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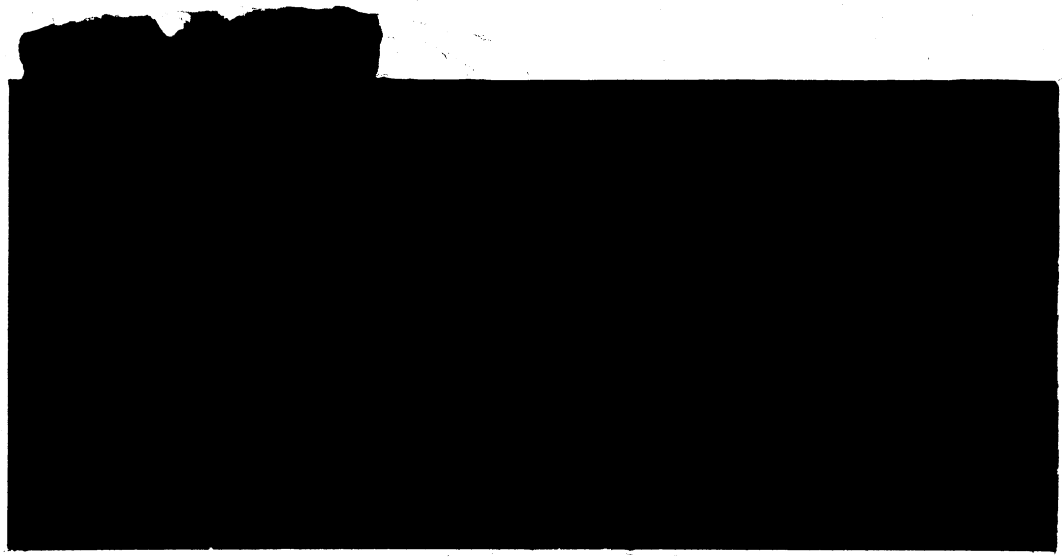

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SELECTING FUNCTIONAL FORMS FOR COST FUNCTIONS: BACKGROUND

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Introduction

The goal in input demand analysis is to generate a system of equations relating factor shares and possibly total expenditures to input prices and output. Regression can then be used to produce estimates of elasticities or test hypotheses of interest. Generally, consideration is limited to systems derived from either production functions or the dual cost or profit functions, and the problem of specifying the functional form is that of approximating these unknown functions.

This paper considers the choice of functional form for cost functions. Advantages over the production function approach are well known, and it is anticipated that particular functional forms would perform the same for production or profit functions, should other considerations lead to their use. Comparison of functional forms is, therefore, limited to cost functions for U. S. agricultural production.

It is assumed that the relevant data for four input categories are correctly measured. The problems of upward-sloping factor supplies, endogenous output, and bias due to the aggregation of inputs are set aside. The remaining problem is then to select a functional form to approximate the unknown cost function; Shephard's Lemma then can be used to provide expressions for factor shares by differentiation and a seemingly unrelated regressions framework for estimation (Zellner).

The focus will be on flexible functional forms and on attempts to generalize commonly used functional forms. Specifically, the generalized Box-Cox (Berndt and Khaled) and logarithmic Fourier flexible form (Gallant, 1982) are used in the estimation of a demand system for four agricultural inputs: capital, intermediate inputs, labor, and land. The paper proceeds as follows: First, the flexible functional forms are reviewed, focusing on the two alternative definitions of flexibility; experience to date with these flexible forms is then summarized; the application is presented; and, finally, the paper concludes with some observations and suggestions for further study.

Flexible Functional Forms

From the neoclassical production function and the assumption of cost minimization, factor demands are based on the necessary conditions for optimization. These are readily obtained for simple production functions such as the Cobb-Douglas; however, factor demands are difficult to determine when the production technology is complex. It is preferable, therefore, to attack the problem of obtaining an expenditure system for factors of production in a manner which preserves the complexity of the structure of input decisions yet simplifies the derivation. The use of duality theory and cost or profit functions has provided this alternative.

In the context of production studies, duality refers to the result that optimal input levels implied by a production function and necessary conditions are equivalently obtained by the minimization of a (dual) total cost function. According to Shepard's Lemma, these optimal input levels are obtained by differentiation of the cost function with respect to the input prices. This produces a set of factor demands with all of the properties of the technology underlying the cost function. Use of the dual cost function approach thus avoids the complexity of the primal problem; yet, it does not involve the sacrifice of consistency with the production technology inherent in ad hoc specifications of demand systems.

The theory of the firm is sufficient to indicate the relevant explanatory variables (prices of inputs and the level of output); the remaining problem is to choose a functional form. Varian, Diewert (1974), and McFadden illustrate that the cost function contains all of the information about the production technology present in the conventional production function. Furthermore, every cost function implies a well-behaved production technology; the logical approach in demand analysis is then to proceed directly to the cost function without prior regard to a functional form for the production technology. Because of the ease in generating the expenditure system, the cost function is a desirable starting point; one hopes to avoid restricting the types of technologies bracketed by the factor demands through generality in specification.

Generality has involved a particular attribute, flexibility, or the ability to approximate any unknown cost function. Specifically, flexibility is based in approximation theory, most often with a Taylor series interpretation. The idea in that case would be to provide the first- and second-order partial derivatives of an unknown function at some point. Let us designate this property "local flexibility," following Barnett (1983), and introduce the locally flexible functional forms for cost functions. We then examine the argument provided by Diewert (1974) for the flexibility of the generalized Leontief cost function. Finally, this section of the paper concludes with some criticisms of the empirical usefulness of locally flexible functional forms and the alternative approach by Gallant (1981, 1982).

The Translog

Christensen, Jorgensen, and Lau propose the translog as an approximation to unknown cost or production functions. Expressed as a second-order polynomial in logarithms of input prices and output, it is a generalization of the Cobb-Douglas (which is linear in logarithms). While it is not "self-dual" as is the Cobb-Douglas, the translog cost function which satisfies certain regularity conditions (Diewert, 1974) does correspond to a well-behaved production technology.

As is the case with all of the flexible functional forms, the advantage of the translog is that arbitrary configurations for the matrix of elasticities of substitution are possible. The translog generalizes the Cobb-Douglas case of elasticities of substitution equal to one by adding the second-order terms so that

$$\sigma_{ij} = 1 + \frac{\gamma_{ij}}{S_i S_j} \quad i \neq j$$

and

$$\sigma_{ii} = 1 + \frac{\gamma_{ii}}{S_i^2} - \frac{1}{S_i},$$

where γ_{ij} is the coefficient of 1/2 ln P_i ln P_j and S_i is the fitted share of input i. Clearly, the translog allows variation in these elasticities across input pairs or prices; and, unlike the Cobb-Douglas or constant elasticity of substitution (CES), it permits complementarities.

The translog is expressed as

$$\ln C = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln P_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln P_i \ln P_j + \beta \ln Y$$

$$+ \left(\frac{\theta}{2}\right)(\ln Y)^2 + \sum_{i=1}^{n} \delta_i \ln P_i \ln Y + \tau t + t \sum_{i=1}^{n} \tau_i \ln P_i$$

where ln C is the log of total costs, ln P_i is the log of the ith input price, ln Y is the log of output, t is time, and n is the number of inputs.[1]

In order for the translog to satisfy linear homogeneity in input prices, the following restrictions are imposed on the estimated parameters:

$$\sum_{i=1}^{n} \alpha_i = 1,$$

$$\sum_{j=1}^{n} \gamma_{ij} = 0,$$

$$\sum_{i=1}^{n} \delta_i = 0,$$

and

$$\sum_{i=1}^{n} \tau_i = 0.$$

Symmetry in the matrix of elasticities of substitution is generally imposed as well, involving the additional restrictions

$$\gamma_{ij} = \gamma_{ji},$$

for each (i, j) combination.

Factor shares are obtained by differentiating with respect to each logged input price so that

$$S_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \ln P_j + \delta_i \ln Y + \tau_i t, \qquad i = 1, \ldots, n.$$

The appearance of output introduces nonhomotheticity; restricting each δ_i to equal zero produces the linear expansion path of the homothetic case. Further restricting θ to equal zero produces a technology which is homogeneous of degree $(1/\beta)$. Setting β equal to one would then yield constant returns to scale.

The terms involving time are meant to capture the presence of technical change. If there is a sufficient number of observations, variation in expenditures and factor shares can be explained not only by relative prices and scale effects but by technical change (Lopez, Berndt and Khaled)--this is assumed to occur through relating the dependent variables to time. The term τ_i indicates technical change bias; as τ_i is greater than (equals or is less than) zero, technical change is said to be factor i-using (i-neutral, i-saving). Hicks-neutral technical change (at rate τ) occurred if each τ_i equals zero.

Binswanger (1974a, 1974b) applied the translog cost function in his study of technical change although he included time effects only in the absence of scale effects. Other examples in agricultural economics include Ray, who considered two outputs (crops and livestock); Brown and Christensen, who compare short-run and long-run substitution possibilities by introducing fixed factors; and Ball and Chambers, who examine the structure of the meat-processing sector in the United States.

The Generalized Box-Cox

Of the class of second-order polynomial flexible functional forms, the generalized Box-Cox is the most general to date. It includes the generalized Leontief and generalized square-root quadratic forms as special cases, and the translog appears as a limiting case. In various forms it has been applied by Denny, Kiefer, and Berndt and Khaled. The presentation here is taken from Berndt and Khaled.

The expression for total costs is

$$C = [1 + \lambda\ G(P)]^{1/\lambda}\ Y^{\beta(Y,P)}$$

with

$$G(P) = \alpha_0 + \sum_{i=1}^{n} \alpha_i P_i(\lambda) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} P_i(\lambda) P_j(\lambda),$$

P = the vector of input prices,

$$\beta(Y, P) = \beta + \left(\frac{\theta}{2}\right) \ln Y + \sum_{i=1}^{n} \delta_i \ln P_i,$$

and

$$P_i(\lambda) = \frac{P_i^{\lambda/2} - 1}{\lambda/2}.$$

With the restriction that the cost function is linearly homogeneous in input prices, the following restrictions are introduced:

$$\sum_{i=1}^{n} \alpha_i = 1 + \lambda \alpha_0,$$

$$\sum_{j=1}^{n} \gamma_{ij} = \left(\frac{\lambda}{2}\right) \alpha_i$$

and

$$\sum_{i=1}^{n} \delta_i = 0.$$

The cost function then reduces to

$$C = \left[\left(\frac{2}{\lambda}\right) \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} P_i^{\lambda/2} P_j^{\lambda/2}\right]^{1/\lambda} Y^{\beta(Y,P)}.$$

The term $\beta(Y, P)$ involves interactions between prices and output, therefore producing a nonhomothetic technology. Homotheticity requires that each δ_i be zero as in the translog case, and homogeneity of degree $(1/\beta)$ again follows when θ equals zero.

To introduce technical change into the generalized Box-Cox cost function, Berndt and Khaled multiply by the term

$$e^{T(t,P)}$$

where

$$T(t, P) = \left(\tau + \sum_{i=1}^{n} \tau_i \ln P_i\right) t.$$

The analogy to the translog specification is evident; τ_i represents technical change bias and so on.

