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RECONCILING THE VON LIEBIG AND DIFFERENTIABLE CROP PRODUCTION FUNCTIONS

Although econometricians usually assume crop yields smoothly respond to variations in inputs, there is agronomic literature that assumes the relationship is of a linear-response-and-plateau nature. On homogeneous plots, some agronomic experiments show that a von Liebig (fixed-proportions) production function best predicts yields for many crops (Lanzar and Paris; Grim; and Grim, Paris, and Williams). This production function assumes that a plant needs fixed relative proportions of various inputs in order to grow; if even one input is below its required proportion, it acts as a limiting nutrient on the plant's growth. For instance, if a plant needs water and nitrogen in a ratio of 2:1 and is receiving exactly 2 units of nitrogen and 1 of water, then giving the plant 3 units of water but only 1 of nitrogen will not increase growth; neither will adding 1.5 units of nitrogen but only 2 of water. In contrast, econometricians usually estimate responses for whole fields or farms (or even larger aggregates) and use functions (such as translog) that have positive elasticities of substitution.

If, as the evidence indicates, agricultural production functions do operate on this "limiting nutrient" concept at the plant level, then estimation of these production functions at the plant level with continuous concave functional forms will not provide correct results. However, the estimation of production functions at the level of a whole field, farm, or county, and the use of inputs other than those that the plant directly utilizes is a different matter. Across any large area, there are nonuniformities in the distribution of inputs. For instance, an irrigation system may not deliver water uniformly (Elliott et al.) or soil characteristics may vary (Nielsen, Biggar, and Erh).

Therefore, plants in different areas in the field will be limited by different limiting values of inputs. These nonuniformities can make a smooth function an appropriate choice for estimation and they always decrease yield. Similarly, small phenotypic differences in plants may cause nonuniform growth which also leads to a smooth yield function. Finally, some inputs, labor, for instance, are only used to make nutrients available to plants and are not directly used by the plants themselves. Production functions estimated with these inputs included could also be expected to be of the smooth rather than the von Liebig functional form. In all these cases, the von Liebig production function will not explain the aggregate yield for the inputs applied, even though it will explain an individual plant's growth. Thus, a smooth aggregate production function can be reconciled with the plant's fixed proportions technology.

This paper explores some of the implications of an underlying von Liebig production function when there is a stochastic aspect in crop response. The second section gives the evidence for a lack of uniformity in applied inputs. The following section presents the case of two inputs, one distributed uniformly across a field and the other with a probability density function describing its spread, and shows that the aggregate production function for the field is an increasing concave function in the uniform input. The production function is also shown to be decreasing when the distribution of the random input is subject to a mean-preserving spread. The fourth section provides the functional forms implied for the production function when the distribution of the random input follows a Pareto, uniform, or exponential distribution. Section five explores the form for the production function when there is randomness in the plant's response. The case of economic inputs that are not directly utilized by the plant are explored in the sixth section which is followed by the conclusions.]

Studies of Nonuniformity

Both the soils science literature and the economics literature include studies of nonuniformity of agricultural inputs. Several studies have looked specifically at the variability in soil or water conditions and how to model it. Elliott et al. compared the uniform, normal, and beta distributions to see which best fits the actual water distribution for overlapping sprinkler systems. They found that the beta distribution best accounted for the distribution of water though the uniform and normal, as more practical distributions for calculating the efficient quantities of irrigation water, may be used in some circumstances. A soil science study by Nielsen, Biggar, and Erh analyzed the spatial variation in soil-water characteristics in a "homogeneous" field and found that water content in the soil is distributed normally, while hydraulic conductivity and soil-water diffusivity are distributed lognormally. Additionally, they note that "even seemingly uniform land areas manifest large variations in hydraulic conductivity values" (p. 257). Day was interested in analyzing the actual distribution underlying farm yield data with applied nitrogen as a control variable. With data from experimental plots, he used the Pearson system of probability density functions to assess the best-fitting distribution. As did Elliott et al., he found that the beta distribution best characterized the distribution of crop yields.

