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ESTIMATION AND TESTING IN DEMAND SYSTEMS
WITH CONCAVITY CONSTRAINTS

by

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DEPARTMENT OF AGRICULTURAL AND
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ABSTRACT

A method for imposing or testing curvature restrictions in demand systems is suggested using Bayesian inference and inequality constrained estimation. The approach makes use of Monte Carlo integration and the approach suggested by Geweke (1986). The result is an inequality constrained estimate of the parameter vector for a demand system, plus an estimate of the probability that the inequality restrictions hold. Application to the U.S. manufacturing data of Berndt and Wood using the translog cost function illustrates the method.

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1. Introduction

The development of flexible functional forms for demand systems makes it possible to represent arbitrary technologies or preferences. This generality has come at a cost, however, in that demand systems derived from flexible cost or indirect utility functions need not be consistent with optimizing behavior. Restrictions on the parameters of the demand system remedy this situation when estimates are inconsistent with symmetry or homogeneity.

Unfortunately, experience has shown that estimated demand systems may also violate curvature restrictions, such as concavity of a cost function. Curvature restrictions involve inequality restrictions on parameters and make estimation difficult (e.g. Jorgensen and Fraumeni (1981); Gallant and Golub (1984); Hazilla and Kopp (1985); Morey (1986)). While Diewert and Wales (1987) suggested alternative flexible forms which could satisfy curvature restrictions globally, this does not solve the problem of testing such restrictions, and requires that one abandon simpler and more familiar forms such as the translog (Christensen, Jorgensen, and Lau (1973)) or generalized Leontief (Diewert (1971)).

The result is that flexible forms are chosen partly by the signs of estimated elasticities. If the inequality restrictions are violated in one model, another is often tried. This is analogous to sequential pre-testing, and the statistical implications of this search process are well documented (Judge and Bock (1978)).

This paper examines the problems of estimation and inference with curvature restrictions on demand systems. Following Geweke (1986), we make use of a Bayesian approach to finding an inequality-constrained estimator, in the context of imposing concavity on a translog cost function. Treating the implied constraints on the parameter space as (diffuse) prior information, we show that estimation and inference are straightforward using an alternative objective, the minimization of expected loss. While the approach is Bayesian, no prior information is required beyond the restrictions for curvature, such as concavity of the cost function. The result of the procedure is a posterior distribution for parameters that is conditional on the observed data and makes possible probability statements about parameters and associated hypotheses. The key elements are an estimated parameter vector consistent with the inequality restrictions and a proportion which can be interpreted as the probability that the inequality restrictions hold.

In section 2, more details are given concerning the translog and the necessary restrictions for concavity. Section 3 characterizes the problem of estimating parameters subject to inequality restrictions. An application using the Berndt and Wood (1975) data set is presented in section 4, and section 5 concludes the paper with a summary of results.

2. The Translog Cost Function

Following Diewert and Wales (1987), the translog cost function can be written as a function of output y, time t (to capture possible technical change), and prices P_i $(i=1,\ldots,n)$, where n is the number of inputs:

$$\ln C = \alpha_0 + \sum_i^n \alpha_i \ln P_i + \alpha_Y \ln Y + \alpha_t t$$
$$+ \sum_i^n \sum_j^n \gamma_{ij} \ln P_i \ln P_j + \sum_i \phi_i \ln y \ln P_i$$
$$+ \sum_i^n \tau_i t \ln P_i + 0.5*(\alpha_{yy} \ln y^2 + \alpha_{tt} t^2 + 2*\alpha_{ty} t \ln y)$$

In virtually all applications, the equality restrictions

$$\gamma_{ij} = \gamma_{ji}\ (\forall i\neq j),\quad \sum_j^n \gamma_{ij}=0\quad (\forall i),\quad \sum_i^n \tau_i=0,\ \sum_i^n \phi_i=0,\ \text{and}\ \sum_i^n \alpha_i=1$$

have been imposed so that the cost function satisfies symmetry and linear homogeneity in prices. The 21 unconstrained parameters for the case of four inputs are $\underline{\Theta}=(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_y,\alpha_t,\gamma_{11},\gamma_{12},\gamma_{13},\gamma_{22},\gamma_{23},\gamma_{33},\tau_1,\tau_2,\tau_3,\phi_1,\phi_2,\phi_3,\alpha_{tt},\alpha_{yy},\alpha_{ty})'$. Additional restrictions on the trend or output coefficients can be imposed to restrict the nature of technical change or the expansion path of the underlying technology.

