. The 2SAE performs quite well relative to the GME, especially when the variances differ. However, under our specification of the model, namely under possible unequal location coefficients between samples, the estimator becomes biased and has an unbounded risk as the location specification error grows.

The OLS is also biased. The relative risk of this estimator is zero when the variances are equal, but becomes unbounded as $\log \tau$ differs from zero. The relative risk of GME is highest at the origin for all values of $\tau$, because the estimator recognizes the possibility that $\underset{\sim}{\alpha}_1 - \underset{\sim}{\alpha}_2 \neq 0$ and applies the least squares procedure to each sample separately. Thus the GME remains in the class of unbiased estimators. Finally, the relative risk of the 2SPE is higher than the relative risk of the 2SAE, and when $\log \tau < 0$, its risk lies between the relative risks of OLS and GM. When $\log \tau > 0$ the risk function for the 2SPE crosses the risk functions of the OLS and GM estimators. Since the pretest estimator is actually some combination of three estimators, as one would expect, its relative risk is located between their relative risks. The relative risk of the 2SPE is higher on the right of $\log \tau = 0$ than it is on the left. This results because we used a one-sided critical region for GQ test and, when indeed $\sigma_{11} < \sigma_{22}$, the power of the test is low.

FIGURE 2 Risk functions of estimators



The sizes of both GQ and the CH tests are set at .05. However, since the exact distribution of the W is not known, we used its asymtotic critical value.

The main result of this paper is contained in Figure 2. Not only does the 2SPE perform better than GM at the origin, but it also performs better along the specification errror parameter space. As $\pi'\pi$ grows, $\delta=\pi'\pi/2\sigma$ goes to infinity, and all eight expectations, ($H_i$, $H_i'$, i = 1, 2, 3, 4) go to zero, since the cumulative probabilities for the noncentral chi square random variables vanish at different rates as the noncentrality parameter goes to infinity. Therefore, the risk in Theorem B converges to the risk of GME staying below it over the whole parameter space. That is, $R(\alpha^*(2SPE),\underset{\sim}{\alpha}) \leqq R(\alpha^*(GME),\underset{\sim}{\alpha})$ for all $\pi'\pi$ that were evaluated. Consequently, the uniform superiority of 2SPE over GME for all combinations of ($\tau$, $\pi'\pi$) is conjectured. Consequently by going outside the class of linear, unbiased estimators, we have shown a procedure to uniformly improve on the Gauss Markov estimator.

Of the three estimators that comprise the 2SPE, only the GME is unbiased. Consequently both the 2SAE and the OLSE have unbounded risks as the specification error grows. Therefore, the risk advantages of the 2SAE and the OLSE over the 2SPE at the origin disappear as the location parameter structure of the first sample differs more and more from that of the second sample.

## 6. Summary and Conclusion

In this paper, the exact risk properties of the 2SPE, that is comprised of the ordinary least squares, the Gauss Markov, and the two stage Aitken estimators, is evaluated . Under squared error loss, for the possibly heteroscedastic two sample model, where the location parameters are not necessarily the same, we have shown that the two stage testing and estimation procedure is uniformly superior to GME. Therefore, the 2SPE is recommended over the location and scale parameter spaces.

The OLS and the 2SAE estimators have risk advantages over the GME at the origin. However, both the OLS and the 2SAE estimators are biased, and have unbounded risks. Consequently, as the specification error grows their risk functions cross the GME risk function.

Given this base it would be interesting to replace the Wald test statistic with the Lagrange Multiplier or the Likelihood ratio test statistics and to compare the risks of the two-stage pretest estimators that result. A non-diagonal error variance-covariance matrix often exists in practice. Results for the correlated samples case are discussed in Ozcam (1987). Finally, it should be noted that the assumption of orthonormal regressors can be eliminated, and the risk properties of the 2SPE can be explored using the techniques presented.

## APPENDIX A

To start the risk evaluation of 2SPE, we take the four expectations in (9) two at a time (define them as E1 and E2). Conditioning on the estimated variances $s_{11}$ and $s_{22}$ the first two give

$$\begin{aligned} E1 &= E(I^{(GQ)}[0,c_1]E(I^{(CH)}[0,c_2]||^2|s_{11},s_{22}) \\ &+ E(I^{(GQ)}[0,c_1]E(I^{(CH)}(c_2,\infty)||(\alpha^*(3) - \alpha)||^2| \\ & s_{11},s_{22})) = E(I^{(GQ)}[0, c1]E(I^{(CH)}{}_{(0,c_2)}|| \\ & (\alpha^*(3) - R'R\alpha^*(3)'/2 - \alpha)||^2| s_{11},s_{22})) \\ &+ E(I^{(GQ)}[0, c_1]E(I^{(CH)}(c_2,\infty)||(\alpha^*(3) - \alpha) ||^2| \\ & s_{11}, s_{22})) \end{aligned} \tag{13}$$

Since $\alpha^*(1) = \alpha^*(3) - R'R\alpha^*(3)/2$, where $R = (I_p - I_p)$ is a $(p \times 2p)$ restriction matrix. Dropping the outside expectations and the term $I^{(GQ)}{}_{[0,c1]}$ momentarily for convenience, and expanding the first term in (13) we obtain

$$\begin{aligned} E1 &= (\sigma_{11} + \sigma_{22})p - E(I^{(CH)}{}_{[0,c2]}(\alpha^*(3) - \alpha)'R'R \\ & \alpha^*(3)) + 1/2\ E(I^{(CH)}{}_{[0,c2]}\alpha^*(3)'R'R\alpha^*(3)) \end{aligned} \tag{14}$$