Special cases of the generalized Box-Cox are obtained by fixing the Box-Cox parameter λ. When λ equals one, the Berndt and Khaled specification reduces to the generalized Leontief. The generalized square-root quadratic is obtained by setting λ equal to two. Finally, the translog is a limiting case as λ approaches zero.

Expressions for factor shares in the generalized Box-Cox cost function are produced by differentiation according to Shephard's Lemma. The share of factor i is given by

$$S_i = \frac{P_i^{\lambda/2}\left[\sum_{j=1}^{n} \gamma_{ij} P_j^{\lambda/2}\right]}{\left[\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} P_i^{\lambda/2} P_j^{\lambda/2}\right]} + \delta_i \ln Y + \tau_i t.$$

Substitution elasticities given by Berndt and Khaled (page 1225) can be obtained by differentiating once more and rescaling:

$$\sigma_{ij} = 1 - \lambda + \gamma_{ij} \frac{(P_i P_j)^{\lambda/2}}{S_i S_j} \tilde{c}^{-\lambda} + \lambda \frac{F_j(Y, t)}{S_j}$$

$$+ \lambda \left[1 - \frac{F_j(Y, t)}{S_j}\right] \frac{F_i(Y, t)}{S_i}, \quad i \neq j$$

and

$$\sigma_{ii} = 1 - \lambda + \gamma_{ii} \frac{P_i^{\lambda}}{S_i^2} \tilde{c}^{-\lambda} + \lambda \frac{F_i(Y, t)}{S_i} + \lambda \left[1 - \frac{F_i(Y, t)}{S_i} \right] \frac{F_i(Y, t)}{S_i}$$

$$+ \frac{\lambda}{2} \left[1 - \frac{F_i(Y, t)}{S_i} \right] \frac{1}{S_i} - \frac{1}{S_i},$$

where

$$\tilde{c} = \frac{C}{Y^{\beta(Y,P)} e^{\tau(t,P)}}$$

and

$$F_i(Y, t) = \delta_i \ln Y + \tau_i t.$$

Setting λ equal to zero, these reduce to the translog expressions for elasticities of substitution.

The generalized Box-Cox has not been applied in agricultural production studies except for an earlier version of the present study reported in Chalfant. The special case of the generalized Leontief has been used by Lopez in a study of technical change in Canadian agriculture. Chambers and Vasavada also applied the generalized Leontief to test for asset fixity in U. S. agriculture.

Local Flexibility

Most applications of flexible functional forms have involved either the translog or generalized Leontief. Since each permits unrestricted estimation of elasticities, it is evident that some other criterion is required to select a functional form for empirical applications. Several studies have considered this question using a variety of techniques.

Caves and Christensen use numerical analysis to examine the cases in which each of these two flexible forms can be expected to satisfy the theoretical restrictions from consumer behavior. Prior expectations about likely values for elasticities then can be used to suggest which form might be most successful in satisfying the theoretical restrictions. Taking demand theory as a maintained hypothesis permits the choice of functional form to be based on the ability of each to satisfy the implied restrictions.

Bayesian techniques were used by Berndt, Darrough, and Diewert in their study of Canadian expenditure data. This involved the comparison of likelihoods for the translog, generalized Leontief, and generalized Cobb-Douglas. They concluded that the translog performed best of the forms considered.

Finally, Monte Carlo experiments of the sort presented by Wales; Guilkey and Lovell; and Guilkey, Lovell, and Sickles can be used. These have produced mixed results. Wales finds that both the translog and generalized Leontief successfully approximate a two-good CES expenditure system. The papers by Guilkey and Lovell and Guilkey, Lovell, and Sickles consider approximation of a more general functional form with results less favorable. Guilkey, Lovell, and Sickles suggest that the translog may be the best of a bad lot.

The generalized Box-Cox is of particular interest in that it allows the choice among functional forms to be made by statistical inference. Subject to the validity of the generalized Box-Cox specification as a maintained hypothesis, tests for the more common flexible forms are constructed as tests for specific values for λ. This would seem to solve the problem of model selection since each commonly used form is nested in the generalized Box-Cox.

If, in fact, the unknown model is of the generalized Box-Cox form, then inferences concerning the parameter are valid; and any of the tests from nonlinear regression can be used. All of the flexible functional forms, including the generalized Box-Cox, can be thought of as possible functional forms for cost functions rather than approximations. However, there is reason to suspect the validity of such inferences when flexible functional forms are taken to be only approximations (the more common case). In recent papers White (1980, 1981) has questioned the practice of treating regression coefficients as local approximations of arbitrary functions.[2] This has led to an alternative approach to approximation, suggested by Gallant (1981, 1982). To introduce these results, it is worth reviewing Diewert's (1974) discussion of the properties of local approximations.

The essence of the cost function approach has already been pointed out. Any function of input prices which meets certain regularity conditions (Diewert, 1974) corresponds to some well-behaved production technology and contains all of the information about returns-to-scale and substitution possibilities. A twice-differentiable function for total costs is then suggested to approximate the true cost function; Diewert suggests a criterion for the approximation--second-order local approximation.

Diewert's definition of a second-order local approximation g to the true function g* is that, at some price vector P*,

$$g(P^*) = g^*(P^*),$$

$$\left[\frac{\partial\, g(P)}{\partial\, P_i}\right]\Bigg|_{P^*} = \frac{\partial\, g^*(P)}{\partial\, P_i}\Bigg|_{P^*} \qquad \forall_i,$$

and

$$\frac{\partial^2\, g(P)}{\partial\, P_i\, \partial\, P_j}\Bigg|_{P^*} = \frac{\partial^2\, g^*(P)}{\partial\, P_i\, \partial\, P_j}\Bigg|_{P^*}$$

for each (i, j) combination. Barnett (1983) has established that this is equivalent to the definition used in mathematics.[3]

It is instructive to consider the theorem and proof provided by Diewert (1974) to establish the flexibility of the generalized Leontief (unit) cost function.[4] The theorem (Diewert, 1974, page 115) may be stated as follows:

Given an arbitrary unit cost function C* which satisfies the conditions of a well-behaved technology and in addition is twice continuously differentiable at $P^* >> 0_n$, where

$$C_i^* \equiv \frac{\partial C^*(P^*)}{\partial P_i}$$

for all i and

$$C_{ij}^* \equiv \frac{\partial^2 C^*(P^*)}{\partial P_i \, \partial P_j}$$

for all i, j, then there exists a generalized Leontief unit cost function C(P) defined by

$$C(P) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} P_i^{1/2} P_j^{1/2}$$

which provides a second-order approximation to C* at the point

$$P^* = (P_1^*, P_2^*, \ldots, P_n^*).$$

By exploiting Euler's theorem, it can be shown (Diewert, page 159) that unit costs C*(P*) plus the derivatives C_{ji}^* (for $1 \leq i < j \leq n$) and C_{ii}^* (for 1 = 1, 2, ..., n) are completely determined by the n first-order partial derivatives C_i^* and the n(n - 1)/2 second-order partial derivatives C^*_{ij}, for $1 \leq i < j \leq n$. Differentiating the approximating generalized-Leontief form and equating the results produces

$$\sum_{j=1}^{n} b_{ij} \left(\frac{P^*_j}{P^*_i} \right)^{1/2} = C^*_i \qquad i = 1, 2, \ldots, n$$

and

$$\frac{1}{2} b_{ij}(P^*_i P^*_j)^{-1/2} = C^*_{ij} \qquad 1 \leq i < j \leq n,$$

a system of n plus n(n - 1)/2 equations in as many unknowns. The b_{ij}'s ($i \neq j$) are determined by the latter set; these, in turn, imply the b_{ii}'s through the first n equations. As Diewert concludes, "at the point $P^* \gg 0_n$, $C(P^*)=C^*(P^*)$ and the first and second order partial derivatives of C and C* will coincide."

Local flexibility is, therefore, established by verifying that, for any price vector P* and arbitrary cost function C*, the system of equations described by Diewert can be solved. For the system of equations to be satisfied for any C*, it should be possible to solve these equations when each is independent. This allows approximation of cost functions with any configuration of second derivatives and translates into a statement about the number of parameters required in the approximating function. That is, local flexibility is satisfied if the approximating function has enough parameters to permit all possible values of second derivatives--enough parameters to make each equation (derivative) independent.

Perhaps this is best illustrated by example--the Cobb-Douglas is not locally flexible as the elasticity of substitution must be one; second derivatives are implied by the first derivatives. The translog generalization of the Cobb-Douglas achieves local flexibility by the addition of the second-order terms--any values for substitution elasticities can be obtained in the translog form.

It is important to note that, while the translog features additional parameters by extending the Cobb-Douglas to the second-order and that this produces the second-order Taylor series interpretation, Diewert's definition of flexibility does not require a quadratic form or a Taylor approximation--only a twice-differentiable expression. Also, Diewert's proof does not require that there be the same number of parameters in the approximating form C(P) as derivatives of interest in the "true" form C*(P). That is, while the second derivatives turn out to be

$$\frac{1}{2} b_{ij}(P_i^* P_j^*)^{-1/2}$$

and

$$\gamma_{ij}$$

for the generalized Leontief and translog, respectively, it does not change Diewert's result if they are of a more general form such as

$$\frac{1}{2} b_{ij}(P_i^* P_j^*)^{-1/2} + \rho Y$$

or

$$\gamma_{ij} + \delta_{ij} P_i P_j .$$

A second-order local approximation can still be obtained.

One never actually <u>solves</u> the system of equations, so these expressions would simply permit more generality in the specification. The statistical formulation must be such that a unique set of parameters for the flexible functional form is produced in estimation. If so, then we can conclude that any expression with enough parameters to allow independent estimation of each element in the matrix of second-order partial derivatives qualifies as a (locally) flexible functional form.

Reflecting on Diewert's proof of local flexibility, it is evident that to repeat such a process in applied work requires knowledge of the true function, in which case approximation is unnecessary. Empirical practice is instead to use regression techniques to estimate the parameters of the flexible form, and local flexibility then represents a criterion to select a specification for regression. Since it involves only a statement about the number of parameters, it is, unfortunately, not a very selective criterion. Perhaps this explains the focus on the locally flexible form as a Taylor approximation.

Essentially, the commonly used locally flexible functional forms are quadratic forms expressed in terms of some transformation of price--logged prices in the case of the translog, square roots of prices in the generalized Leontief, and so on. Treating the terms in the remainder as third- and higher order derivatives, the coefficients can, of course, be related to the unknown function's derivatives. In the case of the translog, at the point of scaling it is believed that

$$\ln C \text{ is estimated by } \alpha_0,$$

$$\frac{\partial \ln C}{\partial \ln P_1} \text{ by } S_1 = \alpha_1,$$

$$\frac{\partial \ln C}{\partial \ln P_2} \text{ by } S_2 = \alpha_2,$$

and so on. The flexible form is in this fashion interpreted as a Taylor series approximation.