The remaining property required for a well-behaved technology is that the cost function must be concave in the n input prices. The matrix of second derivatives with respect to these prices must be negative semi-definite, which implies inequality restrictions involving both the γ_{ij} parameters and the observed data.

A sufficient condition for concavity in the translog is that the matrix $\Gamma = \{(\gamma_{ij})\}$ $(i,j=1,\ldots,n)$ must be negative semi-definite. Jorgensen and Fraumeni (1981) reparameterized the translog using a Cholesky factorization of Γ and obtained concavity using inequality restrictions on the new parameters. However, the statistical behavior of the estimated parameter vector is difficult to determine. Furthermore, Diewert and Wales (1987, page 48) noted the tendency of this procedure to cause Γ to be "too negative semi-definite", limiting the usefulness of the translog in cases where these inequality restrictions are not satisfied by unconstrained estimates.

The condition which is necessary and sufficient for concavity is that the matrix

$$\Gamma - [\mathrm{diag}(\underline{s}) - \underline{ss}']$$

must be negative semi-definite, where $\underline{s}$ is the vector of factor shares

(Diewert and Wales (1987)). Note that for positive shares, Γ itself need not be negative semi-definite because the second part of the expression will be, as long as factor shares are positive.

3. Inequality Constrained Estimation

The usual approach to estimation of demand systems with inequality constraints could be characterized as follows. First, one estimates the parameters of a chosen flexible form, such as the translog, taking into account equality restrictions of the sort described above. If concavity is satisfied, then the constraint is not binding, the property in question is assumed to be consistent with the data, and no further restrictions are imposed. No attempt is made to assign a probability to the concavity hypothesis, since it holds at the estimated parameter vector, usually the maximum likelihood estimate.

Alternatively, the inequality restrictions may be violated and must be imposed somehow. By remaximizing the likelihood function over the same parameter space, subject to inequality restrictions on $\underline{\Theta}$, one obtains the best-fitting parameters which also satisfy concavity. Interpretation of this procedure follows that used in testing equality restrictions--is the decrease in the value of the likelihood function a significant one?--even if the statistical behavior does not.[1] Geweke (1986) argued that the existence of restrictions on the parameter space in the form of inequalities could be treated as prior beliefs about the parameters of the model. These could be informative, or in the absence of any other information, diffuse priors could be used to summarize this evidence. In the former case, some

[1] Although see Dufour (1987) and Wolak (1987) for results concerning the behavior of conventional tests in the linear regression model.

probability density function $p(\underline{\Theta})$ would be used to summarize prior information; in the latter case, $p(\underline{\Theta}) \propto c$. Either type of prior information, when combined with the sample likelihood function and observed data $\underline{y}$ using Bayes Theorem, yields a posterior likelihood function:

$$p(\underline{\Theta}|\underline{y}) \propto p(\underline{\Theta}) * l(\underline{\Theta}|\underline{y}).$$

It is defined only over the restricted parameter space. This function summarizes all available information about the parameters of the model, including the inequality restrictions. Point estimates, depending on the underlying objective, can be obtained using this posterior distribution. Making use of the corner solution in cases where the restrictions are binding--the result when constrained maximum likelihood estimation is used--corresponds to a loss function which gives rise to the dictum "use the β with the largest likelihood value." However, this outcome need not lead to a plausible estimate, nor is it the only possible objective. With a linear loss function, the median of the posterior distribution minimizes expected loss, and if it is quadratic, one should use the mean. As Geweke (1986, page 128) notes, "the inequality constrained maximum likelihood estimator is not likely to be especially interesting, since it may well lie at one end-point of its distribution (whether that distribution is the posterior, or the sampling theoretic distribution)."