The CH test statistic for our orthonormal heteroscedastic model is

$$\begin{aligned} CH &= (n-2p)(X_1'Y_1 - X_2'Y_2)'(X_1'Y_1 - X_2'y_2)'\ (X_1'Y_1 - X_2'Y_2)/2\ p\ s \\ &= (n-2p/2\ p\ s)(R\alpha^*(3))'(R\alpha^*(3)) \end{aligned} \tag{15}$$

where $s=(n_1-p)s_{11} + (n_2-p)s_{22}$, and $n=n_1+n_2$. Define $w = R\alpha^*(3) = X_1'Y_1 - X_2'Y_2$ and $\sigma = \sigma_{11} + \sigma_{22}$. Then $w/\sqrt{\sigma}$ is distributed as $N(\pi/\sqrt{\sigma}, I_p)$. Inserting the value of the CH statistic in (15) we obtain

$$\begin{aligned} E1 &= \sigma p - \sigma/2\ E(I^{(w'w/\sigma)}{}_{[0,c2^*]}w'w/\sigma) \\ &+ \sqrt{\sigma}\ \pi'\ E(I^{(w'w/\sigma)}{}_{[0,c2^*]}\ w/\sqrt{\sigma}) \end{aligned} \tag{16}$$

where $c_2^* = c_2 2ps/((n-2p)\ \sigma)$. Now using the theorems in Judge and Bock (1978, p. 321), we obtain

$$E1 = \sigma\ p - (\sigma/2)\ p\ pr\ (\chi^2{}_{p+2,\ \delta} < c_2^*) - \pi'\pi/2\ pr(\chi^2{}_{p+4,\ \delta} < c_2^*)$$

$$+ \pi'\pi \; pr(\chi^2_{p+2,\ \delta} < c_2^*) \qquad (17)$$

where $\chi^2_{(h,\delta)}$ is a noncentral chi squared random variable with h degrees of freedom and the noncentrality parameter $\delta = \pi'\pi/2\sigma$. This completes the evaluation of the conditional expectation of the first two terms in (9).

We now turn to the derivation of the conditional expectation of the last two terms in (9).

$$\begin{aligned} E2 = {} & E(I^{(GQ)}_{(c_1,\infty)} I^{(W)}_{[0,c_3]} \,||\, (\alpha^*(2) - \alpha) \,||^2) + \\ & E(I^{(GQ)}_{(c_1,\infty)} I^{(W)}_{(c_3,\ \infty)} \,||\, (\alpha^*(2) - \alpha)||^2) = \\ & E(I^{(GQ)}_{(c_1,\infty)} \sigma p - 2E(I^{(GQ)}_{(c_1,\ \infty)} \\ & E(I^{(W)}_{(0,c_3)} (\alpha^*(3) - \alpha)' M \alpha^*(3) | S_{11}, S_{22})) \\ & + E(I^{(GQ)}_{(c_1,\ \infty)} E(I^{(W)}_{[0,c_3]} \alpha^*(3)'M'M\alpha^*(3) \\ & | S_{11}, S_{22})) \end{aligned} \qquad (18)$$

In (18), the second equality follows because $\alpha^*(2) = \alpha^*(3) - M\,\alpha^*(3)$, i.e., 2SAE can be written in terms of GM. $M = \begin{bmatrix} (1-\theta)I_p & \\ & \theta I_p \end{bmatrix}$, where R'R is a (2px2p) matrix. Again we drop the outside expectations and the term $I^{(GQ)}_{(c_1,\infty)}$ momentarily for simplicity, and we get

$$\begin{aligned} E2 = {} & \sigma p - 2E(I^{(W)}_{[0,c3]} (\alpha^*(3) - \alpha)' \Sigma^{-\frac{1}{2}} Q'Q\Sigma^{\frac{1}{2}} \\ & \begin{bmatrix} (1-\theta)I_p & \\ & \theta I_p \end{bmatrix} \Sigma^{-\frac{1}{2}} Q'Q\Sigma^{\frac{1}{2}} R'R\Sigma^{\frac{1}{2}} Q'Q\Sigma^{-\frac{1}{2}} \\ & \alpha^*(3)) + E(I^{(W)}_{[0,\ c_3]} \alpha^*(3)' \Sigma^{-\frac{1}{2}} Q'Q\Sigma^{\frac{1}{2}} R'R \\ & \Sigma^{\frac{1}{2}} Q'Q\, \Sigma^{-\frac{1}{2}} \begin{bmatrix} (1-\theta)^2 I_p & \\ & \theta^2 I_p \end{bmatrix} \Sigma^{-\frac{1}{2}} Q'Q\, \Sigma^{\frac{1}{2}} R'R \\ & \Sigma^{\frac{1}{2}} Q'Q\, \Sigma^{-\frac{1}{2}} \alpha^*(3)) \end{aligned} \qquad (19)$$

In (19), $Q = \begin{bmatrix} -\sqrt{\sigma_{11}}/\sqrt{\sigma} I_p & \sqrt{\sigma_{22}}/\sqrt{\sigma} I_p \\ \sqrt{\sigma_{22}}/\sqrt{\sigma} I_p & \sqrt{\sigma_{11}}/\sqrt{\sigma} I_p \end{bmatrix}$, is a (2px2p) symmetric orthogonal matrix that diagonalizes $\Sigma^{\frac{1}{2}} R'R\, \Sigma^{\frac{1}{2}}$ (2px2p). Thus

$Q\ \Sigma^{\frac{1}{2}}\ R'R\ \Sigma^{\frac{1}{2}}\ Q' = \begin{bmatrix} \sigma I_p & \\ & 0_p \end{bmatrix}$ where $\Sigma^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\sigma_{11}} I_p & \\ & \sqrt{\sigma_{22}} I_p \end{bmatrix}$ is a (2px2p)

symmetric matrix. Inserting these values in (19), we obtain (define $m = Q\ \Sigma^{-\frac{1}{2}}\ \alpha*(3)$ and $Q\ \Sigma^{-\frac{1}{2}}\ \alpha = u$)

$$E2 = \sigma p - 2\sigma E\left(I^{(W)}_{[0,c3]}\ (m-u)'\ A \begin{bmatrix} I_p & \\ & I_p \end{bmatrix} m\right) +$$

$$\sigma^2\ E\left(I^{(W)}_{[0,c3]}\ m' \begin{bmatrix} I_p & \\ & 0_p \end{bmatrix} B \begin{bmatrix} I_p & \\ & 0_p \end{bmatrix} m\right) \qquad (20)$$

where $A = Q\ \Sigma^{\frac{1}{2}} \begin{bmatrix} (1-\theta) I_p & \\ & \theta I_p \end{bmatrix} \Sigma^{-\frac{1}{2}}\ Q'$ and $B = Q\ \Sigma^{-\frac{1}{2}} \begin{bmatrix} (1-\theta)^2\ I_p & \\ & \theta^2 I_p \end{bmatrix} \Sigma^{-\frac{1}{2}} Q'$.