A number of articles assume some underlying distribution for a random variable causing nonuniformity (either water is not distributed uniformly or land quality is not homogeneous) and analyze the impacts of nonuniformity on optimal water use, yields, and other economic variables. Zaslavsky and Buras are motivated in their analysis by the trade-off between increased yields and increased costs that irrigation uniformity implies. In order to analyze how crop yield would respond to nonuniformity in water application, they estimated average yield from applied water by a Taylor series expansion and demonstrated how their model can be used. Seginer assumed one underlying von Liebig crop-water production function, but seasonal water use was assumed to be applied with a uniform distribution. He then derived the economically optimal water application as a function of the Christiansen Uniformity Coefficient (CUC) (Christiansen) and applied the model to cotton production in Israel. Warrick and Gardner similarly assumed a homogeneous von Liebig production function; but they distinguished between applied water and effective water, assumed several combinations of distributions for the two types of water, and then performed Monte Carlo simulations to analyze the effects of these distributions on yields. Feinerman, Bresler, and Dagan assumed no specific distribution but looked at the effect of nonuniformity on optimal water application for two different crop-water production functions: one with yield that peaked and then declined with increasing water application and the linear-response-and-plateau (LRP) von Liebig formulation. In the former case, nonuniformity decreased optimal water applications while, in the latter case, nonuniformity might either increase or decrease optimal water use. Letey, Vaux, and Feinerman applied this model to cotton (which has the first production function) and corn (for an example of the second production function). Assumed levels of the CUC were used to represent different degrees of nonuniformity in their simulated analyses of the effects of nonuniformity on optimal levels of applied water, yield, and profits. Feinerman, Letey, and Vaux analyzed the efficient levels of applied water when land quality is a random variable under different attitudes toward risk. They derived approximate solutions for risk neutrality and risk aversion with a Taylor series expansion and compared them to a deterministic solution. They also derived an analytical solution relating the variance in soil quality to yield when a Mitscherlich yield function is assumed.

Thus, there is ample evidence for nonuniformity and much empirical work on the consequences. Below we derive analytic results to complement this work.

A Model for Aggregating the von Liebig Function with a Random Input

Assume that a plant requires two inputs in fixed proportions in order to grow. One of the inputs, x_1, is assumed to be distributed evenly across the field. The other input, x_2, is distributed with a continuous probability density function, $f(x_2)$, and corresponding cumulative density function, $F(x_2)$; it is assumed that $0 \leq x_2 < \infty$. Let Y = yield for a field and a_0, a_1, b_0, and b_1 be fixed parameters: a_0, $b_0 \leq 0$ (that is, a_0 and b_0 indicate that minimum values of the nutrients may be necessary before any growth can occur) and a_1, $b_1 > 0$ (a_1 and b_1 represent the marginal growth as the input increases). If x_2 is identical across the whole field, then the true production function for that field is

$$(1) \qquad Y = \min\,(a_0 + a_1 x_1,\ b_0 + b_1 x_2).$$

Even if more than two inputs to production are necessary, this form still applies since only one input can be binding at a time.

However, across a field, input x_2 is not found uniformly and, therefore, plants in the field will not respond identically to homogeneous application of input x_1. Define $\bar{x}_2$ as the level of x_2 where the plant receives the exact relative proportions that it needs, i.e., $a_0 + a_1x_1 = b_0 + (b_1\bar{x}_2)$, or

(2) $$\bar{x}_2 = \frac{a_0 - b_0}{b_1} + \frac{a_1}{b_1} x_1 = c_0 + c_1x_1$$

where c_0 is defined to be $(a_0 - b_0)/b_1$, and c_1 is defined to be a_1/b_1. Thus, for $x_2 < \bar{x}_2$, x_2 is the limiting factor; for $x_2 \geq \bar{x}_2$, x_1 is limiting. Yield over the field then becomes

(3) $$Y = \int_0^{c_0+c_1x_1} (b_0 + b_1x_2)\, f(x_2)\, dx_2 + \int_{c_0+c_1x_1}^{\infty} (a_0 + a_1x_1)\, f(x_2)\, dx_2.$$

The first term sums up the yields on all land areas where the level of x_2 is the binding constraint. The second term sums up the remaining area, with ax_1 determining production on that land.

