The Firm Under Price and Output Uncertainty: The State-Theoretic Approach

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1. Introduction

Ever since Sandmo's initial analysis (1971), the problem of the output decisions of the firm under uncertainty has been widely studied. Sandmo's approach has been applied to problems including price stabilization and futures markets (Danthine, 1978; Feder, Just and Schmitz 1980), and analogous models have been developed for a range of decision problems including self-protection against environmental hazards (Lewis and Nickerson 1989). Feder (1977) presents strong versions of Sandmo's main results for a general class of objective functions.

Sandmo assumed expected-utility maximization, and most applied work based on his approach has maintained this assumption. It is well-known that decision-makers systematically violate the assumptions of the expected-utility hypothesis, and the primary argument for maintaining it is analytic convenience. But the robustness of the analysis remains in doubt.

While the limitations of the expected-utility hypothesis are generally well recognized, the problems relating to the technology of production under uncertainty have been less remarked, but turn out to be more fundamental.The clearest way to recognize these shortcomings is to note the stark contrast between the representation of technology used in the literature on production under uncertainty and that used in the general-equilibrium theory of uncertainty. In the latter, production is typically represented in terms of the state-space representation of uncertainty developed by Arrow (1953) and Debreu (1952). By contrast, both the Sandmovian model of production under uncertainty as well as most principal-agent and moral hazard models (e.g., Mirrlees (1974), Holmstrom (1979), Harris and Raviv (1979), and Grossman andHart (1983)) are normally presented in terms of a stochastic production function formulation that degenerates to a family of random variables indexed by effort input.

This divergence reflects a fundamental difference in the approach to specifying the production technology. Arrow and Debreu deal with the most primitive (and, therefore, general) technology specification, i.e., a production set. This representation is quite convenient for the purposes of proving the existence of competitive equilibria under uncertainty and in the analysis of securities-market equilibria. However, until the development of the modern axiomatic approach to production analysis (Shephard, 1970; McFadden, 1978), few tools were available to permit the use of set-theoretic representations of production technologies in the analysis of firm behavior. Consequently, most applied analysis of firm behavior was based on the older idea of a production function. And when issues relating to firm-level stochastic production began to be seriously considered by theorists, they naturally based their analysis on the related notion of a stochastic production function. The widespread success of the axiomatic approach in creating the superstructure of duality theory and its many applications, however, suggests the possibility of an extension of this analysis to a state-contingent production technology as in the work of Arrow and Debreu.

In this paper, the problem of the firm under uncertainty is analyzed using a general state-contingent production technology that has already proved useful in analyzing hedging behaviour (Chambers and Quiggin, 1997) and principal-agent problems (Chambers and Quiggin, 1996; Quiggin and Chambers, 1998). This approach has the natural advantage that it is more general and flexible than the stochastic production function approach. Therefore, it allows a thorough examination of the consequences for producer decisionmaking of placing different structural restrictions upon the stochastic technology. This flexibility opens the door to a number of comparative-static results which are not available in the more restrictive stochastic-production function approach. The approach is also very easy to use largely because it permits the application of standard theoretical constructs to the analysis of producer decisionmaking under uncertainty. Moreover, it carries with it the side-benefit of making the expected-utility hypothesis superfluous in considering productive decisionmaking under uncertainty. Just as the Arrow-Debreu proof of competitive equilibrium only requires the assumption of convex preferences, the state-contingent approach advocated here allows one to rely on a very general specification of preferences that has the expected-utility model and many others as special cases.

The paper is organized as follows. Section 1 deals with general multi-input, multi-output state-contingent production technologies and with their dual analogues, effort-cost and revenue-cost functions. Section 2 develops the idea of a production risk premium. Section 3 deals briefly with objective functions. Sections 4, 5 and 6 present comparative static results for the special cases of risk neutral, maximin and generalized Schur-concave objective functions. Finally, concluding comments are offered.