Partition m and u into two (px1) vectors, $m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ and $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and A and B into four submatrices of dimensions (pxp) each, i.e., $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ $B = \begin{bmatrix} B_1 & B_2 \\ B_2' & B_4 \end{bmatrix}$. Note that m is distributed as Normal $(u, I_{2p})$, since $\Phi_m = Q\Sigma^{-\frac{1}{2}}\Phi_{\alpha*(3)}\Sigma^{-\frac{1}{2}}Q' = I_{2p}$. Where $\Phi_{(\cdot)}$ is the variance-covariance matrix of (°). (Also note that $u_1 = \pi/\sqrt{\sigma}$.) Now using these partitioned values, (20) becomes

$$E2 = \sigma p - 2\sigma E\left(I^{(W)}_{[0,c_3]}\ (m_1'm_1\ A_1 + m_2'm_1\ A_2 - u_1'\ m_1\ A_1 - u_2'\ m_1 A_2)\right) + \sigma^2 E\left(I^{(W)}_{[0,c_3]}\ (m_1'\ m_1\ B_1)\right) \qquad (21)$$

The evaluation of the risk function now requires a reformulation of the Wald test statistic (W), which appears in the argument of the indicator functions.

$$W = (R\alpha*(3))'\ R\alpha*(3)/d = \alpha*(3)'\ \Sigma^{-\frac{1}{2}}\ Q'Q\ \Sigma^{\frac{1}{2}}\ R'R\ \Sigma^{\frac{1}{2}}\ Q'Q\ \Sigma^{-\frac{1}{2}}\ \alpha*(3)/d = m' \begin{bmatrix} I_p\sigma & \\ & 0_p \end{bmatrix} m/d = m_1'm_1\sigma/d \qquad (22)$$

We can now place this new equivalent value of (W) in the argument of the Indicator functions with $c_3* = c_3 d/\sigma$. $(d=s_{11}+s_{22})$

$$E2 = \sigma p - 2\,\sigma E(I^{(m_1'm_1)}_{[0,c3*]}(m'_1{}'m_1A_1 + m_2'm_1A_2$$

$$- u_1'm_1A_1 - u_2'm_1A_2)) + \sigma^2 E(I^{(m_1'm_1)}_{[0,c_3*]}$$

$$m_1'm_1B_1) \tag{23}$$

Using the independence of $m_1$ and $m_2$, we see that the second term in the first bracket in (23) cancels the fourth term. Also straightforward matrix multiplication gives $A_1 = ((1-\theta)\sigma_{11} + \theta\sigma_{22}/\sigma)I_p$ and $B_1 = (1+2\theta^2 - 2\theta/\sigma)I_p$, hence

$$E2 = \sigma p + \{-2\ ((1-\theta)\sigma_{11} + \theta\sigma_{22}) + \sigma(1+2\theta^2 - 2\theta))\}$$

$$E(I^{(m_1'm_1)}_{[0,c3*]}(m_1'm_1)) + 2\ u_1'(1-\theta)\sigma_{11} + \theta\sigma_{22})$$

$$E(I^{(m_1'm_1)}_{[0,\ c3*]}m_1) \tag{24}$$

Referring to the theorems in Judge and Bock (1978, p. 321), we obtain the final form of the evaluation of the inside expectations in (9).

$$E2 = \sigma p + (-2\ ((1-\theta)\sigma_{11} + \theta\sigma_{22}) + \sigma(1+2\theta^2 - 2\theta))$$

$$\{p\ pr(\chi^2_{p+2,\delta} < c_3*) + \pi'\pi/\sigma\ pr(\chi^2_{p+4,\ \delta} < c_3*)\}$$

$$+ 2\ \pi'\pi/\sigma\ ((1-\theta)\sigma_{11} + \theta\sigma_{22})\ pr(\chi^2_{p+2,\ \delta} < c_3*) \tag{25}$$

Putting E1 in (17) and E2 in (25) together, we finish the risk evaluation of the 2SPE for the inside expectations. Hence the conditional risk of 2SPE (conditioned on $s_{11}$ and $s_{22}$) is

$$R(\alpha*(2SPE),\ \alpha|s_{11},s_{22}) = \sigma\ p - \sigma/2p\ pr(\chi^2_{(p+2,\ \delta)} < c_3*)$$

$$+ \pi'\pi/2\ pr(\chi^2_{(p+4,\delta)} < c_3*)$$

$$+ \pi'\pi\ pr(\chi^2_{(p+2,\delta)} < c_3*)$$

$$+ \{-2\ ((1-\theta)\sigma_{11} + \theta\sigma_{22})$$

$$+ \sigma(1+2\theta^2-2\theta)\}[p\ pr(\chi^2_{(p+2,\delta)} < c_3*)$$

$$+ \pi'\pi/\sigma pr(\chi^2_{(p+4,\delta)} < c_3*)]$$

$$+ 2\ \pi'\pi/\sigma\ ((1-\theta)\sigma_{11}$$

$$+ \theta\sigma_{22})\ pr(\chi^2_{(p+2,\delta)} < c_3*) \tag{26}$$

and the unconditional risk, recovering the outside expectations that we dropped earlier, is

$$R(\alpha^*(2SPE), \alpha) = \sigma\, p - E(I^{(GQ)}_{[0, c_1]}\{\sigma/2\; p\; pr(\chi^2_{(p+2,\ \delta)} < c_2^*)$$
$$+ \pi'\pi/2\; pr(\chi^2_{(p+4, \delta)} < c_2^*)$$
$$- \pi'\pi\; pr(\chi^2_{(p+2, \delta)} < c_2^*)\} + E(I^{(GQ)}_{(c_1,\ \infty)}$$
$$(-2((1-\theta)\sigma_{11} + \theta\sigma_{22}) + \sigma(1+2\theta^2-2\theta))$$
$$\{p\; pr(\chi^2_{(p+2, \delta)} < c_3^*) + \pi'\pi/\sigma\; pr(\chi^2_{(p+4, \delta)} < c_3^*)\})$$
$$+ E(I^{GQ)}(c_1,\ \infty)\; 2\; \pi'\pi/\sigma\; ((1-\theta)\sigma_{11} + \theta\sigma_{22})$$
$$pr(\chi^2_{(p+2,\ \delta)} < c_3^*)) \qquad (27)$$

This completes the derivation of the conditional risk function of 2SPE. The next appendix gives the unconditional risk of the 2SPE.