The response of this aggregate function to different applications of x_1 can be found by differentiating equation (3) with respect to x_1:

(4) $$\frac{dY}{dx_1} = a_1[1 - F(c_0 + c_1x_1)] \geq 0.$$

Thus, this aggregate function slopes upward, flattening off when the random variable has hit its maximum value.

The shape of this function can be further examined by taking the second derivative:

(5) $$\frac{d^2Y}{dx_1^2} = -c_1a_1\, f(c_0 + c_1x_1) \leq 0$$

which is strictly negative for $f(c_0 + c_1x_1) \neq 0$. Therefore, since the aggregate production function is an increasing concave function of inputs of

x_1, using aggregate production functions with these properties, such as the Cobb-Douglas or quadratic, does not conflict with the assumption of an underlying von Liebig production function.

This result is consistent with Perrin's findings on the effects of increased information on crop profitability. Using known information about the soil characteristics in an area, he calculated optimal fertilizer inputs, yield, and profit values for both a quadratic and an LRP von Liebig crop production function; he made similar calculations when only average soil characteristics were known. If more precise soil qualities were known, the LRP specification gave slightly higher profits; however, if only the averaged data were available, the quadratic specification gave much higher profits. Since knowledge of more details about exact soil characteristics implied that particular land types could be identified, distinguished, and treated differently, the von Liebig specification more accurately reflected the underlying reality. In contrast, when only averaged data were available, a functional form with a positive decreasing slope better reflected the actual aggregate production function.

It is possible to examine the effects of increasing the randomness of x_2 through application of a mean-preserving spread, as described by Rothschild and Stiglitz. A mean-preserving spread, a way of increasing riskiness more general than increasing variance, involves taking probability density from the middle of a distribution and increasing the density in the tails in a way that leaves the mean of the distribution unchanged.

Rothschild and Stiglitz define the problem as follows: Assume there exist two probability density functions, g(x) and f(x), $0 \leq x \leq 1$, with the same mean, but $g(x)$ has a larger "spread" than $f(x)$; let the associated cumulated distribution functions be $G(x)$ and $F(x)$. The mean-preserving spread, $s(x)$, is the difference between the two density functions so that $s(x) = g(x) - f(x)$. Define $S(x) = \int_0^x s(u)\, du = G(x) - F(x)$. $S(x)$ starts at 0, goes from positive to negative, and ends at 0. Intuitively, $S(x)$ indicates that the difference in the cumulative distribution functions, $G(x)$ and $F(x)$, is positive at first (since $G(x)$ has more weight in its tails) but must become negative at some point since $F(1) = G(1) = 1$.

The function $T(x)$ is defined as $\int_0^x S(y)\, dy$; $T(x)$ has the properties that $T(0) = T(1) = 0$ (since, at these points, there is no difference between the distribution functions); but $T(x) \geq 0$ for all x, since its "slope," $S(x)$, goes from positive to negative and ends up at 0.