The Bayesian approach to constraints on demand systems, then, plus a quadratic loss function, requires finding the mean of the posterior density function for the parameters. While this is not difficult conceptually, in complicated cases, exact formulas are not available. The necessary integrals over the truncated region of the parameter space could not be solved in any but the most trivial of cases.

In a series of papers, Geweke (1986) and others (Kloek and van Dijk (1980); Griffiths (1987)) have pursued this approach to inference using Monte Carlo integration of functions that are too complicated to evaluate analytically or numerically. The method has been demonstrated for inequality restrictions in single-equations by Geweke (1986), Griffiths (1987), and Judge et al. (1988, Chapter 20) and a procedure exists for imposing such restrictions in SHAZAM (White, 1987).

In the demand systems case, the concavity test requires determining whether the matrix of second derivatives of the cost function--or the matrix of substitution elasticities--is negative semi-definite. A common way of testing this is to compute the eigenvalues of either matrix. All of the non-zero eigenvalues must be negative for the concavity condition to hold. Any positive eigenvalue means that the concavity condition is rejected; this probability of rejection is the value of interest in inference. However, the distribution of the coefficients in demand systems is too complex to permit computation of this probability using analytic methods. The distribution of the eigenvalues is even more complicated.

Monte Carlo integration is therefore the only feasible approach. It begins with the estimation of the unrestricted demand system. Estimates of the coefficents and the corresponding covariance matrix are obtained. These are used along with the normality assumption to generate a large number of replications by the Monte Carlo method. For each replication, the matrix of elasticities is obtained, and the eigenvalues are computed. The eigenvalue check for concavity is then performed for each draw.

The probability that the concavity restriction holds is obtained by calculating the proportion of Monte Carlo replications satisfying the eigenvalue test. Estimates of the parameters consistent with the concavity restriction are obtained by computing the mean of the coefficient estimates for all replications where the eigenvalue test is satisfied.[2]

The Bayesian truncated posterior approach is different from the methods used by Diewert and Wales (1987), who directly imposed restrictions on the coefficients of the flexible forms they examined. Here, the restrictions are imposed on the posterior distribution of the coefficients. This method yields an easy test of the inequality hypotheses, in addition to estimates of the parameters.

In the next section, this procedure is applied using the translog cost function. Since elasticities depend on relative prices (through the observed factor shares), for the translog, a different set of restrictions must be used to impose concavity at every data point. We discuss both imposing concavity at the mean of observed factor shares and at each data point.

4. A Concave Cost Function for U.S. Manufacturing

The data from Berndt and Wood (1975) are used in this section to obtain a concave translog cost function using Monte Carlo integration. These data include total input costs (TIC), an index of output (Q), and factor shares and price indices for capital (K), labor (L), energy (E), and materials (M) inputs for U.S. manufacturing for the years 1947-1971. The same data were used by Berndt and Khaled (1979), Diewert and Wales (1987) and many others

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[2] It is necessary to check whether concavity holds at the mean of these data points, unless the region of the parameter space consistent with concavity can be shown to be a convex set. Otherwise, there is no guarantee that the mean of parameter vectors consistent with the restriction will itself satisfy concavity. It will prove much easier simply to verify that concavity holds at the mean, since elasticities using those parameter values are likely to be of interest anyway. We thank Charles Blackorby for this point.

in studying the properties of flexible forms.