APPENDIX B

To free the risk in (27) from the expectation terms, we distinguish 4 types of expectations:

i) $H1 = E[I^{(GQ)}_{[0,c_1]}\ pr(\chi^2_{(h,\delta)} < c_2^*)]$

ii) $H2 = E[I^{(GQ)}_{[0,c_1]}\ pr(\chi^2_{(h,\delta)} < c_3^*)]$

iii) $H3 = E[I^{(GQ)}_{(c_1,\ \infty)}\ \theta\ pr(\chi^2_{(h,\delta)} < c_3^*)]$

iv) $H4 = E[I^{(GQ)}_{(c_1,\ \infty)}\ \theta^2\ pr(\chi^2_{(h,\delta)} < c_3^*)]$

for h =(p+2, p+4.)

We present the derivation of (i) here, and leave the derivations of (ii), (iii), and (iv) to the Appendixes C, D, and E, since the derivations are more or less similar.

The derivation of (i):

We follow procedures similar to the ones outlined in a paper by Lauer and Han (1972) which derives formulas for the computation of the joint distribution of certain ratios of $\chi^2$ random variables.

Define $GQ = Y_1 = s_{11}/s_{22} = X_1/gX_2$

$$Y_2 = s_{11}\ V_1 + s_{22}\ V_2 = \sigma_{11}X_1 + \sigma_{22}\ X_2$$

where $g = V_1/\tau V_2$, $V_i = n_i - p$ $(i=1,2)$ and $X_1$ and $X_2$ are two independent chi square random variables. Also write the stochastic critical value $c_2^*$ as the product of the random component and the fixed component, i.e., $c_2^* = (c_2 2pY_2)/((n-2p)\sigma) = rY_2$ where $r = (c_2 2p)/((n-2p)\sigma)$. The H1 becomes

$$H1 = \int_0^\infty \int_0^{c_1} f(Y_1,\ Y_2)\ pr(\chi^2_{(h,\delta)} < r\ Y_2)\ dY_1\ dY_2 \qquad (28)$$

where $f(\cdot)$ is the joint distribution of ratios of two independent chi square random variates. We use the following transformation with the Jacobian $J=1/\sigma_{22}$.

$$z_1 = X_1 \text{ and } z_2 = \sigma_{11}X_1 + \sigma_{22}X_2 = Y_2 \qquad (29)$$

Then H1 becomes

$$H1 = \int_{Area}\int 1/\sigma_{22}\, y(z_1, z_2 - \sigma_{11}z_1/\sigma_{22})\, pr(\chi^2_{(h,\delta)} < r\, z_2)\, dz_1\, dz_2 \quad (30)$$

where $y(\circ)$ is the joint distribution of two independent chi square variates, Area = $\{z_1, z_2 > 0,\ z_1(\sigma_{22} + c_1 g\, \sigma_{11}) < c_1 g\, z_2$, or $z_1 < b_1 z_2$ with $b_1 = c_1 g/(\sigma_{22} + c_1 g\sigma_{11})\}$. Writing out the joint density $y(\circ)$ ,

$$H1 = \int_0^\infty \int_0^{b_1 z_2} k^*/\ \sigma_{22}^{q_2+1}\ z_1^{q_1}(z_2 - \sigma_{11}z_1)^{q_2} \exp(-\tfrac{1}{2}(z_1 + (z_2 - \sigma_{11}z_1/\sigma_{22})))$$

$$pr(\chi^2_{h,\delta} < r\, z_2)\, dz_1\, dz_2 \quad (31)$$

where $q_i = v_i/2 - 1$, $k^* = (\Gamma(v_1/2)\ \Gamma(v_2/2)\ 2^{(v_1+v)2/2})^{-1}$, for $i = 1, 2$. Assume $v_i$ is an even integer, then $(z_2 - \sigma_{11}z_2/\sigma_{22})^{q2}$ can be expanded as a binomial since $q_2$ is an integer when $v_i$ is even. Take $\sigma_{11} \neq \sigma_{22}$, otherwise the argument of the exponential function is zero in (31), and integrate $z_1$ out directly.

$$H1 = \int_0^\infty \int_0^{b_1 z_2} k^*/\ \sigma_{22}^{q_2+1} \sum_{i=0}^{q_2} \binom{q_2}{i} (-\sigma_{11})^{q_2-i}\ z_1^{q_1+q_2+i}$$

$$z_2^{\,i}\quad \exp(-z_1(1-\tau)/2)\quad \exp(-z_2/2\sigma_{22})$$

$$pr(\chi^2_{(h,\delta)} < r\, z_2)\, dz_1\, dz_2 \quad (32)$$

Make the following transformation, $m_1 = b_2 z_1$ and $m_2 = z_2$ where $b_2 = (1-\tau)/2$. Then

$$H1 = \int_0^\infty \int_0^{b_1 b_2 m_2} k^*/\ \sigma_{22}^{q_2+1} \sum_{i=0}^{q_2} \binom{q_2}{i} (-\sigma_{11})^{q_2-i}\ (1/b_2)^{q_1+q_2-i+1}$$

$$m_2^{i} \exp(-m_2/2\sigma_{22})\ m_1^{q_1+q_2-i}\quad \exp(-m_1)\, pr(\chi^2_{(h,\delta)} < r\, m_2)\, dm_1\, dm_2 \quad (33)$$

