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MATHEMATICAL APPENDICES FOR: RECONCILING THE VON LIEBIG
AND DIFFERENTIABLE CROP PRODUCTION FUNCTIONS

by

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Appendix I

This appendix will show that increased dispersion in the inputs lowers output. Let $F_1(\varepsilon_1)$ and $H_1(\varepsilon_1)$ be CDFs for ε_1 and let Y_{F_1} and Y_{H_1} be the aggregate output when $\varepsilon_1 \sim F_1$, and $\varepsilon_1 \sim H_1$, respectively.

Theorem: If H_1 is more risky than F_1 in the sense of a mean-preserving spread, then $Y_{F_1} \geq Y_{H_1}$.

Proof: Let $z' = (z_1', z_2', z_3')$ and $z'' = (z_1'', z_2'', z_3'')$ be two different 3-vectors. Yield is $y(z) = \min(z_1, z_2, z_3)$. For $t \in [0, 1]$,

$$y[z't + z''(1 - t)] = \min [tz_1' + (1 - t)z_1'', tz_2' + (1 - t)z_2'', tz_3' + (1 - t)z_3''].$$

Let the j th argument of the min function be its least element so

$$\begin{aligned} y[z't + z''(1 - t)] &= tz_j' + (1 - t)z_j'' \geq t \min(z_1', z_2', z_3') + (1 - t) \min(z_1'', z_2'', z_3'') \\ &= ty(z') + (1 - t)y(z''). \end{aligned}$$

Thus, the function y is concave.

Next, use Theorem 2 of Rothschild and Stiglitz: If H_1 is more risky than F_1 , then $E_{H_1} U(x) \leq E_{F_1} U(x)$ for every concave function U .

In particular, for every z_2 and z_3 ,

$$\Phi_{H_1}(z_2, z_3) \equiv \int y(a_0 + a_1 x_1 + \varepsilon_1, z_2, z_3) dH_1 \leq \int y(a_0 + a_1 x_1 + \varepsilon_1, z_2, z_3) dF_1 \equiv \Phi_{F_1}(z_2, z_3).$$

Therefore,

$$Y_{H_1} = \iiint y \, dH_1 \, dF_2 \, dF_3 = \iint \phi_{H_1}(b_0 + b_1 x_2 + \varepsilon_2, P + \varepsilon_3) \, dF_2 \, dF_3$$

$$\leq \iint \phi_{F_1}(b_0 + b_1 x_2 + \varepsilon_2, P + \varepsilon_3) \, dF_2 \, dF_3 = Y_F.$$

So $Y_{H_1} \leq Y_{F_1}$, which was to be shown.

Appendix II

Within the class of distributions that

(1) are expandable in Taylor series about any of their points that has positive probability weight, with the radius of convergence of the Taylor series so large as to encompass all points with positive probability,¹ or

(2) represent nonstochastic variables, which is to say, $F(\epsilon) = 0$ if $\epsilon < 0$ and $F(\epsilon) = 1$ otherwise,

a quadratic aggregate function implies

- (1) an underlying rectangular distribution for one input,
- (2) a nonstochastic distribution for one of the other inputs, and
- (3) any distribution for the remaining input that leaves it nonbinding everywhere.

If all of F_1 , F_2 , and F_3 are nonrandom, then the aggregate function is LRP, which is certainly not quadratic. Since at least one of the variables must be truly random, let F_1 be the one with positive density on the narrowest range. That is, there are numbers k_0 and k_1 , possibly \pm infinity, so that $f_1(\epsilon_1) = 0$ whenever $y - A = \epsilon_1 < k_0$ or $y - A = \epsilon_1 > k_1$. By assumption, F_1 is expandable in Taylor series on that range k_0 to k_1 , so f_1 , the density, exists and is also expandable in Taylor series.