2. Production under uncertainty

The idea that production under uncertainty may be represented simply as a special sort of multi-output production was first developed by Arrow and Debreu. To make this explicit, suppose that the states of nature are given by the set $\Omega = \{1, 2, ..., S\}$, let $\mathbf{x} \in \Re_+^N$ be a vector of inputs committed prior to the resolution of uncertainty and let $\mathbf{z} \in \Re_+^{M \times S}$ be a vector of state-contingent outputs. So, if state $s \in \Omega$ is realized (picked by 'Nature'), the observed output is an M-dimensional vector $\mathbf{z}^s$, obtained as the projection of $\mathbf{z}$ onto $\Re_+^{M \times \{s\}}$.

We represent the production technology in the form of an input correspondence which maps matrices of state-contingent outputs into sets of inputs that are capable of producing that state-contingent output matrix. Formally, it is defined by

$$X(\mathbf{z}) = \{\mathbf{x} \in \Re_+^N : \mathbf{x} \text{ can produce } \mathbf{z} \in \Re_+^{MxS}\}.$$

Intuitively, $X(\mathbf{z})$, which we typically refer to as the *input set*, is best identified with everything on or above an isoquant for the state-contingent technology. We impose the following axioms on the input correspondence. These directly parallel axioms placed on non-stochastic, multi-output technologies (Färe, 1988) and prove sufficient to ensure a duality between the input corrrespondence and the effort-cost function developed below:

X.1 $X(\mathbf{0}_{MxS}) = \Re_{+}^{N}$ (no fixed costs), and $\mathbf{0}_N \in X(\mathbf{z})$ for $\mathbf{z} \geq \mathbf{0}_{MxS}$ and $\mathbf{z} \neq \mathbf{0}_{MxS}$ (no free lunch).

X.2 $\mathbf{z}' \leq \mathbf{z} \Rightarrow X(\mathbf{z}) \subset X(\mathbf{z}')$.

X.3 $\mathbf{x}' \geq \mathbf{x} \in X(\mathbf{z}) \Rightarrow \mathbf{x}' \in X(\mathbf{z})$.

X.4 $\lambda X(\mathbf{z}) + (1-\lambda)X(\mathbf{z}') \subset X(\lambda\mathbf{z} + (1-\lambda)\mathbf{z}') \qquad 0 < \lambda < 1.$

X.5 $X(\mathbf{z})$ is closed and nonempty for all $\mathbf{z} \in \Re_{+}^{MxS}$.

The first part of X.1 says that doing nothing is always feasible, while the second part of X.1 says that realizing a positive output in any state of nature requires the committal of some inputs. X.2, free disposability of state-contingent outputs, says that if an input combination can produce a particular matrix of state-contingent outputs then it can always be used to produce a smaller matrix of state-contingent outputs. X.3 implies that inputs have non-negative marginal productivity. X.4 tells us that the state-contingent technology is convex, and intuitively it leads to diminishing marginal productivity of inputs. X.5 is a technical assumption.

2.1. Stochastic production function technologies

Just as technology represented by a production function is a special case of the general representation of technology in terms of productions sets, existing models of production under uncertainty are special cases of the analysis presented here. The simplest case is that of Sandmo: Revenue is stochastic and thus represented by the state-contingent revenue vector, $\mathbf{r} \in \Re_+^S$, defined by $\mathbf{r} = z\mathbf{p}$ where $\mathbf{p} \in \Re_+^S$ is a vector of state-contingent prices for the scalar output, z, determined by a non-stochastic production function, $z = f(\mathbf{x})$. Imposing weak disposability of output, then

$$X(z) = \{\mathbf{x} : z \leq f(\mathbf{x})\}.$$

The case of production uncertainty generated by a scalar-valued stochastic production function is only slightly more complex. Using a function $f(\mathbf{x}, s)$ to represent the interaction of the choice of inputs $\mathbf{x}$ with the state of nature s, then

$$X(\mathbf{z}) = \cap_s \{\mathbf{x} : z_s \leq f(\mathbf{x}, s)\}, \tag{2.1}$$

where z_s is the scalar-valued output that occurs in state s.