by successive integration by parts

$$H1 = \int_0^\infty k^*/\sigma_{22}^{q_2+1} \sum_{i=0}^{q_2} \binom{q_2}{i} (-\sigma_{11})^{q_2-i} (1/b_2)^{q_1+q_2-i+1}$$

$$[(q_1+q_2-i)!\ m_2^i \exp(-m_2/2\sigma_{22}) - \sum_{j=0}^{q_1+q_2-i}$$

$$((q_1+q_2-i)!/(q_1+q_2-i-j)!)\ (b_1b_2)^{q_1+q_2-i-j}$$

$$m_2^{q_1+q_2-j} \exp(-m_2(1/2\sigma_{22}+b_1b_2))]$$

$$pr(\chi^2_{(h,\delta)} < r\, m_2)\ dm_1\ dm_2 \tag{34}$$

We can write the cumulative probability of a noncentral $\chi^2_{(h,\delta)}$ variable in terms of the probabilities of a poisson random variable and the cumulative probability of a central $\chi^2_{(h+2k)}$, where k follows a poisson distribution, i.e.,

$$pr(\chi^2_{(h,\delta)} < rm_2) = \sum_{k=0}^{\infty} \exp(-\delta)\ \delta^k/k! \int_0^{rm_2} u(\cdot)d\chi^2 \tag{35}$$

Where $u(\cdot)$ is the density of a central chi square with h+2k degrees of freedom. Furthermore, using the expression in Abromowitz and Stegun (1972, p. 941, 26.4.21), for the cumulative probability of a central $\chi^2$ variable, assuming h is even

$$pr(\chi^2_{h,\delta} < rm_2) = \sum_{k=0}^{\infty} \exp(-\delta)\ \delta^k/k!\ (1 - \sum_{y=0}^{h/2+k-1}$$

$$\exp(-m_2r/2)(m_2r/2)^y/y!) = (1 - \sum_{k=0}^{\infty} \exp(-\delta)\delta^k/k! \sum_{y=0}^{h/2+k-1}$$

$$\exp(-m_2r/2)(m_2r/2)^y/y!) \tag{36}$$

Inserting this value in (34), carrying out the multiplication inside the integral and then integrating to gamma functions we obtain the final form of H1.

$$
\begin{aligned}
H1 = {} & k^*/\sigma_{22}^{q_2+1} \sum_{i=0}^{q_2} \binom{q_2}{i} (-\sigma_{11})^{q_2-i} (1/b_2)^{q_1+q_2-i+1} \\
& \{(q_1+q_2-i)!\ \Gamma(i+1)/(1/2\sigma_{22})^{i+1} - \sum_{j=0}^{q_1+q_2-i} ((q_1+q_2-i)! \\
& /(q_1+q_2-i-j)!)\ (b_1 b_2)^{q_1+q_2-i-j} (\Gamma(q_1+q_2-j+1)/ \\
& (1/2\sigma_{22}+b_1 b_2)^{q_1+q_2-j+1}) - \sum_{k=0}^{\infty} \exp(-\delta)\delta^k/k! \sum_{y=0}^{h/2+k-1} \\
& (q_1+q_2-i)!/y!\ (r/2)^y\ \Gamma(i+y+1)/(r/2+1/2\sigma_{22})^{i+y+1} \\
& + \sum_{k=0}^{\infty} \delta^k \exp(-\delta)/k! \sum_{j=0}^{q_1+q_2-i} (q_1+q_2-i)!/(q_1+q_2-i-j)! \\
& (b_1 b_2)^{q_1+q_2-i-j} \sum_{y=0}^{h/2+k-1} (r/2)^y/y!\ (\Gamma(q_1+q_2-j+y+1)/ \\
& (1/2\sigma_{22}+b_1 b_2+r/2)^{q_1+q_2-j+y+1}\}
\end{aligned}
\tag{37}
$$

This completes the evaluation of H1. The derivation of H2, H3 and H4 are shown in Appendices C, D and E, respectively.

Substituting these values in (27), we obtain the unconditional risk of 2SPE (note that $\sigma = \sigma_{11} + \sigma_{22} = (1+\tau)\sigma_{22}$).

$$
\begin{aligned}
R(\alpha^*(2SPE), \alpha) = {} & \sigma_{22}\ (1+\tau)\ p + (\pi'\pi - \sigma_{22}\ (1+\tau)p/2)H1 - \pi'\pi/2 \\
& H1' - 2\ p(\sigma_{11}\ H2 - \sigma_{11}\ H3 + \sigma_{22}\ H3) - 2 \\
& \pi'\pi/\sigma_{22}(1+\tau)\ (\sigma_{11}\ H2' - \sigma_{11}\ H3' + \sigma_{22}\ H3') + \sigma_{22} \\
& (1+\tau)\ p\ (H2 + 2\ H4 - 2\ H3) + \pi'\pi\ (H2' + 2\ H4' - 2\ H3') \\
& + 2\ \pi'\pi/\sigma_{22}\ (1+\tau)\ (\sigma_{11}\ H2 - \sigma_{11}\ H3 + \sigma_{22}\ H3)
\end{aligned}
\tag{38}
$$

where the Hi are the evaluated values of the expectations with $\chi^2$ random variables with p+2 degrees of freedom and Hi' are the values of expectations with $\chi^2$ variables with p+4 degrees of freedom. (i=1,2,3,4)