For application to the problem at hand, define the yield function Y_g and Y_f such that

$$(6) \quad Y_g = \int_0^{c_0+c_1x_1} (b_0 + b_1x_2)\, g(x_2)\, dx_2 + \int_{c_0+c_1x_1}^{\infty} (a_0 + a_1x_1)\, g(x_2)\, dx_2$$

and Y_f is equivalent to Y in equation (3). Here it is assumed that $0 \leq x_2 < \infty$. If $s(x_2) = g(x_2) - f(x_2)$ is a mean-preserving spread, then

$$(7) \quad Y_g = Y_f + \int_0^{c_0+c_1x_1} (b_0 + b_1x_2)\, s(x_2)\, dx_2 + \int_{c_0+c_1x_1}^{\infty} (a_0 + a_1x_1)\, s(x_2)\, dx_2 .$$

Evaluating this integral yields

$$(8) \qquad Y_g = Y_f - T(c_0 + c_1x_1) \leq Y_f$$

since $T(c_0 + c_1x_1) \geq 0$. Intuitively, if the random input is distributed with more weight in the tails, then less will grow where it is the limiting factor in plant growth since more of the field has low levels of that input; where the nonrandom input is limiting, having more of the field with high levels of the random input will not increase plant growth. Therefore, increasing the randomness of the random input in the field will decrease yield.

Analytical Solutions for Aggregated Functions With a Random Input

Thus, even though an individual plant may actually grow via a von Liebig production function, in the aggregate a smooth concave function may provide a better approximation for actual crop yields. Indeed, it is even possible, in some cases, to derive a specific functional form from a particular probability density function for the randomly distributed input and vice versa.

Some of the economics literature on exact aggregation performs similar analyses. Houthakker assumed a number of individual fixed-proportion firms with different relative proportions. Assuming that all firms would produce efficiently and that they would only produce when profits were nonnegative, he found that a Pareto distribution for the relative proportions would yield an exact Cobb-Douglas aggregate function. Levhari extended Houthakker's analysis to find the distribution function underlying an aggregate constant-elasticity-of-substitution (CES) production function when the individual firms have fixed proportions. Sato more generally sought the distributions that would permit aggregation of individual CES production functions into CES or Cobb-Douglas (a special case of CES) aggregate functions when all firms are operating efficiently. He found that a Pareto distribution would lead to aggregation into a Cobb-Douglas and he described the conditions that lead to the existence of distributions for other aggregations to CES.

The following analyses follow in this tradition but with several differences. First, this study concentrates on the production function itself rather than imposing economic conditions such as the nonnegative profits requirements. Additionally, the above articles assume that each individual production unit is operating efficiently, while the following analyses derive their results from the fact that many parcels of land are not "on the knife-edge" where $ax_1 = bx_2$. Given these significant differences, though, some of the results are very similar. For simplicity, we consider cases where $a_0 = 0 = b_0$, $a = a_1$, and $b = b_1$.

The Cobb-Douglas Aggregate Function

If it is assumed that the aggregate production function is Cobb-Douglas in the uniform input (x_1), what probability density function is appropriate for $f(x_2)$? Assume that the aggregate function is

(9) $$Y = kx_1^{\alpha}.$$

Then, obviously,

(10) $$\frac{dY}{dx_1} = \alpha k x_1^{\alpha-1}.$$

Equation (4) gives the formula for dY/dx_1 from the general formulation. Equating these gives

(11) $$F(x_2)|_{cx_1} = 1 - \frac{\alpha}{a} k x_1^{\alpha-1} .$$

Note that, at this point, $cx_1 = x_2$. The goal is to find a density function $f(x_2)$ such that equation (6) holds. If

(12) $$F(x_2) = 1 - \alpha k a^{-\alpha} b^{\alpha-1} x_2^{\alpha-1} ,$$

then

(13) $$f(x_2) = (1 - \alpha) \alpha k a^{-\alpha} b^{\alpha-1} x_2^{\alpha-2} = g x_2^{\beta}$$

where $g = (1 - \alpha) \alpha k a^{-\alpha} b^{\alpha-1}$ and $\beta = \alpha - 2 < 0$. This form can be seen to be that of the Pareto distribution. Substitution of this function back into the aggregation equation, (3), does return a Cobb-Douglas function.