2.2. The effort-cost function

Our concern is not with the input vector $\mathbf{x}$ *per se*, but with its corresponding cost, given by $\mathbf{w} \bullet \mathbf{x}$, where $\mathbf{w} \in \Re^N_{++}$ is a vector of non-stochastic input prices. The definition of the cost function for a general multi-output production technology may be applied directly to the case of production under uncertainty (Chambers and Quiggin, 1992, 1998). That is, the *effort-cost function* is defined by:

$$c(\mathbf{w}, \mathbf{z}) = \min\{\mathbf{w} \bullet \mathbf{x} : \mathbf{x} \in X(\mathbf{z})\}$$

if $X(\mathbf{z})$ is non-empty and ∞ otherwise. The effort-cost function plays a central role in analysis of production under uncertainty. Its properties are standard[1].

Properties of the Effort-cost function $\mathbf{c}(\mathbf{w}, \mathbf{z})$

C.1.$c(\mathbf{w}, \mathbf{z})$ is positively linearly homogeneous, nondecreasing, concave, and continuous on $\Re^N_{++}$;

C.2. (Shephard's Lemma) If an unique solution exists to the minimization problem, $c(\mathbf{w}, \mathbf{z})$ is differentiable in $\mathbf{w}$, and its gradient in $\mathbf{w}$ equals the vector of cost minimizing derived demands; and if $c(\mathbf{w}, \mathbf{z})$ is differentiable in $\mathbf{w}$, an unique

[1]The derivation of these properties is considered in detail in Chambers and Quiggin (1997; 1998) and is completely analogous to the derivation of parallel properties for cost functions for non-stochastic technologies.

solution exists to the cost minimization problem, and the gradient of $c(\mathbf{w}, \mathbf{z})$ equals the vector of cost minimizing demands;

C.3. $c(\mathbf{w}, \mathbf{z}) \geq 0, c(\mathbf{w}, \mathbf{0}_{MxS}) = 0$, and $c(\mathbf{w}, \mathbf{z}) > 0$ for $\mathbf{z} \geq \mathbf{0}_{MxS}, \mathbf{z} \neq \mathbf{0}_{MxS}$;

C.4 . $\mathbf{z}^o \geq \mathbf{z} \Rightarrow c(\mathbf{w}, \mathbf{z}^o) \geq c(\mathbf{w}, \mathbf{z})$;

C.5. $c(\mathbf{w}, \mathbf{z})$ is convex over $\Re_+^{MxS}$

Here C.1 and C.2 follow from X.5 and the principle of optimization, C.3 from X.1, C.4 from X.2, and C.5 from X.4.

Note, in particular, that, where state-contingent production is determined by a stochastic production function, the cost function will generally not be smoothly differentiable in the state-contingent outputs. From (2.1), it follows immediately that in that case[2]:

$$
\begin{aligned}
c(\mathbf{w}, \mathbf{z}) &\geq Max_{s=1,2,\ldots,S} \{Min \{\mathbf{wx} : z_s \leq f(\mathbf{x}, s)\}\} \\
&= Max_{s=1,2,\ldots,S} \{c^s(\mathbf{w}, z_s)\}
\end{aligned}
$$

where $c^s(\mathbf{w}, z_s)$ is the cost function associated with the *ex post* production tech-

[2] Chambers and Quiggin (1998b) develop the effort-cost function for the stochastic-production function technology in the case where the state space is continuous.

nology and is defined by:

$$c^s(\mathbf{w}, z_s) = Min\,\{\mathbf{wx} : z_s \leq f(\mathbf{x}, s)\}\,.$$

2.3. The revenue-cost function

Denote by $\mathbf{p} \in \Re^{M\times S}_{++}$ the matrix of state-contingent output prices. When s occurs the vector of s-contingent prices is denoted $\mathbf{p}^s$. In this paper, the price vector will be interpreted in an *ex post* sense, so that $\mathbf{p}^s$ is the set of spot prices that will prevail in the event that state s occurs. The state-contingent revenue vector $\mathbf{r} = \mathbf{pz} \in \Re^S_+$ has elements of the form $\mathbf{p}^s \bullet \mathbf{z}^s$. In all cases we consider, producers will be concerned with state-contingent revenue rather than output *per se*, and it is useful to consider the *revenue-cost function*

$$C\left(\mathbf{w}, \mathbf{r}, \mathbf{p}\right) = \min\left\{c\left(\mathbf{w}, \mathbf{z}\right) : \sum_m p_{ms} z_{ms} \geq r_s, s \in \Omega\right\}$$

if there exists a feasible state-contingent output array capable of producing $\mathbf{r}$ and ∞ otherwise. Both $c(\mathbf{w}, \mathbf{z})$ and $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ are homogeneous of degree 1 in $\mathbf{w}$,while $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ is homogeneous of degree 0 in $\mathbf{r}$ and $\mathbf{p}$. In the analysis that follows, we presume that the effort-cost function satisfies properties C.1-C.5 The properties of $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ that follow from these are (Chambers and Quiggin, 1998):