APPENDIX C

In this appendix we derive the expectation

$$H2 = E(I^{(GQ)}{}_{(c_1,\infty)} pr(\chi^2{}_{(h,\delta)} < c_3^*))$$

Define $Y_1 = s_{11}/s_{22} = X_1/gX_2 = (GQ)$,

$$Y_2 = s_{11} + s_{22} = \sigma_{11}/n_1-p\ X_1 + \sigma_{22}/n_2-p\ X_2$$

with $g=v_1/v_2$, $i=1,2$ and $v_i=\sigma_{ii}/n_i-p$, where $X_i$ are independent $\chi^2$ random variables. Consider the following transformation

$$z_1 = X_1,$$

$$z_2 = v_1\ X_1 + v_2\ X_2$$

Also write $c_3^* = = c_3(s_{11} + s_{22})/\sigma = r\ z_2$ (a stochastic component $z_2$ and a fixed component $r = c_3/\sigma$). Then H2 becomes

$$H2 = \int\int_{Area} 1/v_2\ h(z_1,\ (z_2-v_1z_1)/v_2)\ pr(\chi^2{}_{(h,\delta)} < r\ z_2)dz_1\ dz_2 \tag{39}$$

where Area = $\{z_1,z_2 : z_1,z_2 > 0,\ z_1(v_2 + c_1\ g\ v_1) > c_1\ g\ z_2,\ z_1 > b_1\ z_2$ with $b_1 = c_1g/(v_2 + c_1gv_1),\ z_2 > z_1\ v_1\}$ and $h(\cdot)$ is the joint distribution of two independent $\chi^2$ variates. Define $q_i = n_i-p/2 - 1$ (assume $q_i$ is integer as before $i=1,2$). Then writing the $h(\cdot)$ density out and expanding the $((z_2-v_1z_1)/v_2)^{q_2}$ term, we have

$$H2 = \int_0^\infty \int_{b_1z_2}^{z_2/v_1} k^*/\ v_2^{q_2+1} \sum_{i=0}^{q_2} \binom{q_2}{i} (-v_1)^{q_2-i}\ z_1^{q_1+q_2-i}$$

$$z_2^i\ \exp(-z_2/2v_2)\ \exp(-z_1(1-\tau((n_2-p)/(n_1-p))/2)$$

$$pr(\chi^2{}_{(h,\delta)} < r\ z_2)\ dz_1\ dz_2 \tag{40}$$

Making the change of variable, $m_1 = b_2z_1$ and $m_2 = z_2$ where $b_2 =$ $(1-\tau((n_2-p)/(n_1-p))/2)$, we have

$$H2 = \int_{0}^{\infty} \int_{b_1 b_2 m_2}^{b_2 m_2 / v_1} k^*/ v_2^{q_2+1} \sum_{i=0}^{q_2} \binom{q_2}{i} (-v_1)^{q_2-i} (1/b_2)^{q_1+q_2-i+1}$$

$$m_2^{i} \exp(-m_2/2v_2)\, m_1^{q_1+q_2-i} \exp(-m_1)\, pr(\chi^2_{h,\delta} < r\, m_2)\, dm_1\, dm_2 \quad (41)$$

By successive integration by parts,

$$H2 = \int_0^{\infty} k^*/ v_2^{q_2+1} \sum_{i=0}^{q_2} \binom{q_2}{i} (-v_1)^{q_2-i} (1/b_2)^{q_1+q_2-i+1} m_2^{i}$$

$$\exp(-m_2/2v_2) \{ \sum_{y=0}^{q_1+q_2-i} ((q_1+q_2-i)!/(q_1+q_2-i-y)!)$$

$$\{ \exp(-b_1 b_2 m_2)(b_1 b_2 m_2)^{q_1+q_2-i-y} - \exp(-b_2 m_2/v_1)$$

$$(b_2 m_2/v_1)^{q_1+q_2-i-y} \} \}$$

$$pr(\chi^2_{(h,\delta)} < r\, m_2)\, dm_2 \quad (42)$$

If we write the cumulative probablity of $\chi^2_{(h,\delta)}$ in terms of poisson probabilities, use the formula in (36) from Abromowitz and Stegun, carry out multiplication inside the integral, and also pass the infinite integral through the summations, we obtain

$$H2 = k^*/ v_2^{q_2+1} \sum_{i=0}^{q_2} \binom{q_2}{i} (-v_1)^{q_2-i} (1/b_2)^{q_1+q_2-i-1}$$

$$\{ \sum_{y=0}^{q_1+q_2-i} ((q_1+q_2-i)!/(q_1+q_2-i-y)) \{ (b_1 b_2)^{q_1+q_2-i-y}$$

$$\int_0^{\infty} \exp(-m_2(1/2v_2+b_1 b_2))\, m_2^{q_1+q_2-y} dm_2 - (b_2/v_1)^{q_1+q_2-i-y}$$

$$\int_0^{\infty} \exp(-m_2(1/2v_2+b_2/v_1)) m_2^{q_1+q -y} dm_2\} + \sum_{k=0}^{\infty} \exp(-\delta) k^{\delta}/k! \sum_{y=0}^{q_1+q_2-i}$$

$$(q_1+q_2-i)!/(q_1+q_2-i-y)!) \sum_{j=0}^{h/2+k-1} (r/2)^j/j! \{-(b_1b_2)^{q_1+q_2-i-y}$$

$$\int_0^\infty m2^{q_1+q_2-y+j} \exp(-m_2(r/2+b_1b_2+1/2v_2))\, dm_2 + (b_2/v_1)^{q_1+q_2-i-y}$$

$$\int_0^\infty m_2^{q_1+q_2-y+j} \exp(-m_2(r/2+b_2/v_1+1/2v_2))\, dm_2 \} \} \qquad (43)$$

Finally integrating the gamma functions, we obtain

$$H2 = k^*/v_2^{q_2+i} \sum_{i=0}^{q_2} \binom{q_2}{i} (-v_1)^{q_2-i} (1/b_2)^{q_1+q_2-i+2}$$

$$\{ \sum_{y=0}^{q_1+q_2-i} ((q_1+q_2-i)!/(q_1+q_2-i-y)!) \; \Gamma(q_1+q_2-y+1)$$

$$\{(b_1b_2)^{q_1+q_2-y+1}/(1/2v_2+b_1b_2)^{q_1+q_2+1-y} - (b_2/v_1)^{q_1+q_2-i-y}/$$

$$(1/2v_2 + b_2/v_1)^{q_1+q_2-y+1} \} + \sum_{k=0}^{\infty} \delta^k \exp(-\delta)/k! \sum_{y=0}^{q_1+q_2-i}$$

$$(q_1+q_2-i)!/(q_1+q_2-i-y)! \sum_{j=0}^{h/2+k-1} (r/2)^j/j! \{ -((b_1b_2)^{q_1+q_2-i-y}/$$

$$(r/2+b_1b_2+1/2v_2))^{q_1+q_2-y+j+1} + ((b_2/v_1)^{q_1+q_2-i-y}/$$

$$(r/2+b_2/v_1+1/2v_2))^{q_1+q_2+j-y+1} \} \} \qquad (44)$$

This completes the evaluation of H2.