Since, for Cobb-Douglas, it is usually assumed that $0 < \alpha < 1$, then this function is downward-sloping and convex to the origin. It thus suggests that the probability of low values of the random input is much higher than the probability of high values of it (though, for the integral of the density function to be 1, it must be bounded below). If empirical evidence suggests that such a functional form is a reasonable approximation to the distribution of the random input in a particular case, then a Cobb-Douglas aggregate production function may provide an accurate representation of yield as a function of the nonrandom input.

The Quadratic Aggregate Function and the Uniform Distribution

This analytic result will be used to demonstrate the process of moving from the assumption of a distribution to the aggregate production function. As noted previously, Elliott *et al.* found that a uniform distribution can, in some circumstances, serve as a good approximation to the distribution of water from sprinkler irrigation. The following analysis will show that the assumption of a uniform distribution results in a quadratic aggregate production function.

Assume now that $f(x_2) = 1/(h - g)$, $g \leq x_2 \leq h$. Three possibilities arise: $g \leq cx_1 \leq h$, $cx_1 \leq g \leq h$, and $g \leq h \leq cx_1$. Note that the only interesting result arises when $g \leq cx_1 \leq h$; in the other two cases, one of the two terms in the integral in equation (3) drops out and either x_1 or x_2 is completely limiting.

Incorporating this distribution into equation (3) yields

$$Y = \int_g^{cx_1} bx_2\left[\frac{1}{h - g}\right] dx_2 + \int_{cx_1}^{h} ax_1\left[\frac{1}{h - g}\right] dx_2$$

(14)
$$= \frac{bx_2^2}{2(h - g)}\Bigg|_g^{cx_1} + \frac{ax_1x_2}{h - g}\Bigg|_{cx_1}^{h}$$

$$= \left[\frac{1}{2(h - g)}\right]\left\{-\frac{a^2}{b} x_1^2 + 2ahx_1 - bg^2\right\}$$

which is a quadratic function in x_1. This functional form was used by Hexem and Heady in their estimation of yields from inputs of water and nitrogen. Reversing the process, as demonstrated with the Cobb-Douglas function, will convert a quadratic function to the assumption of a uniform distribution.

The Negative Exponential Aggregate Function and the Exponential Distribution

The final analytical result that will be shown is for the exponential distribution, $f(x_2) = \lambda e^{-\lambda x_2}$, $0 < x_2 < \infty$. Then

(15)
$$Y = \int_0^{cx_1} bx_2\lambda e^{-\lambda x_2} dx_2 + \int_{cx_1}^{\infty} ax_1\lambda e^{-\lambda x_2} dx_2 .$$

The first integral can be integrated by parts, yielding $b\{(1/\lambda) - [cx_1 + (1/\lambda)] e^{-\lambda x_2}\}$. The second integral can be integrated directly, yielding $ax_1 e^{-\lambda cx_1}$. Thus, with this distribution,

$$Y = \left(\frac{b}{\lambda}\right)\left(1 - e^{-\lambda c x_1}\right) \tag{16}$$

which takes the form of a negative exponential function. It should be noted that the exponential distribution has the same characteristics as the Pareto distribution.

The advantage of these analytic solutions is that, if the distribution of a random input can be approximated by one of the distributions above, then an analyst is justified in estimating an aggregate production function with the characteristics economists consider desirable.

von Liebig Yield Function with a Random Coefficient

Suppose that, rather than one of the inputs to the production function being distributed randomly across a field, the coefficient on one of the inputs in equation (1)--for example, a_1--is a random variable. This model of non-uniformity may be relevant when, for instance, there are other inputs in the fixed-proportions production function upon which there are no measurements, such as nutrients in the soil, or genetic variability among the plants; it is also relevant when the link between the applied input and the input the plant actually uses is not known with certainty--for instance, applied water does not necessarily equal effective water to a plant. In this case, even if other inputs to the production function are applied in a uniform fashion, the plants may not respond consistently to these fixed inputs. The following discussion will present a model for a von Liebig production function when one of the coefficients is random.