Properties of the Revenue-Cost Function (CR):

CR.1$C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ is positively linearly homogeneous, nondecreasing, concave, and continuous in $\mathbf{w} \in \Re^N_{++}$.

CR.2 Shephard's Lemma.

CR.3 $C(\mathbf{w}, \mathbf{r}, \mathbf{p}) \geq \mathbf{0}$ with equality if and only if $\mathbf{r} = 0$.

CR.4 $\mathbf{r}' \geq \mathbf{r} \Rightarrow C(\mathbf{w}, \mathbf{r}', \mathbf{p}) \geq C(\mathbf{w}, \mathbf{r}, \mathbf{p})$.

CR.5 $\mathbf{p}' \geq \mathbf{p} \Rightarrow C(\mathbf{w}, \mathbf{r}, \mathbf{p}') \leq C(\mathbf{w}, \mathbf{r}, \mathbf{p})$.

CR.6 $C(\mathbf{w}, \mathbf{r}_{-s}, \theta r_{s,}, \mathbf{p}_{-s}, \theta \mathbf{p}_s) = C(\mathbf{w}, \mathbf{r}_{-s}, \theta r_{s,}, \mathbf{p}_{-s}, \theta \mathbf{p}_s), \quad \theta > 0$.

CR.7$C(\mathbf{w}, \mathbf{r}, \mathbf{p}) = C(\mathbf{w}, \mathbf{r}/k, \mathbf{p}/k), \quad k > 0$.

CR.8 $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ is convex in $\mathbf{r}$.

For analytic simplicity, we shall typically assume that $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ is smoothly differentiable in all state-contingent revenues. By assuming a differentiable cost structure, we, therefore, rule out the stochastic production function approach and the non-stochastic production approach of Sandmo.[3]

[3]If production is non-stochastic, then it follows immediately that

$$C(\mathbf{w}, \mathbf{r}, \mathbf{p}) = \mathbf{Max}_{1,2,\ldots S} \left\{ C^f(\mathbf{w}, r_s, p_s) \right\}$$

where

$$C^f(\mathbf{w}, r_s, p_s) = Min \left\{ \mathbf{wx} : p_s f(\mathbf{x}) \geq r_s \right\}.$$

Generally, neither this function or the one corresponding to the stochastic-production function will be everywhere smoothly differentiable in revenues or outputs respectively.

Our main concern will be with the case where prices are interpreted in *ex post* terms. However, the analytical tools presented here may also be applied to the case of a profit-maximizing firm, in the presence of a complete set of state-contingent markets with *ex ante* prices **p**. The resulting constructions may be applied, with appropriate modifications, to the optimization problem faced by a risk-neutral producer under price uncertainty, and adapted further to the case of a producer with preferences characterized by additively separable effort-costs.

3. Effort-cost and Technological Risk

3.1. The certainty-equivalent revenue and the production-risk premium

Just as a risk-averse individual will pay a premium in each state to ensure the certainty outcome, achieving the certainty outcome may prove costly. That is, typically it should cost more to remove production uncertainty and produce the same non-stochastic output in each state than to allow for stochastic production. The intuitive reason is clear: Most people appear averse to taking risk, but producers routinely use stochastic technologies. A plausible conjecture, therefore, is that removing risk is typically costly. If it were not, we'd expect to see, for example, farmers growing all their crops in greenhouses under closely controlled climatological conditions rather than in the open air subject to the vagaries of weather.