Most recently, Diewert and Wales (1987) found that the translog cost function, when fit to these data with a system of share equations (K, L, and E) of the form

$$s_i = \alpha_i + \sum_{i}^{n} \gamma_{ij} \ln P_j + \phi_i \ln y + \tau_i t,$$

produced violations of concavity at six data points.[3] They imposed concavity using the Jorgensen and Fraumeni (1981) approach, but found the quality of results to deteriorate considerably, and so suggested alternative flexible forms (generalized McFadden; generalized Barnett) which could satisfy concavity.

In this section, we explore how the translog can be used along with the inequality-constrained estimator described earlier to obtain results consistent with the concavity restriction. One reason for doing this is to permit the translog to be used for its convenience, without requiring the sacrifice of concavity. However, even when there are no practical considerations to prefer one form or another, estimates of the translog parameters consistent with concavity are useful for comparisons of results from different flexible forms to select the most appropriate form or for testing the concavity restriction.

For simplicity, concavity was checked only at the means of observed shares. First, it was necessary to obtain the unconstrained parameter estimates of the vector of translog parameters, $\underline{\Theta}$. This was done using an iterated seemingly unrelated regressions estimator, obtaining essentially

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[3]This occurs for the years 1949-1953 and 1956 when predicted factor shares are used to calculate substitution elasticities.

the same results as did Diewert and Wales (1983). The coefficient estimates are shown in Table 1. When combined with the estimated covariance matrix these characterize the multivariate normal from which the parameters are hypothesized to be drawn. With completely noninformative priors, this multivariate normal would be the posterior distribution.[4] Combining the sample likelihood function with the inequality restrictions on the parameters--that they lead to a matrix of substitution elasticities with no positive eigenvalues--the posterior density function is the truncated multivariate normal. Its region of support is the region of the parameter space consistent with the concavity restriction.

In principle, we could work directly with such a distribution, finding the mean or median, constructing interval estimates, etc. The difficulty in doing so in the present case illustrates that it is not necessary to be able to solve the appropriate integral to find the mean of the posterior distribution. In fact, one cannot even calculate the region over which it should be solved without extensive computations.[5]

For our purposes, all that is necessary is to be able to perform the Monte Carlo integration. A random sample of 10,000 replications (plus the antithetic replications, giving a total of 20,000 replications) is drawn from the multivariate normal. In this case, that distribution is the 21-variate normal with mean vector given by the estimated parameters in Table 1 and the estimated covariance matrix. This distribution summarizes our

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[4]This treats the estimated covariance matrix as known. For discussions of the Bayesian analysis of Seemingly Unrelated Regression models, see Zellner (1979) or Judge et al. (1985).

[5]Though see Caves and Christensen (1980) or Barnett, Lee, and Wolfe (1985) for examples with fewer goods in which the "regular" regions of various functional forms were constructed.

knowledge about the parameters of the translog.

The method works as follows. For each replication, compute elasticities using the parameters drawn and the mean factor shares. Then check whether the replication corresponds to a violation of concavity or whether the computed elasticities satisfy the restriction. After this is done for each of the replications, the percentage of draws not violating concavity estimates the probability that the restriction holds. For a quadratic loss function, we use the mean of those vectors as the estimated parameter vector.

With this approach, the items of interest are the estimated parameters and standard errors--easily calculated for each parameter estimate as the standard deviation of the replications consistent with concavity, divided by the square root of that number of replications. Estimates of elasticities using these parameter estimates are also of interest. As noted earlier, these elasticities are necessary to verify that concavity holds at the mean of the replications consistent with the restriction; although that seems likely, we know of no easy way to rule out counterexamples. Finally, interval estimates or histograms for the parameters or elasticities can be used to summarize in a probabilistic fashion the information about these values that is contained in the posterior distribution of the parameter vector.

These results are reported in Tables 2 and 3 and in Figure 1. Table 2 contains the summary statistics from the replications, including the estimated mean of the truncated posterior distribution and standard errors. Of the 20,000 replications, 10,062 satisfied the concavity restriction, representing a probability of 0.5031 that the restriction holds. Following

Geweke, a standard error can be calculated using the formula

$$s.e.(\hat{p})=\surd(\hat{p}(1-\hat{p})/n),$$

which is .00354 in this example with n=20,000 replications. When greater precision is desired, more Monte Carlo replications can be included.