By focusing on the error term, Barnett (1983) points out that the Taylor interpretation provides little in the way of support for the locally flexible functional forms. The approximation is valid only within some neighborhood which may not include all of the data. Furthermore, the results he presents from the mathematics literature do not apply to the fixed-order (second-order) approximations used in the common locally flexible functional forms. This leads Barnett to accept Diewert's approach to local flexibility, being the same as the mathematical definition of a second-order local approximation, but to take issue with the common practice of "further limiting the <u>definition</u> of the class of second-order approximations to include only second-order <u>Taylor series</u> expansions."

Of a more crucial nature for the interpretation of the results from standard demand analysis are the findings concerning the actual behavior of a Taylor series approximation in a regression setting. References include Gallant (1979), Kumm, and Barnett (1983); but the first to draw the implications for flexible functional forms appears to be White (1980). Standard practice is to regress factor shares and possibly total expenditures on logged prices and output; regression coefficients are then interpreted as the coefficients in the Taylor series--the derivatives of the unknown function at the point of approximation. Furthermore, the point of approximation is assumed to be the point of scaling which is generally either the sample mean or a particular observation.

However, White (1980) showed that regression does not produce the coefficients of the Taylor series unless the true function is of that form. This is evident, considering the nature of the remainder terms; these can become quite large away from the point of approximation in a Taylor series but approach zero as that point is reached. Regression, on the other hand, attempts to make errors of approximation small over the range of the data.

What does this imply for demand analysis? Essentially, it implies that the estimated parameters of locally flexible functional forms do not consistently estimate parameters of the true function unless the latter is of the approximating class. Elasticities obtained are, therefore, also inconsistent. Diewert's proof then indicates that locally flexible functional forms could take on arbitrary configurations of elasticities--not that they will do so in practice.

The Taylor series interpretation seemed to restrict the class of flexible forms to quadratic forms, with the empirical task remaining being simply to determine the appropriate transformation of the independent variables. However, the observations of White (1980), Barnett (1983), and others indicate that the Taylor approximations have no apparent advantage over any functional form with a sufficient number of parameters to satisfy the definition of local flexibility. Furthermore, the model specification remains as a joint hypothesis in any inferences about elasticities. Whether one should prefer a Taylor approximation or some other locally flexible functional form as a maintained hypothesis is not clear.

White (1981) extends his results to nonlinear models and shows that the least-squares estimator converges to a least-squares approximation to the true model. This approximation is the parameter vector θ^* which minimizes the prediction mean squared error

$$S^2(\theta) = \int [g(z) - h(z, \theta)]^2 \, dF(z),$$

where $g(z)$ is the correct model; $h(z, \theta)$ is the approximating form; $F(z)$ is the distribution function of the vectors z; and integration is over the range of z. That the derivatives of h(•) evaluated at $\hat{\theta}$ and a particular z_0 need not resemble those of g(•) at z_0 is evident; least squares weights errors in predicting the response but not the derivatives of the unknown response function. Furthermore, it weights errors throughout the data rather than only at a point of approximation. As a result, fitted residuals may differ from zero at every point including the point at which the data are scaled. This is in contrast to the Taylor series approximation, which places a large cost on being wrong at the point of approximation, but zero cost of errors elsewhere in the data space.

The Fourier Flexible Form

Each of the flexible forms mentioned thus far presumably satisfies the conditions set forth by White (1980, 1981); given those regularity conditions and the absence of such errors as the improper aggregation of inputs, the estimated parameter vectors are strongly consistent for some asymptotic least-squares approximation of the unknown model. The asymptotic approximation, according to White (1981, page 420), minimizes the prediction mean squared error. However, optimal properties for the estimation of the derivatives of the unknown model (elasticities) do not follow. Essentially, this is due to the fact that the Taylor approximation criterion, which does produce derivatives of the unknown function at the point of approximation, does not require an approximating functional form to emulate the unknown response surface. While Taylor approximations do hold in local neighborhoods, White (1980, 1981) has established that the least-squares technique is inconsistent with such an approximation. Regression with a second-order specification, therefore, cannot be given a Taylor approximation interpretation; the model specification must be a maintained hypothesis.

The problem is that the measure of approximating error used to link the Taylor approximation to the true function is irrelevant in empirical practice. Statistical techniques do not place all the emphasis on not being wrong at one point. However, that is the Taylor criterion for approximation; see Barnett (1983) for a discussion of the details.

An alternative measure of approximating error is suggested by these observations--one which is large when the relevant errors of approximating a cost function are also large. Gallant (1981) suggests the Sobolev norm as the appropriate measure. The Sobolev measure of the distance of an approximating function g(x) from the true g*(x) is

$$||g^* - g||_{m,P,\omega} = \left[\sum_{|\mu|^* \leq m} \int_{\psi} \left| D^{\mu} g^* - D^{u}_{g} \right|^{P} d\omega(x) \right]^{1/P} \qquad 1 \leq P < \infty,$$

where m denotes the largest order derivative of g* which is of interest, D^{μ} denotes partial differentiation, $\omega(x)$ is a distribution function, and ψ is the region of approximation.

Why is the Sobolev norm relevant for demand analysis? Our interest is in approximating not only the unknown function but first and second derivatives; accordingly, one should seek out a measure of errors of approximation which emphasizes derivatives as well as errors in approximating the function. The Sobolev norm does exactly that. To make the Sobolev norm useful for applications, however, requires a flexible functional form which can make it small in empirical applications.

Gallant (1981) shows that an expansion in a Fourier series will approximate closely an arbitrary function, with "close" being measured by the Sobolev norm. Based on this finding, it can be assumed that every unknown cost function has a representation as a Fourier series. The representation improves in quality as more terms are added to the Fourier series so that asymptotically the Sobolev measure of approximation error can be made arbitrarily small. The derivatives of the Fourier series representation then represent the derivatives of the unknown function.

The task of empirical demand analysis can then be shifted to finding the appropriate vector of parameters of the Fourier series approximation. Some vector θ^* exists which makes the Sobolev norm,

$$S(\theta^*) = ||g^* - g||,$$

arbitrarily small; perhaps, this requires an infinite number of parameters. With a limited number of parameters K, some θ^0 would be available to minimize $S_K(\theta)$, where

$$0 \leq S(\theta^*) \leq S_K(\theta^0)$$

because θ^0 represents the parameters of a truncated Fourier series. It follows that applying least squares or some other estimation criterion produces an estimate $\hat{\theta}_0$ such that

$$0 \leq S(\theta^*) \leq S_K(\theta^0) \leq S_K(\hat{\theta}_0).$$

The estimate $\hat{\theta}_0$ is consistent for θ^0 as long as the estimation procedure is a good one; consistent estimation of elasticities has been shown by El Badawi, Gallant, and Souza to follow.[5]

Let us summarize the distinction between the polynomial approximation approach and the Fourier approximation. The goal in demand analysis is to approximate the derivatives of a function as well as the function itself, and a local approximation exists in the form of a second-order Taylor series. The parameters of the Taylor series are the derivatives of the unknown function at the point of approximation. However, because the approximation is local, estimation of the parameters using a global approximation method, such as least squares, generally leads to biased and inconsistent estimates of the derivatives of the unknown function.

Barnett (1983) argues convincingly that there is nothing in the definition of second-order approximation to suggest limiting attention to the class of second-order Taylor series approximations. However, the use of other approximating forms which are locally flexible, such as the generalized Cobb-Douglas (Diewert, 1973), does not improve matters of approximating derivatives unless the correct functional form is used. The problem is one of focusing on a local approximation criterion in obtaining the specification, but estimating its parameters based on a global technique (least squares).[6] Unless the least-squares estimator converges to the true parameter vector, rather than a least-squares approximation, the derivatives of the function, even at the point of approximation, need not be consistently estimated by the derivatives of the locally flexible functional form.[7]

It does not seem promising to focus on estimation techniques which do not emphasize a good fit throughout the data. Rather than "throwing away" information away from the point of approximation, we can focus instead on a global approximation technique consistent with least squares or other estimation procedures. It is not so much that global approximation is preferred to local, although that seems obviously so if global approximation imposes no additional cost in estimation. Instead, the situation is one in which no estimation technique can produce the local approximation.

A global approximation to a function and its derivatives exists in the form of a Fourier series with the quality of approximation improving with additional terms in such a way that approximation errors can be made arbitrarily small over the entire range of the sample. To within irrelevant approximation errors, therefore, unknown functions can be expressed as a Fourier series throughout their domain rather than at a particular point, i.e., all of the information relevant for demand analysis (the function, together with its first and second derivatives) is preserved.

The upshot is that one can fit the unknown model or its identical (for our purposes in demand analysis) Fourier series representation. Not all cost functions can be expressed as a translog; they can be expressed as a Fourier series. Following Barnett's terminology, we can designate this property global flexibility. Of course, the vector of parameters of the Fourier series is unknown. Estimation based on a Fourier series, however, produces a consistent estimator of that parameter vector.

This is no different than what estimation accomplishes in the presence of misspecification according to White--consistent estimation of an approximating function. However, because both the Fourier series approximation and the least-squares estimation procedure attempt to fit the unknown function throughout its domain, the derivatives (and, hence, elasticities) of the Fourier series approximation do resemble those of the unknown function. While the estimated vector of parameters of the approximation need not produce the global minimum for $S_K(\theta^0)$, the θ^0 which does so is consistently estimated. This is all that is ever hoped for in regression settings even when the functional form is not at issue.

The reader is referred to El Badawi, Gallant, and Souza for details concerning the requirements of the estimation procedure and the proof of consistency in estimation of elasticities; we now present the logarithmic version of the Fourier flexible form introduced in Gallant (1982). The discussion is a review, and the reader may wish to consult Gallant (1982) for the complete development or Chalfant and Gallant for an example expressed without the simplifying notation.

Let k_α denote a vector of integers or a multi-index. These can be used to denote partial differentiation so that

$$D^{k_\alpha} g(x) = \frac{\partial^{|k_\alpha|^*} g(x)}{\partial x_1^{k_1} \, \partial x_2^{k_2} \cdots \partial x_n^{k_n}} .$$

where

$$|k_\alpha|^* = \sum_{i=1}^{n} |k_i|$$

is the norm of a multi-index of length n. Multi-indexes are also used to represent the Fourier flexible form.

Let K_T denote the number of parameters, possibly a function of sample size T. The logarithmic version of the Fourier flexible form is

$$g_{K_T}(x|\theta) = u_0 + b' x + \frac{1}{2} x' Cx$$

$$+ \sum_{\alpha=1}^{A} \left\{ u_{0\alpha} + 2 \sum_{j=1}^{J} [u_{j\alpha} \cos(j\lambda_s k_\alpha' x) - v_{j\alpha} \sin(j\lambda_s k_\alpha' x)] \right\}$$

with

$$x = (\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4, \tilde{Y})$$

and scaling factor λ_S defined below,

$$C = -\lambda_S^2 \sum_{\alpha=1}^{A} u_{0\alpha} k_\alpha k_\alpha' ;$$

A, the number of multi-indexes; and J, a positive integer.