## APPENDIX D

In this Appendix, we derive the expectation H3. The change of variables is as in Appendix C. The difference between H3 from H2 is the presence of $\theta = \dfrac{s_{22}}{s_{11}+s_{22}}$ in the expectation. This term increases the powers of $z_2-v_1z_1$ and $z_2$ in (40) by one, since $\theta$ is also $z_2-v_1z_1/z_2$. Allowing for this difference, we obtain

$$H3 = \int_0^\infty \int_{b_1z_2}^{z_2v_1} k^*/v_2^{q_2+1} \sum_{i=0}^{q_2+1} \binom{q_2+1}{i} (-v_1)^{q_2+1-i} z_1^{q_1+q_2+1-i}$$

$$z_2^{i-1} \exp(-z_2/2v_2) \quad \exp(-z_1/2(1-\tau(n_2-p/n_1-p)))$$

$$pr(\chi^2_{(h,\delta)} < c_3^*) \quad dz_1 \, dz_2 \qquad (45)$$

Making the change of variables $m_1 = b_2z_1$, $m_2 = z_2$ and using successive integration by parts as before, we have

$$H3 = \int_0^\infty k^*/v_2^{q_2+1} \sum_{i=0}^{q_2+1} \binom{q_2+1}{i} (-v_1)^{q_2+1-i} (1/b_2)^{q_1+q_2+2-i}$$

$$\Big\{ \sum_{y=0}^{q_1+q_2+1-i} (q_1+q_2+1-i)!/(q_1+q_2+1-i-y)! \; \{(b_1b_2)^{q_1+q_2+1-i-y}$$

$$m_2^{q_1+q_2-y} \quad \exp(-m_2(1/2v_2+b_1b_2)) - (b_2/v_1)^{q_1+q_2+1-i-y}$$

$$m_2^{q_1+q_2-y} \exp(-m_2(1/2v_2+b_2/v_1)) \Big\} \Big\} \; [1 - \sum_{k=0}^{\infty} \exp(-\delta)\delta^k/k!$$

$$\sum_{j=0}^{h/2+k-1} \exp(-rm_2/2) \; (rm_2/2)^j/j!] \; dm_2 \qquad (46)$$

In the following we pass the infinite integral through the summations indexed by y as long as $i=0$, $y\neq q_1+q_2+1$. In other words for $i=0$ the last term in the summation indexed by y requires a different integration technique given in Abromowitz and Stegun. Then we obtain

$$H3 = k^*/v_2^{q_2+1} \sum_{i=0}^{q_2+1} \binom{q_2+1}{i} (-v_1)^{q_2+1-i} (1/b_2)^{q_1+q_2+2-i}$$

$$\{ \sum_{y=0(*)}^{q_1+q_2+1-i} (q_1+q_2+1-i)!/(q_1+q_2-i-y+1)! \quad \Gamma(q_1+q_2+1-y)$$

$$\{ \{ (b_1 b_2)^{q_1+q_2+1-i-y} / (1/2v_2+b_1 b_2)^{q_1+q_2+1-y}$$

$$-(b_2 v_1)^{q_1+q_2+1-i-y} / (1/2v_2+b_2 v_1)^{q_1+q_2+1-y} \} + \sum_{k=0}^{\infty} \exp(-\delta)\delta^k/k!$$

$$\sum_{j=0(*)}^{h/2+k-1} (r/2)^j/j! \quad \Gamma(q_1+q_2+1+j-y) \{ (b_2/v_1)^{q_1+q_2+1-i-y} /$$

$$(r/2+b_2/v_1+1/2v_2)^{q_1+q_2+1+j-y} - (b_1 b_2)^{q_1+q_2+1-i-y}$$

$$(r/2+b_1 b_2+1/2v_2)^{q_1+q_2+1+j-y} \} \} \}$$

$$+ \ldots\ldots\ldots\ldots \qquad (47)$$

In (47) an (*) under the summation means that we pick up all the terms except $y=q_1+q_2+1$ when $i=0$. Now to the expression given in (47) we have to add the value of H3 when $y=q_1+q_2+1$ for $i=0$. Using the integral value in Abromowitz and Stegun we have[1]

$$H3 = \ldots\ldots + k^*/v_2^{q_2+1} (-v_1)^{q_2+1} (1/b_2)^{q_1+q_2+2}$$

$$(q_1+q_2+1)! \quad \{ \log((b_2/v_1+1/2v_2)/(b_1 b_2+1/2v_2)) +$$

$$D \exp(-\delta) + \sum_{k=1}^{\infty} \exp(-\delta) \; \delta^k/k! \; \{ D \exp(-\delta)$$

$$+ \sum_{j=2}^{h/2+k-1} (r/2)^j/j! \quad \{ \Gamma(j)/(r/2+b_2/v_1+1/2v_2)^j$$

$$- \Gamma(j)/(r/2+b_1 b_2+1/2v_2)^j \} \} \} \qquad (48)$$

[1] Note that $\int_0^{\infty} (\exp(-ax) - \exp(-bx))/x \, dx = \log(b/a)$.

Where $D = \log((r/2+b_1 b_2+1/2v_2)/(r/2+b_2/v_1+1/2v_2) + r/2 \{(1/(r/2+b_2/v_1+1/2v_2)) - (1/(r/2+b_1 b_2+1/2v_2))\}$. Putting (47) and (48) together we obtain H3.