Table 3 reports elasticities calculated using these parameter estimates for selected data points. These were obtained by using the parameter estimates from Table 2 in the usual translog formulas, along with the predicted budget shares for 1947 and 1971 and the mean observed shares.[6] It is interesting to note that using these parameters, concavity holds not only at the mean values for shares but at all 25 data points in the Berndt and Wood data set. Thus, while some of the 10,062 replications consistent with concavity at the mean would not satisfy it at all 25 data points, the <u>mean</u> of those replications is consistent with concavity everywhere. This suggests that imposing the restriction at all 25 data points will not greatly reduce the probability that the restriction holds.[7]

Figure 1 contains histograms which represent the posterior distributions for own-elasticities of substitution. These were computed from the substitution elasticities saved from each replication satisfying concavity, a subset of all 20,000 replications, and for the entire set, shown for comparison. The truncation of own elasticities of substitution at

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[6] An alternative set of elasticities could be obtained by using the coefficient estimates given by the posterior mean to predict factor shares, rather than using the actual data.

[7] We have done a smaller version of the same experiment, in which we imposed concavity at every data point. This increases computing time and the number of inequalties which must be checked by a multiple equal to the number of data points, but not the complexity of the problem. We find that there is a decrease in the probability that concavity holds of approximately 10 percent, with a relatively small number of replications.

zero is apparent from the histograms, but these also illustrate that many cases in which negative values are obtained for some of the elasticities still violate concavity. Similar summaries could be generated for the parameter estimates or the eigenvalues, but the elasticities have the advantage of being unit-free and interpretable.

Table 3 indicates that our results are not subject to the criticism of Diewert and Wales that constrained estimates are too negative semi-definite. Of course, this is due to the use of the necessary restriction, rather than the sufficient one of Jorgensen and Fraumeni. We also tested the latter restriction, but found that the probability that concavity holds dropped to zero--none of the 20,000 replications drawn earlier satisfy the stronger restriction. This illustrates the difference between the two approaches--for the Berndt and Wood case, the strong restriction dramatically understates the compatibility of the data with the concavity restriction.

5. Conclusions

This paper has shown how, using a Bayesian approach, inequality constraints can be imposed in demand systems. Information from unconstrained estimation of the parameters of the demand system is combined with inequality restrictions to obtain a posterior distribution that summarizes all information about the parameters and conditions the analysis on the actual outcomes observed in the sample. Under a quadratic objective, the mean of this distribution yields a parameter vector that minimizes expected loss.

Application to the Berndt and Wood data set shows that the method yields both a probability that the restriction holds and estimates of the coefficients. The approach does not produce corner solutions as estimates and, while it is somewhat computer-intensive, it is very intuitive and easy to apply, so it provides a nice alternative to the nonlinear programming problems needed to impose inequality constraints in previous applications.

Of interest in future applications will be results using other demand systems and other data sets. Using the same data set, we have found in limited simulations that the probability that concavity holds appears much higher with a generalized-Leontief demand system. Such results will be of use in the problem of selecting appropriate flexible forms for particular applications.

It will also be of interest to examine the normality assumption and the role of the covariance matrix. Here, we have appealed to asymptotic theory, treating the posterior distribution as the multivariate normal. As noted by Zellner, this is conditional on the estimated covariance matrix; the posterior distribution for both the parameter vector and variance matrix is otherwise more complicated. Only in simpler settings, such as two equations with identical design matrices can exact properties be obtained. This method could be used with no greater difficulty if Monte Carlo replications were generated from a more complicated posterior distribution. We chose to use the multivariate normal to be consistent with the way the conventional estimates are interpreted.

One may doubt the initial normality assumption as well. For instance, because of restrictions on the range of error distributions in share systems, the Dirichlet was used by Woodland and the lognormal by Rossi. In that case, the method we have suggested could be applied with alternative density functions or even those estimated from bootstrap-generated simulations.