Choice of A and J determines the number of parameters, K_T. A reasonable choice for four inputs is to set the largest norm of the included multi-indexes at 3; then A = 19. With J = 1, this produces a parameter vector with length 63. However, due to the homogeneity restriction, 10 of these are not free parameters. Nine are redundant terms in the matrix C, and one is the element in the vector b corresponding to the deleted input share.

Gallant (1982) proves that it is sufficient to consider only those multi-indexes involving contrasts among the input prices when using $g_{K_T}(\cdot)$ to approximate a linearly homogeneous cost function. The 19 multi-indexes for the Fourier flexible form estimated in this paper are illustrated in table 1.

Prior to estimation, it is necessary to rescale the data--the Fourier cost function is periodic, so data must fall within $(0, 2\pi)$. Rescaling is accomplished by the following procedure.

First, from each member of the logged series of exogenous variables, subtract the minimum of that series and then add some ε, say, 10^{-5}. The result is a rescaled set of exogenous variables, between ε and ∞. Next, rescale any covariates such as output by the scalar

$$\lambda_C = \frac{\log\,(\text{maximum rescaled price}) + \varepsilon}{\log\left(\dfrac{\text{maximum value of the covariate}}{\text{minimum value of the covariate}}\right) + \varepsilon} .$$

Finally, all data are rescaled by

$$\lambda_S = \frac{6}{\log\,(\text{maximum rescaled price}) + \varepsilon} .$$

Table 1. Scaling Factors and Multi-Indexes for the Fourier Flexible Form ($K_T = 63$)

$$\tilde{P}_1 = \ln P_1 - \ln(0.69108) + \varepsilon$$

$$\tilde{P}_2 = \ln P_2 - \ln(0.89755) + \varepsilon$$

$$\tilde{P}_3 = \ln P_3 - \ln(0.69348) + \varepsilon$$

$$\tilde{P}_4 = \ln P_4 - \ln(0.40279) + \varepsilon$$

$$\tilde{Y} = \lambda_C[\ln Y - \ln(0.69880) + \varepsilon]$$

$$\lambda_C = \frac{\ln(2.36809) - \ln(0.40279) + \varepsilon}{\ln(1.33735) - \ln(0.69880) + \varepsilon} = 2.72907$$

$$\lambda_S = \frac{6}{\ln(2.36809) - \ln(0.40279) + \varepsilon} = 3.38709$$

$$\varepsilon = 10^{-5}$$

Multi-indexes ($|k|^* \leq 3$)

α	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
k_1	0	0	0	1	0	1	1	0	0	0	0	0	1	0	1	1	1	1	1
k_2	0	1	0	-1	1	0	0	0	1	0	1	1	-1	1	-1	0	0	0	0
k_3	0	-1	1	0	0	0	-1	1	-1	1	-1	0	0	0	0	-1	0	-1	0
k_4	0	0	-1	0	-1	-1	0	-1	0	-1	0	-1	0	-1	0	0	-1	0	-1
k_5	1	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1

The resulting time series of exogenous variables is restricted to the (0, 2π) interval. This is the vector x described previously. Scaling factors for the agricultural production data are also presented in table 1.

Differentiation of the logarithmic Fourier flexible form produces share equations of the form

$$\nabla g_{K_T}(x|\theta) = b - \lambda_S \sum_{\alpha=1}^{A} \left\{ u_{0\alpha} \lambda_S k_\alpha' x + 2 \sum_{j=1}^{J} j[u_{j\alpha} \sin(j \lambda_S k_\alpha' x) + v_{j\alpha} \cos(j \lambda_S k_\alpha' x)] \right\} k_\alpha,$$

where ∇ denotes the gradient vector formed by differentiating with respect to the logged input prices. An expenditure system is then formed, for estimation with the seemingly unrelated regressions technique, with the cost function and n - 1 of the share equations. This system is linear in the parameters. Using estimated parameters, elasticities of substitution are obtained from expressions for second derivatives contained in Gallant (1982).

Prior Experience With the Generalized Box-Cox and Fourier Flexible Forms

The generalized Box-Cox was used by Berndt and Khaled in their study of the U. S. manufacturing sector. They apply the cost function to the capital, labor, energy, and materials (KLEM) data of Berndt and Wood to reconsider the question of energy-capital complementarity without the translog assumption. The complementarity finding is supported still; the translog is not. They find that models near the translog case of $\lambda = 0$ are rejected as is the generalized square-root quadratic. The generalized Leontief cannot be rejected.

In the production context, prior experience with the Fourier flexible form consists of an application by Gallant (1982) to the same data and a Monte Carlo experiment by Chalfant and Gallant.[8] Gallant's application to the Berndt and Wood data permits a comparison of the logarithmic Fourier flexible form with the generalized Box-Cox; the findings do not permit the energy-capital complementarity conclusion. In the case of a homothetic technology, the estimated elasticity of substitution between energy and capital for 1959 is -4.2067, but the standard error is 3.4998. In the nonhomothetic case, this elasticity is estimated as 0.6613. Finally, Gallant and Golub present a case in which the cost function is restricted to be quasi-convex at the 1959 prices--the elasticity becomes 1.1704.[9]

The Monte Carlo results of Chalfant and Gallant indicate that the logarithmic Fourier form succeeds in obtaining an unbiased estimate of the partial elasticity of substitution. This finding holds for hypothetical technologies throughout the range of the generalized Box-Cox parameter from 0 to 2. Because it approximates the entire range of the generalized Box-Cox with negligible bias, the Fourier flexible can, therefore, be expected to approximate any unknown function which would be well approximated by the locally flexible functional forms. Given the less favorable results obtained by Guilkey and Lovell and Guilkey, Lovell, and Sickles, these results suggest that the Fourier flexible form offers an attractive alternative for cost function studies.

Of course, the utility function of the hypothetical researcher Chalfant and Gallant have in mind includes only one argument--unbiasedness. In applications it is likely that stable estimates of the elasticities of substitution will also be desirable. Given the possible variability of the Fourier flexible form estimates, due to the increased number of parameters, one would prefer further experience with specific data sets to obtain some notion of the extent of variability.[10] If there appears to be a trade-off between unbiasedness and stability for fixed data sets, further research along the lines of Gallant and Golub is suggested concerning restrictions on the curvature of flexible forms. Gallant (1982) also suggests using the approach of smoothness penalties (Wahba).

For the Monte Carlo experiment of Chalfant and Gallant, the production technology is known to exist since it was used to generate the data. Elasticities were evaluated only at the mean price vector; it will be useful in Monte Carlo contexts to consider the behavior of estimated elasticities throughout the data. This will indicate the extent to which unstable elasticities are due to the specification and the extent to which the data simply were not generated from a well-behaved technology.

Application to U. S. Agricultural Production

The flexible functional forms described previously are now applied to the estimation of elasticities of substitution for U. S. agricultural production. Specifically, we consider the extent to which it matters which flexible form is used. This section includes a description of the estimation procedure and the data followed by empirical results. We conclude with some observations and suggestions for further study.

Estimation

The estimation procedures used are iterative versions of seemingly unrelated regressions [Zellner, Gallant (1975)]. The SYSNLIN procedure of the Statistical Analysis System (SAS) was used for the generalized Box-Cox in an iterative version of the seemingly unrelated nonlinear regressions estimation technique described by Gallant (1975). FORTRAN code was used for the Fourier form which is linear in the unknown parameters. The seemingly unrelated regressions technique is preferred because contemporaneous correlation is likely among the residuals in demand equations and because it permits convenient imposition of across-equation restrictions.

In demand systems with n goods, it is well known that the covariance matrix is singular, requiring deletion of one equation from the system. In order that results be invariant to the choice of equation for deletion, the contemporaneous covariances across equations are recalculated and the parameter vector updated until convergence.

Capps provides an excellent description of the estimation of demand systems, comparing the iterative seemingly unrelated regressions and maximum-likelihood methods in an application of the linear expenditure system. Barnett (1976) discusses the conditions under which iterative seemingly unrelated regressions is equivalent to maximum likelihood in the nonlinear case; for the linear case, equivalence of the two approaches was established by Kmenta and Gilbert.

The Data

Four inputs are used in this study--capital, intermediate inputs, labor, and land. While the U. S. Department of Agriculture reports expenditures for a large number of inputs, it would require an extremely long time series to fit an expenditure system to more than four or five inputs. Therefore, it is necessary to aggregate. A brief description of the data follows. For a complete description and listing of the variables, see Appendix A.

Capital

Expenditures on capital services are the sum of reported expenditures for repairs and operation of all motor vehicles and machinery, total depreciation on motor vehicles and other machinery and equipment, petroleum fuel and oils, machine hire and custom work, electricity, estimated taxes on farm capital, and estimated interest expense.[11] An index of the rental rate for capital was constructed by dividing expenditures by an index of the quantity of capital used.

Intermediate Inputs

The expenditures on this class consist of feed, seed, and livestock purchases plus expenditures on agricultural chemicals (fertilizer, lime, and pesticides). A price index for intermediate inputs was constructed. Since a pesticide price index is only available for a part of the sample period, it could not be used. Instead, expenditures on agricultural chemicals were divided by an index of agricultural chemicals used to produce a price index for that portion of the intermediate inputs expenditures. A Tornquist price index for intermediate inputs was then constructed using the four categories and associated price indexes. A base year of 1967 was used.

Labor

Expenditures on labor services consist of total farm wages including contract labor and the value of perquisites. The index of wage rates provides the corresponding input price.

Land

Expenditures on land consist of repairs, maintenance, and depreciation of service buildings and structures, estimated tax payments, and estimated interest payments. The corresponding rental rate for land is obtained by dividing by the number of acres in farms.

The Generalized Box-Cox Cost Function

The generalized Box-Cox cost function was estimated using the SYSNLIN procedure of SAS (Release 82.3). The estimated parameters are reported in table 2. The weighted error sum of squares is 144, indicating convergence of the estimated parameters.[12]

One interesting finding is that the estimated λ is approximately 2.4. This is far from the translog case ($\lambda \rightarrow 0$) as well as the generalized Leontief. In fact, this application of the generalized Box-Cox suggests that the generalized square-root quadratic would be more appropriate for this agricultural production example.

The generalized Box-Cox was reestimated subject to three restrictions on the parameter λ. These were $\lambda = 0.1$ (a case near the translog), $\lambda = 1$ (generalized Leontief), and $\lambda = 2$ (generalized square-root quadratic); weighted error sums of squares were 161.92, 197.75, and 150.51, respectively. Using the likelihood ratio test (Burguete, Gallant, and Souza), all three restrictions are rejected by comparing the differences in weighted error sum of squares from 144 (the unrestricted result) with the 5 percent critical value, $\chi_1^2 = 3.84$.

Homotheticity was also imposed, by restricting each ϕ_i to equal zero. For the homothetic generalized Box-Cox, the weighted error sum of squares rises to 335.39. The homotheticity restriction, therefore, is rejected by the data. It turns out that, if homotheticity is imposed, the generalized square-root quadratic fits better than in the nonhomothetic case, as the estimated λ is now 1.993972, with standard error 0.1072029.