APPENDIX E

In this appendix we examine the last expectation H4 needed for the evaluation of the 2SPE. Transformation is the same as in Appendix C. The difference between H4 from H3 is that $\theta = s_{22}/s_{11}+s_{22}$ is now squared. Remembering that $\theta = z_2 - v_1 z_1/z_2$, and allowing for this difference, H4 can be written now as

$$H4 = \int_0^\infty \int_{b_1 z_2}^{z_2 v_1} k^*/v_2^{q_2+1} \sum_{i=0}^{q_2+2} \binom{q_2+2}{i} (-v_1)^{q_2+2-i} z_1^{q_1+q_2+2-i}$$

$$z_2^{i-2} \exp(-z_1/2(1-\tau((n_2-p/n_1-p)))$$

$$\exp(-z_2/2v_2)\ pr(\chi^2_{(h,\delta)} < c_3^*)\ dz_1\ dz_2 \qquad (49)$$

Making the change of variable $m_1 = b_2 z_1$, $m_2 = z_2$ as before, and using successive integration by parts

$$H4 = \int_0^\infty k^*/v_2^{q_2+1} \sum_{i=0}^{q_2+2} \binom{q_2+2}{i} (-v_1)^{q_2+2-i} (1/b_2)^{q_1+q_2+3-i}$$

$$\{ \sum_{y=0}^{q_1+q_2+2-i} (q_1+q_2+2-i)!/(q_1+q_2+2-i-y)!\ \{(b_1 b_2)^{q_1+q_2+2-i-y}$$

$$m_2^{q_1+q_2-y} \exp(-m_2(1/2v_2+b_1 b_2)) - (b_2/v_1)^{q_1+q_2+2-i-y}$$

$$m_2^{q_1+q_2-y} \exp(-m_2(1/2v_2+b_2/v_1))\ \}\ \} pr(\chi^2_{(h,\delta)} < r\ m_2)\ dm_2 \qquad (50)$$

Next we will pass the infinite integral through the summations and integrate the gamma functions as long as i=0, $l \neq q_1+q_2+1$, and $l \neq q_1+q_2+2$, and i=1, $l \neq q_1+q_2+1$. Put differently, for i=0 the last two terms, and for i=1 the very last term in the summations indexed by l require different integration methods that we will write separately. Consequently,

$$H4 = k^*/v_2^{q_1+1} \sum_{i=0}^{q_2+2} \binom{q_2+2}{i} (-v_1)^{q_2+2-i} (1/b_2)^{q_1+q_2+3-i}$$

$$\{ \sum_{y=0(*)}^{q_1+q_2+2-i} ((q_1+q_2+2-i)!/(q_1+q_2+2-i-y)!)$$

$$\{ \{\Gamma(q_1+q_2+1-1)\ (b_1/b_2)^{q_1+q_2+2-i-y} / (1/2v_2 + b_1 b_2)^{q_1+q_2+1-y}$$

$$- (b_2/v_1)^{q_1+q_2+2-i-y} / (1/2v_2+b_2/v_1)^{q_1+q_2+1-y} \}$$

$$+ \sum_{k=0}^{\infty} \exp(-\delta)\ k^{\delta}/k! \sum_{j=0}^{h/2+k-1} (r/2)^j/j!$$

$$\Gamma(q_1+q_2+1+j-y)\ \{(b_2/v_1)^{q_1+q_2+2-i-y} /$$

$$(r/2+b_2/v_1+1/2v_2)^{q_1+q_2+1+j-y} - (b_1 b_2)^{q_1+q_2+1-i-y} /$$

$$(r/2+b_1 b_2+1/2v_2)^{q_1+q_2+1+j-y} \} \} \} + \ldots\ldots\ldots \qquad (51)$$

To the expression in (51) we have to add the value of H4 when $i=0$ $y=q_1+q_2+1$ and $y=q_1+q_2+2$, and when $i=1$ $y=q_1+q_2+1$ (these terms of the sums indexed by y are left out in (51). This is indicated by (*) in (51)).

Define

$$d1 = (b_1 b_2 + 1/2v_2) \qquad d3 = (b_1 b_2 + 1/2v_2 + r/2)$$

$$d2 = (b_2/v_1 + 1/2v_2) \qquad d4 = (b_2/v_1 + 1/2v_2 + r/2) \qquad (52)$$

Then H4 becomes

$$H4 = \ldots\ldots + k^*/v_2^{q_2+1} \quad (-v_1)^{q_2+1} \quad (1/b_2)^{q_1+q_2+2}$$

$$(q_1+q_2+2)!\ \{(b_2/v_1-b_1 b_2) + 1/2v_2 \log (d1/d2)$$

$$+ \exp(-\delta)\ F + \sum_{k=1}^{\infty} \exp(-\delta)\delta^k/k!\ \{ \exp(-\delta)\ F$$

$$+ \sum_{j=2}^{h/2+k-1} (r/2)^j/j!\ \{(b_2/v_1\ \Gamma(j)/(d4)^j) - (b_1 b_2)\Gamma(j)/(d3)^j)$$

$$+ (\Gamma(j-1)/d4^{j-1} - \Gamma(j-1)/d3^{j-1}\} \} + k^*/v_2^{q_2+1} (q_2+2) (-v_1)^{q_2+1}$$

$$(1/b_2)^{q_1+q_2+2} (q_1+q_2+1)! \{ \log(d2/d1) + D \exp(-\delta)$$

$$+ \sum_{k=1}^{\infty} \exp(-\delta)\delta^k/k! \{ D \exp(-\delta) + \sum_{j=2}^{h/2+k-1} (r/2)^j/j!$$

$$(\Gamma(j)/d4^j - \Gamma(j)/d3^j) \} \} \qquad (53)$$

Where D is given in Appendix D, and

$$F = r/2 \log(d3/d4) + b_2\ r/v_1\ 2\ d4 - b_1 b_2\ r/2\ d3$$

$$+ d3 - d4 + (r/2 + 1/2v_2) \log (d4/d3) \qquad (54)$$

In (53) we used the following integral value

$$\int_0^{\infty} ( \exp(-ax) + a\ x \exp(-ax) + \exp(-bx) - b\ x \exp(-bx))$$

$$\exp(-cx)/x^2\ dx$$

$$= b - a + c \log (a+c/b+c) \qquad (55)$$

In (55), integrate the first and the third terms by parts and then collect terms and finally use the integral value given in Abromowitz and Stegun (as in footnote 2). Inserting H1, H2, H3, and H4 in the expression (38) for the risk of 2SPE we obtain the analytical unconditional risk of 2SPE as a function of $\tau$ and $\pi$.

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