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Table 1. Estimated Translog Coefficients

Parameter*	Estimate	St. Error	T-Ratio
α_0	0.27150	16.532	0.016423
α_1	0.27034	0.037939	7.1255
α_2	0.41622	0.085991	4.8402
α_3	0.19513	0.015073	12.946
α_y	0.93261	6.3458	0.14696
α_t	0.057111	0.24409	0.23397
γ_{11}	0.034405	0.0040311	8.5349
γ_{22}	0.13876	0.047464	2.9235
γ_{33}	0.015034	0.0060254	2.4951
γ_{12}	0.012729	0.0090091	1.4129
γ_{13}	-0.0078071	0.0016745	-4.6624
γ_{23}	0.0081668	0.0098706	0.82739
τ_1	0.0012273	0.00033507	3.6630
τ_2	-0.00020166	0.0011903	-0.16943
τ_3	0.00077853	0.00027661	2.8146
ϕ_1	-0.040733	0.0072614	-5.6095
ϕ_2	-0.031028	0.016463	-1.8847
ϕ_3	-0.028742	0.0025787	-11.146
α_{tt}	0.0011512	0.0018481	0.62289
α_{yy}	0.0010494	1.2179	0.00086161
α_{ty}	-0.012290	0.046919	-0.26194
γ_{14}	-0.039327	0.011970	-3.2854
γ_{24}	-0.15966	0.052901	-3.0180
γ_{34}	-0.015394	0.0081792	-1.8821
γ_{44}	0.21438	0.059845	3.5822

Log of Likelihood Function = 447.561

* 1=capital, 2=labor, 3=energy, 4=materials. Estimates of parameters involving materials were obtained through the restrictions.

Table 2. Results of Constrained Estimation

Negativity and Concavity

20000 replications -- 10062 satisfied proportion = 0.50310

Precision of Proportion= 0.00354

parameter	average	stdev	variance	precision
α_0	0.13248	16.384	268.42	0.16333
α_1	0.26988	0.037795	0.0014285	0.00037679
α_2	0.41682	0.086751	0.0075258	0.00086483
α_3	0.19570	0.016802	0.00028231	0.00016750
α_y	0.98298	6.2887	39.548	0.062693
α_t	0.056290	0.24197	0.058548	0.24122
γ_{11}	0.033241	0.0037434	0.000014013	0.000037318
γ_{22}	0.10797	0.034264	0.0011740	0.00034158
γ_{33}	0.014516	0.0051030	0.000026040	0.000050872
γ_{12}	0.0074911	0.0071817	0.000051577	0.000071596
γ_{13}	-0.0082594	0.0016000	0.000002560	0.000015951
γ_{23}	0.0065574	0.0094656	0.000089597	0.000094364
τ_1	0.0013382	0.00032169	0.000000103	0.000003207
τ_2	0.00047204	0.00097147	0.000000944	0.000009685
τ_3	0.00081576	0.00021890	0.000000048	0.000002182
ϕ_1	-0.040664	0.0072401	0.000052419	0.000072178
ϕ_2	-0.031232	0.016595	0.00027541	0.00016544
ϕ_3	-0.028851	0.0031815	0.000010122	0.000031717
α_{tt}	0.0011389	0.0018338	0.0000033629	0.000018282
α_{yy}	-0.0080414	1.2069	1.4567	0.012032
α_{ty}	-0.012151	0.046513	0.0021634	0.00046369

Table 3: Estimated Elasticities

Using $\hat{\theta}$, the unconstrained maximum likelihood estimate

Observation	σ_{11}	σ_{22}	σ_{33}	σ_{44}
1947	-5.92	-0.78	-13.93	-0.03
1971	-5.17	-0.79	-13.62	-0.07
mean	-5.67	-0.80	-13.83	-0.05