Elasticities of substitution in the generalized Box-Cox were calculated for each pair of goods at each data point. These are reported, for the 1967

Table 2. Nonlinear Iterative Seemingly Unrelated Regressions Estimates--Generalized Box-Cox

Free parameters	Estimate	Standard error
λ	2.383083	0.162252*
γ_{11}	-0.108727	0.0300246*
γ_{12}	0.179068	0.0182710*
γ_{13}	0.094014	0.0151879*
γ_{14}	0.125590	0.0164128*
γ_{22}	-0.011709	0.0111982
γ_{23}	0.048004	0.0113066*
γ_{24}	0.146048	0.0140102*
γ_{33}	-0.098169	0.0102197*
γ_{34}	0.060593	0.0055438*
γ_{44}	0.027934	0.0043416*
β	0.694451	0.0261009*
θ	0.135344	0.2284396
ϕ_1	0.138634	0.0508670*
ϕ_2	0.208051	0.0488330*
ϕ_3	-0.176970	0.0263920*

*Significant at the 5 percent level.

observation, in table 3.[13] Estimated elasticities for the various restricted cases are also reported.

The effect of imposing the various restrictions on the generalized Box-Cox can be seen in the table. First of all, the own-elasticities take on the expected negative sign in each case. This is true for all years except for σ_{11} in two years in the case of $\lambda = 0.1$. All signs appear stable at the point of approximation, with only σ_{12}, σ_{13}, and σ_{24} in the case where $\lambda =$ 0.1 and σ_{34} in the homothetic case changing sign from the maintained model.

On balance, the generalized Box-Cox appears to fit well. Plots of predicted values against the observed data suggest that the estimated parameter vector is a global minimum. Furthermore, the results are plausible--the model satisfies the theoretical restriction of negative own-elasticities at all data points. Inputs appear to substitute for each other, although σ_{14} is negative for the years prior to 1956, and σ_{23} is negative after 1974; see Appendix B for the unrestricted generalized Box-Cox results for all observations.

Unrestricted Fits of the Special Cases

We have presented elasticities of substitution estimated from various special cases of the generalized Box-Cox. These are obtained by using the covariance matrix from the generalized Box-Cox to estimate each restricted version and then using parameter estimates and predicted shares and expenditures to calculate elasticities. Referral to table 3 indicates that, at least for the generalized Leontief and generalized square-root quadratic, while we reject the restriction on λ, elasticities seem somewhat unaffected. The translog and homothetic generalized Box-Cox appear a little less similar to the unrestricted generalized Box-Cox.

Table 3. Estimated Elasticities of Substitution: Generalized Box-Cox

	Generalized Box-Cox	Homothetic generalized Box-Cox ($\phi_i = 0\ \forall i$)	Generalized Leontief ($\lambda = 1$)	Generalized square-root quadratic ($\lambda = 2$)	Near-translog ($\lambda = 0.1$)
σ_{11}	- 2.366	- 1.220	- 2.728	- 2.590	-0.266
σ_{12}	0.889	0.179	1.601	1.166	-0.085
σ_{13}	2.745	1.589	1.835	2.562	-0.222
σ_{14}	0.216	0.414	0.106	0.177	0.365
σ_{22}	- 0.911	- 1.316	- 1.645	- 1.172	-0.409
σ_{23}	0.308	2.016	0.274	0.208	1.757
σ_{24}	0.109	0.564	0.250	0.175	-0.030
σ_{33}	-11.303	-10.616	-11.932	-11.729	-8.755
σ_{34}	0.759	- 0.352	1.686	1.128	0.959
σ_{44}	- 0.503	- 0.796	- 0.827	- 0.648	-0.543

In practice, one does not fit the special cases for λ as restrictions; rather, they are maintained hypotheses. For comparison of the performance of flexible forms, then it is of interest to estimate them using the same iterative technique as for the generalized Box-Cox.[14] Table 4 contains the results for the three flexible forms which are special cases of the generalized Box-Cox as well as the case near the translog with $\lambda = 0.1$.

As unrestricted models, the translog and near-translog cases no longer feature negative own-elasticities of substitution at all observations. The translog does not produce negative own-elasticities of substitution at all observations. Positive signs are obtained for 18 of 36 years for σ_{11} and 23 of 36 years for σ_{22}. A similar result occurs with the case $\lambda = 0.1$.

Also evident from table 4 is that the flexible forms do not appear as similar as when they were estimated as restrictions in the generalized Box-Cox. The difference is quite evident in the generalized Leontief and generalized square-root quadratic which had very similar elasticities in table 2. Now they differ by more. Each has negative own-elasticities at each observation. However, cross-elasticities vary between the forms. The elasticity of substitution between capital and labor, σ_{13}, does not even have the same sign in the two models.

Should we consider this variation large? Of course, it depends on the context; perhaps it would not matter in some contexts which flexible form is used. However, we have little to tell us whether we should expect more or less variation in other examples. Clearly, we should consider this question further.

Each of the locally flexible forms is able to attain arbitrary values at a particular observation. The fact that they can vary in the values produced suggests we should seek some criterion to determine which form is appropriate.

Table 4. Estimated Elasticities from Unrestricted Fits of Special Cases of the Generalized Box-Cox

	Translog	λ = 0.1	Generalized square-root quadratic	Generalized Leontief
σ_{11}	-0.026	0.492	- 2.800	-0.741
σ_{12}	-0.850	-0.782	1.252	0.308
σ_{13}	0.603	-0.783	2.661	-0.390
σ_{14}	0.681	0.581	0.221	0.425
σ_{22}	0.088	-0.045	- 1.189	-0.746
σ_{23}	2.356	2.497	0.038	1.563
σ_{24}	-0.048	-0.020	0.176	0.023
σ_{33}	-9.956	-7.405	-11.006	-8.596
σ_{34}	0.017	0.320	1.034	1.290
σ_{44}	-0.522	-0.553	- 0.658	-0.751

Prior expectations about elasticities will help. Also, the generalized Box-Cox can be used to select among the various forms. In our example, it rejects each; but as it qualifies as a functional form in its own right, perhaps this is of no consequence.

On the other hand, the generalized Box-Cox can only select from among the locally flexible forms which have a Taylor series interpretation. As was pointed out above, Barnett (1983) has shown that restriction to the Taylor class is inappropriate. Hence, we would be unable to rule out forms such as the generalized Cobb-Douglas or Barnett's Laurent series approach. The generalized Box-Cox then only partially solves the functional form problem--we must restrict attention to the forms based on a Taylor series.

The Fourier Flexible Form

Gallant's (1982) logarithmic Fourier flexible form represents a different approach to determining the appropriate functional form. Rather than choosing from among a set of locally flexible forms which is potentially large, the Fourier form is a means of minimizing the average specification bias throughout the data. With a sufficient number of parameters in the Fourier series, average bias can be made arbitrarily small.

In this application, a 63-parameter specification has been used. No attempt has been made to test alternative specifications with the exception of a restriction to homotheticity. The estimated parameters and standard errors are reported in table 5.

These parameters are of little interest unless one is attempting to choose alternative parameter settings. Then significance of included parameters is of interest in the same fashion as when choosing a lag length. However, having our estimated parameters, we can compute elasticities of substitution. These are given for the 1967 prices in table 6 and for all years in Appendix B.

Table 5. Estimated Parameters and Standard Errors of the Fourier Flexible Form

Parameter	Estimate	Standard error	Parameter	Estimate	Standard error
θ_1	-0.75904	0.034883	θ_{32}	0.00049	0.00052811
θ_2	0.22832	0.037517	θ_{33}	-0.00167	0.00043620
θ_3	0.34590	0.022486	θ_{34}	0.0	0.0
θ_4	0.17784	0.034405	θ_{35}	0.00064	0.00036469
θ_5	0.24794	0.013070	θ_{36}	0.00068	0.00039540
θ_6	0.38069	0.086536	θ_{37}	0.0	0.0
θ_7	0.00466	0.010040	θ_{38}	0.00292	0.00123047
θ_8	-0.00680	0.00845208	θ_{39}	0.00081	0.00093552
θ_9	-0.00686	0.00505149	θ_{40}	0.0	0.0
θ_{10}	0.00131	0.00650290	θ_{41}	0.00034	0.00022040
θ_{11}	-0.00326	0.00125019	θ_{42}	-5.119×10^{-5}	0.00022878
θ_{12}	-0.00057	0.00046882	θ_{43}	-0.00253	0.00394332
θ_{13}	0.00374	0.00348896	θ_{44}	-0.00222	0.00063494
θ_{14}	0.00096	0.00053991	θ_{45}	-0.00301	0.00060255
θ_{15}	-0.00121	0.00071969	θ_{46}	0.0	0.0
θ_{16}	-0.00743	0.00712296	θ_{47}	-0.00060	0.00078168
θ_{17}	0.00153	0.00168230	θ_{48}	0.00169	0.00117018
θ_{18}	-0.00226	0.00085437	θ_{49}	0.0	0.0
θ_{19}	-0.00410	0.00220881	θ_{50}	-0.00015	0.00174626
θ_{20}	0.00043	0.00027217	θ_{51}	0.00053	0.00149652
θ_{21}	0.00238	0.00054193	θ_{52}	0.0	0.0
θ_{22}	-0.00799	0.00390264	θ_{53}	-0.00032	0.00055076
θ_{23}	-0.00088	0.00067876	θ_{54}	0.00261	0.00057660
θ_{24}	0.00047	0.00151142	θ_{55}	0.0	0.0
θ_{25}	-0.07054	0.040590	θ_{56}	6.173×10^{-5}	0.00048091
θ_{26}	0.03550	0.020780	θ_{57}	0.00062	0.00050166
θ_{27}	-0.00597	0.00363282	θ_{58}	0.0	0.0
θ_{28}	2.509×10^{-5}	0.00253177	θ_{59}	0.00224	0.00083477
θ_{29}	-0.00204	0.00066215	θ_{60}	-0.00022	0.00084061
θ_{30}	0.00275	0.00152127	θ_{61}	0.0	0.0
θ_{31}	0.00615	0.00450607	θ_{62}	0.00275	0.00123795
			θ_{63}	-0.00162	0.00135416

Table 6. Estimated Elasticities of Substitution from the Fourier Flexible Form

	Nonhomothetic	Homothetic
σ_{11}	-0.632	-0.688
σ_{12}	0.081	0.889
σ_{13}	-0.026	-0.396
σ_{14}	0.412	-0.229
σ_{22}	-0.705	-1.775
σ_{23}	0.753	0.578
σ_{24}	0.454	0.892
σ_{33}	0.962	-2.871
σ_{34}	-1.075	0.685
σ_{44}	-0.470	-0.938

As was the case with the translog, the logarithmic Fourier flexible form does not produce uniformly negative estimates of own-elasticities of substitution. In this case, for the year 1967, the own-elasticity of substitution for labor, σ_{33}, is positive. At least, this estimate has a relatively large standard error of 2.21148 so that we would not reject the hypothesis that this value is nonpositive.