As before, let Y = yield and x_1 and x_2 be inputs to production. For simplicity, set a_0 and b_0 equal to 0 and let $\tilde{a}$, b be the parameters of the production function. Assume that x_1 and x_2 are known with certainty but that $\tilde{a}$ is a random parameter with probability density function $f(\tilde{a})$ and cumulative distribution function $F(\tilde{a})$, $0 \leq \tilde{a} \leq \infty$. Also as before, assume that the underlying production function for a plant is

$$Y = \min\,(\tilde{a}x_1,\ bx_2). \tag{17}$$

The knife-edge, where $\tilde{a}x_1 = bx_2$, represents the point where both inputs are binding. If $\tilde{a} < bx_2/x_1$, then x_1 is the binding input; if $\tilde{a} \geq bx_2/x_1$, then the level of x_2 is binding. Thus, the expected yield function, analogous to equation (3), becomes

$$\begin{aligned} Y &= \int_0^{bx_2/x_1} \tilde{a}x_1\ f(\tilde{a})\ d\tilde{a} + \int_{bx_2/x_1}^{\infty} bx_2\ f(\tilde{a})\ d\tilde{a} \\ &= \int_0^{bx_2/x_1} \tilde{a}x_1\ f(\tilde{a})\ d\tilde{a} + bx_2\left[1 - F\left(\frac{bx_2}{x_1}\right)\right]. \end{aligned} \tag{18}$$

In contrast to the previous model, where the "slope" of the yield function was a constant b_1 for $x_2 \leq c_0 + c_1x_1$ but the height of the plateau varied, here the slope varies. As with the case of a random input in equation (3), this function can be shown to be increasing and concave by differentiation:

$$\frac{dY}{dx_1} = \int_0^{bx_2/x_1} \tilde{a}\ f(\tilde{a})\ d\tilde{a}. \tag{19}$$

This expression is nonnegative, as long as it is assumed that $\tilde{a}$ would never go below zero--a reasonable assumption, since the alternative would occasionally require negative inputs of x_1 for the plant to grow. Therefore, the expectation of $\tilde{a}$ over a positive range is, at the very least, nonnegative. Similarly,

(20) $$\frac{dY}{dx_2} = b\left[1 - F\left(\frac{bx_2}{x_1}\right)\right]$$

is nonnegative everywhere, and it is strictly positive for $F(bx_2/x_1) < 1$. The second derivatives are

(21) $$\frac{d^2Y}{dx_1^2} = \frac{-b^2x_2^2}{x_1^3} f\left(\frac{bx_2}{x_1}\right) \leq 0$$

(22) $$\frac{d^2Y}{dx_2^2} = \frac{-b^2x_2}{x_1} f\left(\frac{bx_2}{x_1}\right) \leq 0$$

(23) $$\frac{d^2Y}{dx_1 dx_2} = \frac{b^2x_2}{x_1^2} f\left(\frac{bx_2}{x_1}\right) \geq 0.$$

Since the Hessian for this problem is negative semi-definite, then, as in the case where an input is random, this production function is increasing and concave in both its inputs.

Analytical Solutions for Expected Yield With a Random Coefficient

Analogously to the aggregate functions with a random input, it is possible in the case of a random coefficient to derive some analytical results linking expected yield functions to some probability density functions for the random coefficient. A few results demonstrating this link are given below.