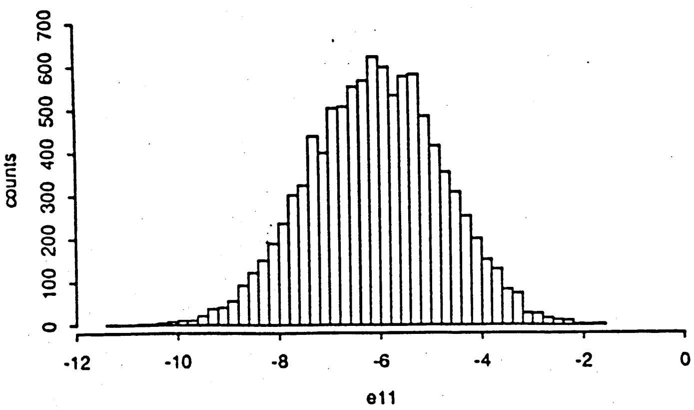
Observation	σ_{12}	σ_{13}	σ_{14}	σ_{23}	σ_{24}	σ_{34}
1947	1.89	-2.13	-0.08	1.73	0.02	0.46
1971	1.86	-2.42	-0.31	1.59	0.12	0.45
mean	1.87	-2.26	-0.17	1.66	0.07	0.45

Using $\bar{\theta}$, the mean of the posterior distribution for θ

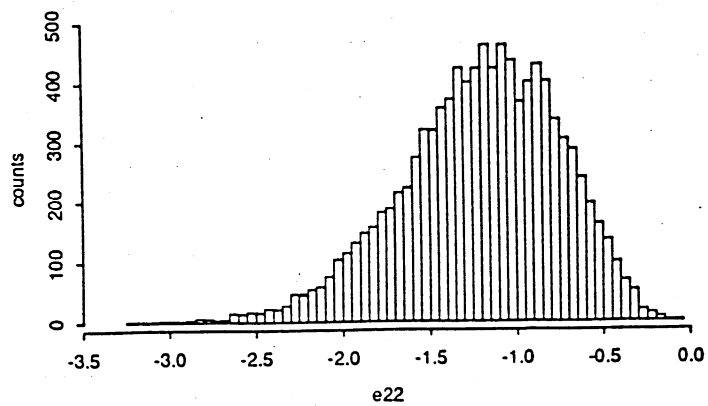
Observation	σ_{11}	σ_{22}	σ_{33}	σ_{44}
1947	-6.29	-1.27	-14.19	-0.15
1971	-5.64	-1.14	-13.86	-0.19
mean	-6.08	-1.21	-14.09	-0.17

Observation	σ_{12}	σ_{13}	σ_{14}	σ_{23}	σ_{24}	σ_{34}
1947	1.52	-2.30	0.11	1.59	0.25	0.55
1971	1.51	-2.62	-0.08	1.48	0.32	0.54
mean	1.51	-2.45	0.03	1.53	0.29	0.54