Referral to Appendix B indicates a number of positive own-elasticities. As this was also the case with the translog, perhaps this is not unexpected. The Fourier form includes the translog as a special case with the sine/cosine terms added. As evidence from the generalized Box-Cox rejects the logged-price specification, it may require additional sine/cosine terms to reduce specification bias to an acceptable level.

Of course, the estimation procedure can be modified to impose curvature restrictions with the same number of parameters. When those are correct restrictions, there is a gain in efficiency.

As was pointed out above, we should perhaps not be too surprised if microeconomic restrictions do not appear to hold for aggregate data. If the goal is to determine whether a well-behaved aggregate technology exists or is to compare functional forms, the unrestricted model seems of interest. For applied work, it is probably more appropriate to impose the restrictions, although the example provided by Gallant and Golub indicates that the programming task is formidable.

The homotheticity restriction is imposed by reestimating the model with no interaction between output and the input prices. This involves deleting the last 12 multi-indexes. The Fourier form with homotheticity imposed features a weighted error sum of squares of 978.937 or an increase over the unrestricted model of 834.940.[15] Comparison with the χ_1^2 value of 3.84 indicates that, again, the restriction to homotheticity is rejected.

Elasticities of substitution were calculated for the homothetic case. For 1967 prices, these are presented in table 3; a negative value is obtained for σ_{33}. The restriction does not avoid the problem of positive own-elasticities--there are 18 out of 144.

Summary and Conclusions

This paper has reviewed the literature on flexible functional forms. Specifically, interest has centered on the model selection problem. Two approaches to choosing among functional forms were outlined.

The generalized Box-Cox of Berndt and Khaled nests some of the previously used locally flexible forms in a more general specification. Tests for special cases are easily constructed by restricting the Box-Cox parameter λ. Also, the generalized Box-Cox qualifies as a functional form in its own right.

The application to agricultural production data indicates that plausible results are obtained for elasticities and that special cases previously used in agricultural production studies (the translog and generalized Leontief) are rejected. While this did not affect estimated elasticities as much in the case where the generalized Box-Cox was maintained, unconstrained estimates using the other locally flexible forms produced fairly wide variation in those elasticities. The practical implication appears to be that changes in λ, with a fixed variance matrix, do not affect elasticities much but that, when the variance matrix is reestimated iteratively, variations in λ do lead to different conclusions about substitution possibilities. Further experience is required to indicate the extent of variation one should expect--both Monte Carlo studies and comparisons with specific applications will be useful.

The logarithmic Fourier flexible form was suggested by Gallant (1982) as an alternative approach to model selection. All cost functions can be expressed in Fourier series form, allowing specification bias to decrease to arbitrarily small levels as additional parameters are included. This is in contrast to the generalized Box-Cox which is usually taken as an approximation. White's (1980, 1981) results indicate that, when the true cost function is not of the generalized Box-Cox class, estimated elasticities are biased, even asymptotically.

The application of the Fourier flexible form to the agricultural production example indicated that it fails to satisfy the desired curvature restrictions on the cost function. Estimated own-elasticities of substitution were positive roughly one-fourth of the time.

There is no reason to expect these restrictions to hold--the aggregate technology could simply fail to satisfy these restrictions. This raises a question: Why is there such a different result for the generalized Box-Cox? I believe that specification tests will help answer this question. Essentially, we would like to know whether the Fourier form is "overfitting" the data or the generalized Box-Cox is "underfitting."

However, even with the aggregate data, for most purposes, elasticities are useless without being consistent with theory. Since the whole point of a cost function approach is to maintain the connection with the theory of the firm, it will be fruitful to explore estimation procedures with the Fourier form which produce the desired signs. Whether this is best accomplished through the Gallant and Golub approach of imposing appropriate restrictions on parameters or by some alternative to the seemingly unrelated regressions technique remains to be seen.

It is beyond the scope of this comparison of flexible forms to take on these questions; but to summarize, it appears that there are a number of questions which remain. A useful way to view these problems is to add at least one argument to the researcher's utility function. Along with unbiased estimates of elasticities, the properties of precision or stability in elasticities are desirable. Neither the appropriate marginal rate of substitution between these two goods nor the extent to which the "market" in flexible forms allows trade-offs to be made has been established. Monte Carlo results have established that unbiasedness does follow with the Fourier flexible form but not with the locally flexible forms; each of these results is consistent with theoretical findings by White (1980, 1981) and Gallant (1981, 1982).

The application presented in this paper indicates that the bias-instability trade-offs can be substantial. The Fourier form, with desirable properties concerning bias, features much greater oscillation in estimated elasticities than does the generalized Box-Cox. As El Badawi, Gallant, and Souza point out, bias minimization and accurate estimation are conflicting objectives.

The usual bias-variance trade-off involves the deletion of variables to reduce variances. A similar effect appears to be present in the variation of estimated elasticities from year to year. The degree of variability will vary from application to application as will the appropriate levels of bias and instability. Further research is called for to adjust estimation procedures and model specification to improve the attainable levels of each.

APPENDIX A

VARIABLE NAMES AND DEFINITIONS

YEAR = calendar year, 1945-1980.

CPI = consumer price index, 1967 = 100 (ECIFS 1-1, page 86).

Q = index of agricultural output, 1967 = 100 (ECIFS 1-3, pages 8 and 9).

QC = index of agricultural crops output, 1967 = 100 (ECIFS 1-3, pages 8 and 9).

QL = index of agricultural livestock output, 1967 = 100 (ECIFS 1-3, pages 8 and 9).

QCHEM = index of agricultural chemical inputs, 1967 = 100 (ECIFS 1-3, pages 60 and 61).

QCAP = index of mechanical power and machinery inputs, 1967 = 100 (ECIFS 1-3, pages 60 and 61).

Annual Farm Production Expenses

(millions of dollars)

X1 = feed purchased.

X2 = livestock purchased.

X3 = seed purchased (includes bulbs, plants, and trees).

X6 = fertilizer and lime.

X7 = repairs and operation of capital items (excluding fuel and operator dwellings; the sum of K1 and K2).

X9 = petroleum fuel and oils (for farm business use only).

X14 = total farm wages, including contract labor and the value of perquisites (including Social Security taxes paid by employers under the old-age survivor's insurance provision of the Social Security Act beginning 1951).

X15 = machine hire and custom work (excluding contract labor--not available for the years prior to 1950 and assumed to be zero).

X17 = pesticides.

X18 = electricity.

X19 = other operating expenses (excluding operator dwellings).

X21 = interest on nonreal estate debt.

X22 = interest on real estate debt (excluding operator dwellings).

X24 = taxes (excluding operator dwellings).

X26 = depreciation (excluding operator dwellings; the sum of K4, K5, and K6, plus accidental damage, not available until 1959 as a separate component).

X30 = net rent to all landlords.

K1 = repairs and maintenance of service buildings, other structures, and land improvements (includes fences, windmills, wells, dams and ponds, terraces, drainage ditches, tile lines, other soil conservation facilities, and dwellings not occupied by the farm operators).

K2 = repairs and operation of all motor vehicles and machinery (for farm business use).

K4 = farm depreciation and other capital consumption of service buildings and other structures (in terms of current replacement cost, not original cost; includes fences, windmills, wells and dwellings not occupied by the farm operators).

K5 = total depreciation on motor vehicles (for farm business use; in terms of current replacement cost, not original cost).

K6 = total Depreciation on Other Machinery and Equipment [in terms of current replacement cost, not original cost; excludes minor types of equipment charged to the "other operating expense" category (X19)].

KDEBT = total nonreal estate farm debt outstanding (excluding farm households and CCC loans; ECIFS 1-1, page 116).

LDEBT = total real estate farm debt outstanding (ECIFS 1-1, page 114).

L2 = land in farms, thousands of acres (ECIFS 1-1, page 105).

L6 = farm real estate value, includes land and service structures (ECIFS 101, page 105).

Constructed Variables

KTAX = X24 * [CAPSTK/(CAPSTK + L6)].

KINT = X21 * (CAPSTK/KDEBT).

LTAX = X24 * [L6/(CAPSTK + L6)].

LINT = X22 * (L6/LDEBT).

CAP = expenditures on capital services (K2, K5, K6, X9, X15, X18, KTAX, KINT).

INTER = expenditures on intermediate inputs (X1, X2, X3, X6, X17).

LABOR = wages paid (X14).

LAND = expenditures on land services (K1, K4, LTAX, LINT).

SUM = nominal production expenditures (CAP, INTER, LABOR, LAND).

C = real production expenditures (SUM/CPI).

S1 = capital's share (CAP/SUM).

S2 = intermediate inputs' share (INTER/SUM).

S3 = labor's share (LABOR/SUM).

S4 = land's share (LAND/SUM).

P1 = PCAP (an estimated user cost of capital index (CAP/QCAP).

PCHEM = estimated chemical price index [(X6 + X17)/QCHEM].

P2 = PINTER (Tornquist index of PFEED, PSEED, PLVSTK, and PCHEM).

P3 = PLABOR.

P4 = estimated land rent index (LAND/L2).

<u>Prices Paid by Farmers (1967 = 100)</u>

(Source: Various numbers in the agricultural statistics series)

PFEED = feed price index.

PSEED = seed price index.

PLVSTK = livestock price index.

PLABOR = index of wage rates (the simple average of seasonally adjusted quarterly indexes).

Scaling

Having constructed the four factor shares and prices, plus total expenditures, the data were rescaled for estimation. This usually involves division by either the sample means of each time series or by a particular observation; in this case, 1967 values were used. Real prices and expenditures (deflated by the Consumer Price Index) and output were divided by their 1967 values. The original variables and the final data appear in the following pages.

Sources

U. S. Department of Agriculture. Agricultural Statistics. Washington, D. C., 1967, 1972, 1977, 1980, and 1981.

_______. Economic Indicators of the Farm Sector: Income and Balance Sheet Statistics (ECIFS 1-3). Washington, D. C., 1981 (1980).

_______. Economic Indicators of the Farm Sector: Production and Efficiency Statistics (ECIFS 1-3). Washington, D. C., 1981 (1980).