<u>The Cobb-Douglas Expected Yield Function and the Pareto Distribution</u>

Assume that the expected yield function has the Cobb-Douglas form in its inputs, i.e.,

(24) $$E(Y) = gx_1^{\alpha} x_2^{\beta} .$$

Then, following a procedure analogous to that used when an input is random, differentiate this function with respect to x_2 and set the result equal to equation (20):

(25) $$\frac{\partial E(Y)}{\partial x_2} = \beta g x_1^{\alpha} x_2^{\beta-1} = b\left[1 - F(\tilde{a})\Big|_{bx_2/x_1}\right].$$

Note that, where $\tilde{a} = bx_2/x_1$, $x_1 = bx_2/\tilde{a}$, and substitute that expression in for x_1. The result is

(26) $$F(\tilde{a})\Big|_{bx_2/x_1} = 1 - \beta g b^{\alpha-1} x_2^{\alpha+\beta-1} \tilde{a}^{-\alpha}.$$

Differentiating this function to get the density function yields

(27) $$f(\tilde{\alpha}) = \alpha\beta g b^{\alpha-1} x_2^{\alpha+\beta-1} \tilde{a}^{-\alpha-1} = k\tilde{a}^{\gamma}$$

where $k = \alpha\beta g b^{\alpha-1} x_2^{\alpha+\beta-1}$ and $\gamma = -\alpha - 1 < 0$. As with the random input, a Pareto distribution results from the assumption of an aggregate Cobb-Douglas. Substituting this distribution back into the expected yield equation, (18), does return the Cobb-Douglas form.

The Uniform Distribution

Assume that $\tilde{a}$ is distributed uniformly, with $g \leq \tilde{a} \leq h$. Thus, $f(\tilde{a}) = 1/(h - g)$. Substituting this into equation (15) yields

$$E(Y) = \int_{g}^{bx_2/x_1} \tilde{a}x_1\left(\frac{1}{h - g}\right) d\tilde{a} + \int_{bx_2/x_1}^{h} bx_2\left(\frac{1}{h - g}\right) d\tilde{a}$$

(28)

$$= \frac{a^2 x_1}{2(h - g)}\Bigg|_{g}^{bx_2/x_1} + \frac{abx_2}{h - g}\Bigg|_{bx_2/x_1}^{h}$$

$$= \frac{-2b^2x_2^2 + 2bhx_1x_2 - g^2x_1^2}{2(h - g)x_1} ,$$

which bears a limited resemblance to a quadratic formulation. Using the procedure of deriving the density function from an assumed quadratic expected-yield function does not provide an analytical solution.

The Exponential Distribution

Now let $f(a) = \lambda e^{-\lambda\tilde{a}}$. Using the same procedure as above, the production function becomes

$$E(Y) = \int_{0}^{bx_2/x_1} \lambda x_1\tilde{a}e^{-\lambda\tilde{a}}\, d\tilde{a} + \int_{bx_2/x_1}^{\infty} bx_2\, \lambda e^{-\lambda\tilde{a}}\, d\tilde{a}$$

(29)

$$= \left(\frac{x_1}{\lambda}\right)\left[1 - e^{-\lambda bx_2/x_1}\right]$$

which bears a limited resemblance to the negative exponential form. It thus appears that a link between specific functional forms and distributions does depend on whether the coefficient or the input is random: While the Cobb-Douglas is still related to the Pareto distribution, the exponential and the quadratic distributions are only loosely related to the yield equations derived when the input, not the coefficient, is random.

von Liebig Production Functions with Labor as an Input

This section will explore some of the implications of the von Liebig production function when both inputs can be distributed evenly across the field and a plant receives the correct relative proportions of the inputs but both inputs are functions of labor--for example, the amounts of water and nitrogen that reach a plant depend on people watering and fertilizing it. Analysis of this kind can be useful when the two types of labor should be distinguished, but only aggregate labor data are available.