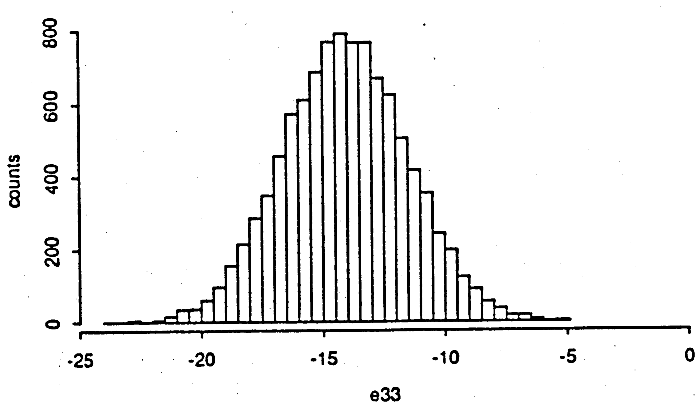
capital, concavity imposed



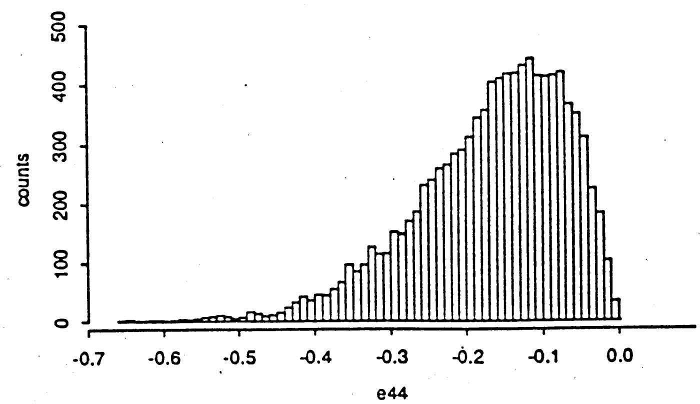
labor, concavity imposed



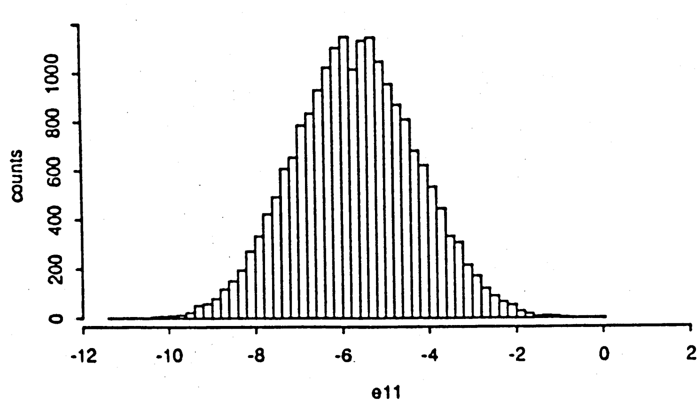
energy, concavity imposed



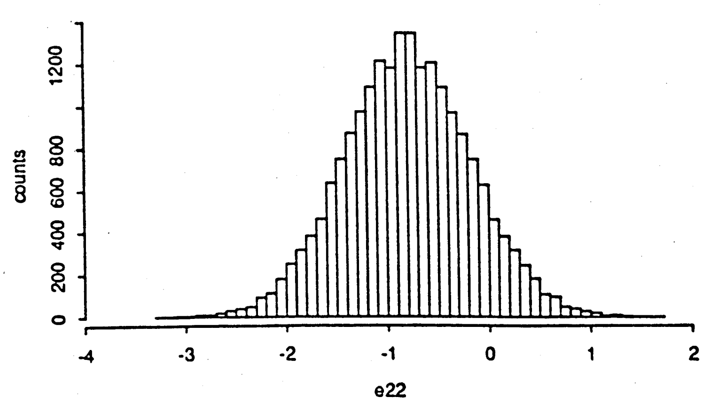
materials, concavity imposed



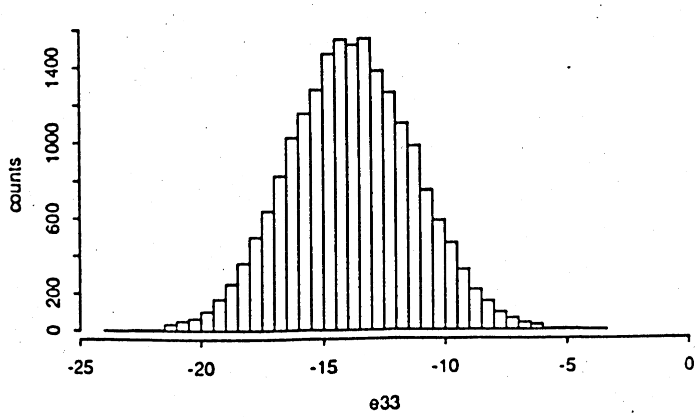
capital, all replications



labor, all replications



energy, all replications



materials, all replications