The ranges over which F_2 and F_3 have positive density (and are expandable in Taylor series) are $[k_{02}, k_{12}]$ and $[k_{03}, k_{13}]$, respectively. These variables are nonstochastic if $k_{02} = k_{12}$ and $k_{03} = k_{13}$, respectively. Let $l_0 = \min(k_{02} + B, k_{03} + P)$ and $l_1 = \min(k_{12} + B, k_{13} + P)$. Define $G^* = F_2^* F_3^*$. Then $G^* = 1$ if $y < l_0$, $G^* = 0$ for $y > l_1$, and, for $l_0 \leq y \leq l_1$, G^* can be expanded in a power series. Define z as a point ($l_0 < z < l_1$ and $k_0 < z < k_1$) around which expansions of both G^* and f_1 can be made, and define \tilde{y} as $y - z$. Then with $D_n G^*$ referring to the n th derivative of G , the Taylor series expansions for G^* is

$$(1) \quad G^* = \sum_{n=0}^{n=\infty} D_n G^*(z) \left(\frac{\tilde{y}^n}{n!} \right) \quad \text{for } l_0 \leq y \leq l_1.$$

For f_1 ,

$$(2) \quad f_1 = \begin{cases} 0 & \text{if } y - A < k_0, \\ \sum_{n=0}^{n=\infty} D_n f_1(z) \left(\frac{\tilde{y}^n}{n!} \right) & \text{if } k_0 \leq \epsilon_1 = y - A \leq k_1, \\ 0 & \text{if } y - A > k_1. \end{cases}$$

The following shows that a quadratic aggregate function requires $l_0 = l_1 = 0$ and F_1 to be a rectangular distribution.

Proving this result requires consideration of all possible arrangements of l_0 , l_1 , k_0 , and k_1 , and examining what they imply about the marginal product of x_1 , which is given by equation (13) in the text. Since the aggregate function is quadratic, the marginal products must be linear in x_1 , x_2 , and P ; or, because A and B are linear in x_1 and x_2 , it must be linear in A , B , and P . That is, $\partial Y / \partial A$ is

$$(3) \quad \int f_1(y - A) G^*(y) dy = e_0 + e_1 A + e_2 B + e_3 P,$$

where the e_i s are constants.

The possible arrangements of the l 's and k 's are most usefully grouped by considering the interval in which both G^* and f_1 are expandable in power series, $J = [m_0, m_1] = \{[k_0 + A, k_1 + A] \cap [l_0, l_1]\}$. There are five cases for J : (1) the null set; (2) the real line; (3) $[m_0, m_1] = [l_0, l_1]$ and $l_i \neq k_i + A$ nor does $l_0 = l_1$; (4) $[k_0 + A, m_1]$ or $[m_0, k_1 + A]$; and (5) the point $l_0 = l_1$. We now show that only the fifth case can result in a quadratic.

Case 1. When J is empty, it is easy to show that response is not quadratic. If G^* is 0 between k_0 and k_1 , then the marginal product is 0, which cannot come from a quadratic aggregate function. Otherwise, $G^* = 1$ between k_0 and k_1 , while $f_1 = 0$ between l_0 and l_1 .

$$(4) \quad \int_{-\infty}^{\infty} f_1 G^* dy = \int_{k_0+A}^{k_1+A} f_1 dy$$

because f_1 is 0 elsewhere on the real line. This integral is 1 since G^* is 1 over this range and f is a density with its integral 1. Since the integral is a constant and not linear, it cannot be the marginal product of a quadratic function. These are just the cases where one constraint binds with probability 1, resulting in a linear response to the binding input, and $\partial y / \partial x_1 = a_1 \int_{-\infty}^{\infty} f_1 G^* dy =$ either a_1 or 0. Thus, J empty gives a constant marginal product which violates the assumption of a quadratic total function.

Case 2. When J is the whole real line, both functions are everywhere expandable in Taylor series. Separate the Taylor series expansion of $f_1 G^*$ around z into its first term and all other terms: $f_1(z)G^*(z) + \Theta(y - A, y - B, y - P)$. Here Θ is a polynomial in its arguments but does not have a constant term. It is just the terms (in powers of $y - A$, $y - B$, and $y - P$ or higher) of the product of the Taylor expansions of f_1 and G^* about z . Using this Taylor series expansion, the linearity of the marginal products requires

$$(5) \quad \int_{-\infty}^{\infty} f_1(y - A)G^*(y)dy = f_1(z)G^*(z)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \Theta dy = e_0 + e_1 A + e_2 B + e_3 P.$$

A solution to this equation is impossible since integrating the constant $f_1(z)G^*(z)$ over the real line gives infinity, which, when added to the integral of Θ , is still infinity and is certainly not equal to a linear function of A , B , and P .