U. S. Agricultural Production Data

CAP	INTER	LABOR	LAND
2628.1	4909	2299	2866.2
2559.6	5380	2532	3412.2
3082.0	6492	2783	3958.3
3938.6	7110	2990	4274.9
4663.2	6130	2806	4463.1
5202.3	6959	2811	4614.2
5961.8	8391	2921	5280.2
6520.3	8217	2857	5682.1
6681.1	6974	2736	5797.4
6897.3	7370	2596	5754.0
7092.2	7370	2615	6091.7
7408.1	7458	2641	6448.4
7834.3	7839	2734	6910.7
8155.5	9183	2842	7258.6
8561.3	9546	2906	8073.9
8652.2	9211	3062	8486.2
8535.9	9804	3192	8917.5
8700.6	10768	3299	9454.3
8852.1	11326	3400	9999.0
8919.9	10882	3483	10633.5
9230.3	11774	3604	11273.7
9755.4	13486	3683	12044.8
10423.6	14111	3723	12967.4
10962.1	14125	3920	13935.0
11558.3	15414	4152	14885.9
12015.5	16675	4340	15702.5
12750.5	18041	4372	16829.6
13215.9	20268	4557	18709.5
14961.5	27823	5167	21694.4
18263.8	29151	6075	27277.1
21927.1	28442	6586	29686.1
25003.6	31196	7510	35026.4
28088.3	31990	7953	42160.6
31619.0	37529	8348	47805.4
38157.8	44001	9429	59841.8
44299.4	45747	10411	70323.5

PSEED	PLVSTK	PFEED	PLABOR	PCAP	PCHEM	PLAND
81	59	81	42	43.367	114.310	21.711
84	67	94	46	43.038	111.269	25.779
96	84	111	49	46.233	116.559	29.826
112	104	118	52	52.412	120.932	32.100
101	92	97	51	56.579	117.743	33.427
97	108	99	50	59.613	124.492	33.206
99	132	111	55	63.380	122.884	37.953
111	110	118	59	66.415	122.447	40.793
102	80	107	61	67.222	113.843	41.593
95	81	107	60	69.397	117.516	41.260
100	80	100	61	70.499	108.185	43.844
88	74	97	63	72.763	100.937	46.598
91	83	95	66	77.876	103.244	50.179
90	102	93	68	81.069	104.827	52.990
86	102	94	72	84.039	103.637	59.133
89	96	92	74	85.006	104.662	62.526
89	96	93	76	86.945	103.480	66.152
92	100	94	78	89.730	103.132	70.640
97	94	98	80	91.306	99.672	75.221
97	84	97	82	91.992	102.939	80.392
100	90	97	86	95.194	103.237	85.715
98	103	101	93	98.155	101.789	92.193
100	100	100	100	100.000	100.000	100.000
104	104	94	108	103.758	95.870	108.252
106	117	96	119	110.536	99.355	116.421
112	122	101	128	115.272	92.783	123.360
124	125	105	134	120.317	96.082	132.374
135	149	106	142	125.331	97.432	148.295
167	192	160	155	137.155	111.931	172.506
215	148	194	178	162.028	163.565	217.244
245	134	187	192	183.449	208.500	242.762
241	154	191	210	210.502	183.106	287.387
261	158	136	226	231.742	173.547	348.606
273	221	183	242	250.839	177.672	396.386
286	293	204	265	294.225	193.893	496.871
309	281	230	287	348.038	226.132	584.446

C	S1	S2	S3	S4
0.57165	0.206902	0.386466	0.180991	0.225642
0.57569	0.184359	0.387501	0.182371	0.245769
0.59157	0.188904	0.397909	0.170576	0.242611
0.61613	0.215066	0.388239	0.163268	0.233427
0.61364	0.258173	0.339381	0.155351	0.247095
0.65896	0.265606	0.355296	0.143517	0.235580
0.70321	0.264333	0.372041	0.129512	0.234114
0.71021	0.280125	0.353018	0.122742	0.244115
0.67195	0.301106	0.314307	0.123307	0.261280
0.68153	0.304955	0.325857	0.114779	0.254408
0.70076	0.306109	0.318098	0.112867	0.262926
0.71337	0.309245	0.311327	0.110246	0.269181
0.72852	0.309435	0.309621	0.107986	0.272958
0.76853	0.297221	0.334668	0.103575	0.264536
0.80821	0.294332	0.328186	0.099907	0.277576
0.80432	0.294178	0.313179	0.104110	0.288534
0.82435	0.280332	0.321977	0.104830	0.292862
0.86279	0.270021	0.334183	0.102384	0.293412
0.88847	0.263854	0.337213	0.101229	0.297704
0.88564	0.262981	0.320829	0.102688	0.313503
0.92105	0.257242	0.328131	0.100440	0.314187
0.97251	0.250336	0.346068	0.094511	0.309085
1.00000	0.252846	0.342292	0.090309	0.314552
0.99966	0.255277	0.328931	0.091286	0.324506
1.01646	0.251212	0.335013	0.090241	0.323534
1.01644	0.246558	0.342170	0.089057	0.322215
1.03974	0.245234	0.346989	0.084088	0.323689
1.09864	0.232878	0.357143	0.080299	0.329681
1.26928	0.214822	0.399492	0.074190	0.311496
1.32645	0.226130	0.360928	0.075217	0.337726
1.30376	0.253077	0.328273	0.076015	0.342633
1.40472	0.253237	0.315954	0.076061	0.354748
1.47269	0.254903	0.290312	0.072174	0.382611
1.55559	0.252343	0.299510	0.066623	0.381523
1.68962	0.251984	0.290571	0.062267	0.395179
1.67855	0.259393	0.267870	0.060961	0.411776

Q	QL	QC	P1	P2	P3	P4	YEAR
0.69880	0.72340	0.72727	0.80458	1.49316	0.77922	0.40279	1945
0.72269	0.70213	0.76623	0.73672	1.52154	0.78632	0.44067	1946
0.69880	0.69149	0.72727	0.69108	1.55534	0.73244	0.44583	1947
0.75904	0.68085	0.83117	0.72694	1.59035	0.72122	0.44522	1948
0.74699	0.71277	0.79221	0.79242	1.39765	0.71429	0.46816	1949
0.73494	0.74468	0.76623	0.82681	1.46809	0.69348	0.46056	1950
0.75904	0.77660	0.77922	0.82108	1.51255	0.70694	0.48782	1951
0.79518	0.78723	0.80519	0.83540	1.46454	0.74214	0.51312	1952
0.79518	0.78723	0.80519	0.83923	1.26669	0.76155	0.51926	1953
0.79518	0.81915	0.79221	0.86208	1.26237	0.74534	0.51255	1954
0.83133	0.84043	0.81818	0.87904	1.20744	0.76060	0.54668	1955
0.83133	0.84043	0.81818	0.89389	1.14046	0.77396	0.57246	1956
0.80723	0.82979	0.80519	0.92379	1.10572	0.78292	0.59525	1957
0.87952	0.84043	0.89610	0.93613	1.12441	0.78522	0.61189	1958
0.89157	0.88298	0.88312	0.96322	1.11622	0.82474	0.67735	1959
0.91566	0.87234	0.93506	0.96963	1.07439	0.83427	0.70491	1960
0.91566	0.91489	0.90909	0.97037	1.06699	0.84821	0.73831	1961
0.92771	0.91489	0.92208	0.99040	1.07348	0.86093	0.77969	1962
0.96386	0.94681	0.95104	0.99668	1.06068	0.87241	0.82029	1963
0.95181	0.96809	0.93506	0.99023	1.02103	0.88267	0.86536	1964
0.96795	0.94681	0.98701	1.00734	1.02347	0.91005	0.90703	1965
0.95181	0.96809	0.94805	1.00982	1.04423	0.95679	0.94854	1966
1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1967
1.02410	1.00000	1.02597	0.99767	0.93727	1.03647	1.03889	1968
1.02410	1.01064	1.03896	1.00989	0.91326	1.08379	1.06031	1969
1.01205	1.05319	1.00000	0.99116	0.89937	1.10060	1.06070	1970
1.10843	1.06383	1.11688	0.99684	0.89755	1.10470	1.09541	1971
1.04699	1.07447	1.12987	1.00025	0.92834	1.13328	1.18352	1972
1.12048	1.05319	1.19481	1.03047	1.17692	1.16454	1.29606	1973
1.06024	1.06383	1.09091	1.09700	1.20691	1.20515	1.47085	1974
1.14458	1.01064	1.20779	1.16903	1.13365	1.19107	1.50597	1975
1.16867	1.05319	1.19481	1.23462	1.07461	1.23167	1.68849	1976
1.20482	1.06383	1.29870	1.27682	0.99250	1.24518	1.92070	1977
1.25301	1.07447	1.32468	1.28372	1.00681	1.23849	2.02859	1978
1.33735	1.10638	1.45753	1.35338	1.02983	1.21895	2.28552	1979
1.24096	1.14894	1.31169	1.41041	0.99189	1.16288	2.36809	1980

APPENDIX B

ESTIMATED ELASTICITIES OF SUBSTITUTION FOR THE GENERALIZED BOX-COX AND FOURIER FLEXIBLE FORMS

Generalized Box-Cox

σ_{11}	σ_{22}	σ_{33}	σ_{44}
-2.2073	-1.6148	-5.087	-0.73682
-1.9525	-1.5876	-5.592	-0.64618
-1.3661	-1.6729	-4.966	-0.64478
-1.9466	-1.5551	-5.205	-0.61114
-2.1966	-1.4799	-5.129	-0.59748
-2.2458	-1.5189	-4.654	-0.62781
-2.1225	-1.4828	-4.874	-0.59040
-2.1214	-1.3962	-5.459	-0.54379
-2.3543	-1.3318	-6.035	-0.52847
-2.4452	-1.3296	-5.734	-0.53092
-2.4814	-1.2521	-6.243	-0.49258
-2.6167	-1.2205	-6.406	-0.48987
-2.8248	-1.2246	-6.060	-0.50199
-2.6541	-1.1470	-6.751	-0.45890
-2.6531	-1.1097	-7.119	-0.46171
-2.6710	-1.0677	-7.560	-0.45801
-2.6476	-1.0553	-7.703	-0.46564
-2.6360	-1.0356	-7.745	-0.46791
-2.5680	-0.9951	-8.324	-0.46652
-2.5397	-0.9803	-9.323	-0.48306
-2.5169	-0.9473	-9.116	-0.48370
-2.4853	-0.9629	-9.155	-0.49329
-2.3656	-0.9108	-11.303	-0.50326
-2.3434	-0.8709	-13.225	-0.51742
-2.3500	-0.8556	-14.694	-0.52525
-2.3200	-0.8550	-15.558	-0.52869
-2.1816	-0.8192	-19.951	-0.53683
-2.1075	-0.8188	-18.947	-0.54331
-1.8875	-0.8523	-17.127	-0.52675
-1.9573	-0.8582	-12.783	-0.53194
-2.0363	-0.8071	-13.806	-0.54270
-2.0641	-0.7736	-13.265	-0.55994
-2.0374	-0.7454	-12.169	-0.57538
-1.9512	-0.7418	-11.642	-0.58119
-1.8716	-0.7353	-6.851	-0.59144
-2.0361	-0.7364	-5.245	-0.57290