Assume that the "true" plant production function takes the form of

$$Y = \min[a_1\ x_1(L_1),\ a_2\ x_2(L_2)] \tag{30}$$

where x_1 and x_2 are inputs; L_1 and L_2 are the amounts of labor used to achieve the levels of x_1 and x_2, respectively; and a_1 and a_2 are parameters. If the farmer is operating efficiently, then L_1 and L_2 will be chosen such that $a_1\ x_1(L_1) = a_2\ x_2(L_2)$. If both inputs are Cobb-Douglas in labor--that is, if $x_1(L_1) = L_1^{\alpha}$ and $x_2(L_2) = L_2^{\beta}$, then $a_1\ x_1 = a_2\ x_2$ implies

$$L_2 = kL_1^{\gamma}, \qquad \text{with } k = \left(\frac{a_1}{a_2}\right)^{1/\beta}, \qquad \text{and } \gamma = \frac{\alpha}{\beta}. \tag{31}$$

If total labor used, $\overline{L}$, equals $L_1 + L_2$, then substituting in the expression for L_2 and totally differentiating yields

$$\frac{dL_1}{d\overline{L}} = (1 + \gamma k L_1^{\gamma-1})^{-1}. \tag{32}$$

Now, since yield is produced efficiently, on the knife-edge (where $a_1x_1 = a_2x_2$), then $Y = a_1x_1(L_1) = a_1L_1^a$, and

$$\frac{dY}{d\bar{L}} = \left(\frac{dY}{dL_1}\right)\left(\frac{dL_1}{d\bar{L}}\right) \tag{33}$$

$$= \frac{\alpha a_1 L_1^{\alpha-1}}{1 + \gamma k L_1^{\gamma-1}} > 0.$$

$$\frac{d^2Y}{d\bar{L}^2} = \frac{d}{d\bar{L}}\left(\frac{dY}{d\bar{L}}\right) = \frac{d}{dL_1}\left[\left(\frac{dY}{d\bar{L}}\right)\left(\frac{dL_1}{d\bar{L}}\right)\right] \tag{34}$$

$$= (1 + \gamma k L_1^{\gamma-1})^{-3}(\alpha\ a_1\ L_1^{\alpha-2})\ \{(\alpha - 1) + (\alpha - \gamma)\ \gamma k L_1^{\gamma-1}\}$$

which is less than 0 since $0 < \alpha < 1$ and $0 < \beta < 1$ for Cobb-Douglas, $\gamma = \alpha/\beta$, and the two terms multiplied by the terms in brackets are positive. Thus, this yield-labor production function is an increasing concave function of the total labor used.

Now, suppose that data are available on the total amount of labor used and on the amount of one of the inputs used--x_1 in the following example. What would be an appropriate function to estimate econometrically, given those two kinds of data? The production function takes the same form as in (14). Substituting in $x_2(L_2)$ for x_2 gives

$$Y = \min(a_1x_1,\ a_2L_2^{\beta}). \tag{35}$$

Further substituting $L_2 = \bar{L} - L_1$ and $L_1 = x_1^{1/\alpha}$ into this function gives

$$Y = \min[a_1x_1,\ a_2(\bar{L} - x_1^{1/\alpha})^{\beta}]. \tag{36}$$

Since it is assumed that the farmer is operating efficiently, then the production function can be estimated as a simultaneous system, with

$$Y = a_1 x_1 + \varepsilon_1$$

$$Y = a_2(\bar{L} - x_1^{1/a})^{\beta} + \varepsilon_2. \quad (37)$$

Note that this result relies on more particular assumptions than before, namely, that both inputs are Cobb-Douglas in labor, that inputs and coefficients are deterministic, and that the farmer is operating on the knife-edge of efficiency.

Summary and Conclusions

Though experimental evidence indicates that the von Liebig crop production function best predicts yields on experimental plots, most larger areas, such as farms, cannot be controlled precisely for uniformity. The question then becomes, what production functions will best predict yield when crop response or inputs are not uniform? The preceding analysis shows that smooth aggregate production functions can be derived from a fixed-proportions function when randomness affects some aspect of that function.

Therefore, if a density function is known (even approximately) for the random element in crop growth, this analysis provides a method for justifying a particular aggregate production function. Additionally, if one aggregate functional form can be shown to predict crop yields much better than another one, then a probability density function for the random element can be derived. Through analyses of this type, the link between the biology of plant growth and aggregate crop production can be strengthened.

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