Case 3. The case $J = [l_0, l_1]$ and not a point is just $A + k_0 < l_0 < l_1 < A + k_1$, which is impossible because it requires one of F_2 and F_3 to have nonzero density only on a range narrower than f_1 , thus violating the assumption that input 1 had the narrowest distribution.

Case 4. When one or both of the boundaries of J is $A + k_i$, $\int_{-\infty}^{\infty} f_1 G^* dy$ can be broken into three integrals, one of which is of the form $\int_{m_0}^{m_1} f_1 G^* dy$, where either $m_0 = A + k_0$, $m_1 = A + k_1$, or both. Since, over this range, $f_1 \neq 0$ and G^* takes values other than 0 or 1, $\partial Y/\partial A$ is

$$(6) \quad \int_{m_0}^{m_1} f_1 G^* dy = \int_{m_0}^{m_1} [f_1(z)G^*(z) + \Theta] dy = (m_1 - m_0)f_1 G^* + \int_{m_0}^{m_1} \Theta(y) dy,$$

where $\Theta(y)$ is either 0 or a polynomial in y of degree 1 or greater. If it isn't zero, the integral of $\Theta(y)$ is of degree two or greater in m_0 and m_1 . Since at least one of the m 's is linear in A , $\int_{m_0}^{m_1} \Theta(y) dy$ would have to be at least quadratic in A . But $\int_{m_0}^{m_1} \Theta(y) dy$ quadratic in A violates the linearity of $\partial Y/\partial A$. Thus neither f_1 nor G^* can have more than one nonzero term in the Taylor expansions, which is to say they must both be the constant functions. (Only one of the three integrals of y is presented here: the rest are either 0 or are integrals of f_1 over some range and do not change the conclusion.)

Thus, f_1 and G^* must both be constants. By assumption, G^* is neither 0 nor 1 over the range of this integral, so it is some other constant. Since the Taylor series expansion is valid for $[l_0, l_1]$, G^* is constant on that interval as well. If it is a constant other than 0 or 1 over this range, it must become one at l_1 , the end of the range. Thus G^* has a jump at l_1 . It has probability mass at l_1 , which is to say its density does not exist at that point. This contradicts the assumption that the distributions are expandable in power series (and therefore have densities) at every point at which

they have probability mass. Therefore, this arrangement also fails to produce linear marginal products and thus a quadratic aggregate function.

Case 5. This leaves only the case $k_0 + A < l_0 = l_1 < k_1 + A$, precisely the case examined in detail in the text. In that case,

$$(7) \quad \int_{-\infty}^{\infty} f_1 G^* dy = \int_{k_0+A}^{l_0} f_1 dy = e_0 + e_1 A + e_2 B + e_3 P.$$

Note the definition of l_0 as $\min(k_{02} + B, k_{03} + P)$. When $l_0 = k_{02} + B$, the integral, (7), is linear in A and B only if f_1 is a constant. If f_1 is constant, then it is the density function for the rectangular distribution. Because $l_0 = l_1$, ϵ_2 must be deterministic. It is a point mass at zero. Since $k_{02} + B < k_{03} + P$, the plateau never binds and its distribution is irrelevant. If $l_0 = k_{03} + P$, then the plateau is deterministic and always binds before B . Thus, one input rectangular, one nonstochastic, and one irrelevant is the only case leading to a quadratic aggregate function. Since one of the inputs or the plateau is irrelevant (that is, one of these will never bind before the other), the aggregate function is quadratic in only two of the plateau and the two inputs. The remaining input or plateau does not enter the function.

Footnotes

¹The assumption on the radius of the Taylor series implies that the function does not have any poles, that is, any asymptotes, within that radius.

Reference

Rothschild, Michael, and Joseph E. Stiglitz. "Increasing Risk: I. A Definition."

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