σ_{12}	σ_{13}	σ_{14}	σ_{23}	σ_{24}	σ_{34}
1.25453	1.39615	-0.25135	0.8904	0.401289	0.60884
1.15093	1.25732	-0.35257	1.0185	0.449224	0.60677
1.17025	1.07495	-0.40293	1.0915	0.508664	0.58739
1.20620	1.03567	-0.39280	0.9175	0.498540	0.48716
1.25880	1.29770	-0.24175	0.7341	0.376937	0.53613
1.33109	1.13417	-0.23192	0.6736	0.390465	0.50689
1.28771	1.15630	-0.26918	0.6969	0.410757	0.48000
1.24227	1.29908	-0.24844	0.6923	0.372742	0.47015
1.21313	1.64269	-0.14262	0.6547	0.273371	0.54850
1.26896	1.62254	-0.11155	0.5874	0.257718	0.51561
1.24445	1.77770	-0.06409	0.5323	0.217041	0.49852
1.23702	1.93391	0.00032	0.4900	0.173005	0.52839
1.26617	2.00275	0.06505	0.4405	0.136412	0.55387
1.24451	2.04065	0.05979	0.3843	0.139105	0.45712
1.19683	2.14323	0.09583	0.3653	0.126640	0.48809
1.17406	2.26709	0.13806	0.3217	0.105153	0.48497
1.14046	2.28858	0.14731	0.3254	0.108191	0.51611
1.12635	2.29283	0.16914	0.2887	0.108564	0.50387
1.09374	2.36643	0.19031	0.2421	0.108819	0.48370
1.04817	2.39658	0.22241	0.2338	0.103710	0.54719
1.01957	2.49514	0.23420	0.2054	0.104202	0.54040
0.96334	2.48241	0.20526	0.3021	0.113894	0.65920
0.88919	2.74511	0.21610	0.3039	0.108857	0.75885
0.83291	3.02301	0.24617	0.3116	0.086300	0.88420
0.79657	3.25101	0.25070	0.3676	0.064181	1.01014
0.75771	3.34617	0.22931	0.4512	0.061463	1.12167
0.77062	3.64257	0.24329	0.3320	0.081091	1.14004
0.72527	3.42277	0.23162	0.3370	0.112052	1.17277
0.75268	2.86300	0.12841	0.3149	0.220052	0.94567
0.70997	2.49073	0.18443	0.2361	0.223656	0.88393
0.76481	2.72504	0.29507	-0.0533	0.185954	0.71239
0.72661	2.70880	0.38707	-0.2350	0.174098	0.68176
0.67022	2.56278	0.47890	-0.4723	0.182124	0.61038
0.66252	2.50600	0.48846	-0.6749	0.210937	0.50047
0.66406	2.35843	0.54685	-1.2537	0.246097	0.07216
0.64270	1.92247	0.62275	-0.9439	0.217512	0.11664

Fourier Flexible Form

σ_{11}	σ_{22}	σ_{33}	σ_{44}
0.814	0.064	-11.312	-0.084
2.691	0.205	-9.215	-0.585
3.333	-0.034	-9.829	-0.552
2.422	0.190	-9.402	-0.872
0.713	0.175	-9.212	-0.895
1.990	0.147	-6.930	-0.821
1.461	0.138	-7.602	-0.795
0.384	-0.003	-6.423	-0.587
-0.696	-0.319	-4.251	-0.866
-0.496	-0.483	-2.635	-0.727
-0.952	-0.770	3.946	-0.488
-1.298	-1.091	4.687	-0.717
-1.347	-1.622	2.392	-1.233
-0.944	-1.028	14.734	-0.124
-1.218	-1.082	13.975	-0.273
-0.989	-0.993	13.315	-0.168
-1.238	-1.048	12.377	-0.303
-1.096	-0.966	13.867	-0.281
-0.607	-0.745	11.243	-0.184
-1.079	-0.847	10.350	-0.372
-0.597	-0.603	5.435	-0.278
-1.432	-1.015	6.820	-0.527
-0.632	-0.705	0.962	-0.470
0.018	-0.606	-3.022	-0.458
0.610	-0.760	-2.942	-0.479
1.292	-1.035	1.165	-0.552
0.793	-0.122	-13.587	-0.567
1.143	-0.116	-3.926	-0.606
1.301	-0.112	-4.194	-0.918
-0.036	-0.180	5.391	-0.718
-1.084	0.191	-22.186	-0.409
-1.665	0.259	-25.775	-0.300
-2.303	-0.205	-28.234	-0.368
-2.393	-0.544	-33.034	-0.242
-1.624	-1.394	-35.394	-0.104
-0.447	-1.405	6.573	-0.539

σ_{12}	σ_{13}	σ_{14}	σ_{23}	σ_{24}	σ_{34}
-1.621	2.376	0.089	2.855	-0.621	1.367
-1.874	0.626	0.316	2.535	-0.396	1.440
-2.053	1.609	-0.258	2.623	-0.431	1.925
-1.870	0.901	0.491	2.403	-0.226	1.208
-1.322	0.882	0.554	2.432	-0.378	1.385
-1.461	-0.609	0.428	2.658	-0.516	1.617
-1.350	-0.173	0.552	2.537	-0.366	1.206
-0.677	-0.012	0.550	1.925	-0.193	0.466
0.041	0.282	0.618	0.911	-0.077	0.533
0.207	-0.437	0.532	1.000	-0.106	0.494
0.836	-0.771	0.407	-0.319	0.096	-0.296
1.099	-0.721	0.502	-0.592	0.251	-0.382
1.148	-0.989	0.680	0.065	0.430	0.043
1.298	-1.776	0.094	-1.611	0.394	-1.318
1.292	-1.271	0.199	-1.685	0.519	-1.582
1.135	-1.117	0.038	-1.577	0.590	-1.649
1.128	-0.826	0.192	-1.481	0.618	-1.743
1.014	-0.951	0.184	-1.496	0.639	-1.914
0.652	-0.582	-0.036	-1.127	0.648	-1.719
0.675	-0.386	0.310	-0.916	0.621	-1.851
0.280	0.267	0.100	-0.414	0.570	-1.439
0.671	-0.162	0.526	-0.291	0.598	-1.692
0.081	-0.026	0.412	0.753	0.454	-1.075
-0.289	-0.497	0.430	1.838	0.303	-0.622
-0.388	-1.871	0.470	2.751	0.258	-0.477
-0.333	-4.067	0.594	3.284	0.288	-0.587
-0.531	-0.929	0.160	2.160	0.027	1.716
-0.614	-2.907	0.509	2.199	0.021	0.860
-0.680	-2.242	0.506	1.315	0.265	0.997
-0.367	-2.516	1.026	0.886	0.203	-0.623
-0.648	4.105	0.507	1.520	-0.083	0.573
-0.472	4.912	0.579	1.611	-0.232	0.452
-0.071	5.532	0.604	1.655	-0.102	0.184
0.267	5.908	0.383	1.867	-0.085	0.300
0.810	3.201	-0.054	3.323	0.053	0.662
0.340	-3.275	0.573	2.076	0.371	-0.352

Footnotes

*The research reported in this paper began as part of the author's Ph.D. thesis at North Carolina State University. Many colleagues there provided valuable assistance, especially Julian M. Alston, A. Ronald Gallant, Paul R. Johnson, Thomas Johnson, and Ann A. McDermed. Charles Cobb of the U. S. Department of Agriculture gave useful information concerning the construction of reported data, and Amor Nolan contributed exceptional word processing. Remaining shortcomings are the author's responsibility.

[1]Generalization to the case of several outputs is straightforward and available in Ray.

[2]A recent note by Byron and Bera argues that White's case was overstated as he only focused on first-order approximations while the translog is a second-order approximation. However, first-order approximations using the translog are the norm for share equations suggesting that, at least in the case of demand analysis, White's arguments are still of importance. Byron and Bera also present a finding similar to White's for the case of regressors which follow trends through the sample period and seems likely for time series studies in demand analysis.

[3]For a description of the stronger criterion sometimes used in demand analysis, see Lau and also Barnett (1983).

[4]A similar description for the generalized Cobb-Douglas appears in Diewert (1973), and an analogous argument follows for any locally flexible functional form.

[5]This involves allowing $K \to \infty$ while $(K/T) \to 0$, where T is the sample size.

[6]The situation can be visualized by drawing a curve tangent to the unknown surface at the the point of approximation and then changing its slope to allow the approximating function to track the data elsewhere around the unknown function. The slope of the global approximation generally would not be that of the tangent curve at the point of approximation or at any other point that can be identified without knowledge of the true functional form; see the examples provided by White (1980).

[7]Barnett (1983) suggests that approximation based on a Laurent series may perform better than the Taylor series approximations because it appears to approximate well over a wider range. This is due to the fact that its remainder term is smaller; recall that Barnett uses the remainder term to illustrate the shortcomings of second-order Taylor approximations. It will be interesting but outside the scope of this paper to consider the quality of approximation using the Laurent approach for the data used in this study.

[8]The Fourier flexible form has also been applied, in consumer demand contexts, by Gallant (1981), Kumm, Ewis and Fisher, and Wohlgenant; see Kiefer for a generalized Box-Cox example.

[9]Because the generalized Box-Cox does not take on the logarithmic Fourier flexible form as a special case and vice versa, it requires something other than the conventional F-test to reject either model. Without prior knowledge of σ_{KE} or some other elasticities, model selection also cannot be based on results. Even acceptance based on satisfying curvature restrictions requires that they be maintained.

Kumm estimated a hybrid of the two forms to test for the restriction to the generalized Box-Cox. This can be done by replacing the translog portion of the Fourier form with the generalized Box-Cox. The advantage of this approach is that, if the compound model can be maintained, conventional F-tests then apply for model selection. The disadvantages are that the number of parameters or nonlinearity of the model can increase and that it is of limited potential when several models are available for comparison.

In Chalfant, I apply the methods of non-nested hypothesis testing to the model selection problem. I hope to have results available soon which apply the tests described to the present application.

[10] In Chalfant, results are presented for an earlier version of the agricultural production data which indicate greater instability of the estimated elasticities of substitution using the Fourier form. My experience with the data of Berndt and Wood produces a similar finding, although I have not compared my results to those with the generalized Box-Cox throughout the data.

Of course, our expectation that substitution elasticities ought to be somewhat stable may be due in part to previous findings. If the functional forms used (e.g., quadratic forms) do not admit any possibility other than smoothness, we should be careful in using past empirical findings to judge the performance of the Fourier flexible form. Especially for highly aggregated data used in this study, oscillating elasticities or failure to display the appropriate signs may be due as much to the nonexistence of the aggregate technology as to the properties of the flexible form.

[11] For both capital and land, interest and tax expense are estimated. Observed interest payments do not pertain to owned land or capital and, therefore, do not include opportunity costs for use of all of the assets of the agricultural sector--only the unowned portions. Reported interest expenses were, therefore, scaled upwards by multiplying, for both capital and land, by the ratio of asset value in that category to corresponding total debt.

As taxes were not reported by component, these were assigned on a share basis to capital and land. Weights used were the ratios of the total value of the capital stock and or real estate, respectively, to the sum of these two variables (see Appendix A for details).

[12]There are 36 observations and four equations, hence, the error sum of squares converges to their product, 144.

[13]1 = capital, 2 = intermediate inputs, 3 = labor, and 4 = land.

[14]SYSNLIN is used for the generalized Leontief and generalized square-root quadratic because they are nonlinear in the parameters; the translog is linear, but to iterate, SYSNLIN is again used. Translog results were not reported using the covariance matrix of the generalized Box-Cox since the two forms involve different dependent variables.

[15]The iterations continued in that case until the error sum of squares reached 143.997, at which point the convergence criterion was met.

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