For the revenue-cost function, $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$, and $\mathbf{r} \in \Re_{+}^{S}$, we define the *(cost) certainty equivalent revenue,* denoted by $e^{c}(\mathbf{r}, \mathbf{p}, \mathbf{w}) \in \Re_{+}$, as the maximum non-stochastic revenue that can be produced at cost $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$, that is,

$$e^{c}(\mathbf{r}, \mathbf{p}, \mathbf{w}) = \sup\{e : C(\mathbf{w}, e\mathbf{1}^{S}, \mathbf{p}) \leq C(\mathbf{w}, \mathbf{r}, \mathbf{p})\},$$

where $\mathbf{1}^{S}$ is the S-dimensional unit vector. By analogy with the risk premium used in the theory of portfolio choice under uncertainty, we define the *production-risk premium* as the difference between mean revenue and the certainty equivalent revenue. Notationally, letting $\bar{\mathbf{r}} \in \Re_{+}^{S}$ denote the vector with the mean of $\mathbf{r}$,

$$\bar{r} = \sum_{k} \pi_{k} r_{k},$$

occurring in each state, then the production risk premium is defined by

$$p(\mathbf{r}, \mathbf{p}, \mathbf{w}) = \bar{r} - e^{c}(\mathbf{r}, \mathbf{p}, \mathbf{w})$$

and satisfies:

$$C(\mathbf{w}, \mathbf{r}, \mathbf{p}) = C(\mathbf{w}, \bar{\mathbf{r}} - p(\mathbf{r}, \mathbf{p}, \mathbf{w})\, \mathbf{1}^{\mathbf{S}}, \mathbf{p}) = C(\mathbf{w}, e^{c}(\mathbf{r}, \mathbf{p}, \mathbf{w})\, \mathbf{1}^{S}, \mathbf{p}).$$

The technology will be called *inherently risky* if producing $\bar{\mathbf{r}}$ is more costly than producing $\mathbf{r}$ and *not inherently risky* if producing $\bar{\mathbf{r}}$ is less costly than producing $\mathbf{r}$. By CR.4, the technology is inherently risky at $\mathbf{r}$ if and only if $p(\mathbf{r}, \mathbf{p}, \mathbf{w})$ is positive, or equivalently if and only if the certainty equivalent revenue is no greater than the mean. Both imply that producing $\bar{\mathbf{r}}$ is more costly than producing the stochastic, $\mathbf{r}$, there are costs to removing uncertainty. This seems the natural state of affairs. However, $p(\mathbf{r}, \mathbf{p}, \mathbf{w})$ may be negative, implying that certainty is less costly than the stochastic output vector, and in this case the technology is not inherently risky at $\mathbf{r}$.[4]

The certainty equivalent may be formally derived by using a directional distance function analogous to David Luenberger's benefit function (1992). The *directional distance function* for the revenue-cost function is defined (Chambers,

[4] Chambers and Quiggin (1997) provide an example of such a technology that is closely related to the generalized Schur concave preference structure introduced below.

Chung, and Färe, 1996):

$$\vec{D}(C, \mathbf{r}, \mathbf{w}, \mathbf{p}; \mathbf{g}) = \max\{\theta \in \Re : C(\mathbf{w}, \mathbf{r} + \boldsymbol{\theta}\mathbf{g}, \mathbf{p}) \leq C\} \quad \mathbf{g} \in \Re_+^S$$

if there exists a $\theta \in \Re : \mathbf{r} + \boldsymbol{\theta}\mathbf{g}$ such that $C(\mathbf{w}, \mathbf{r} + \boldsymbol{\theta}\mathbf{g}, \mathbf{p}) \leq C$ and $\inf \left\{\theta \in \Re : \mathbf{r} + \theta\mathbf{g} \in \Re_+^S\right\}$ otherwise. Here $\mathbf{g}$ is a reference vector of revenues. In words, $\vec{D}(C, \mathbf{r}, \mathbf{w}, \mathbf{p}; \mathbf{g})$ is the maximal translation of the state-contingent revenue vector in the direction of $\mathbf{g}$ that keeps the translated revenue vector less costly than C.

Among other things, it is easy to show that if state-contingent revenues are freely disposable, then

$$\vec{D}(C, \mathbf{r}, \mathbf{w}, \mathbf{p}; \mathbf{g}) \geq 0 \Leftrightarrow C(\mathbf{w}, \mathbf{r}, \mathbf{p}) \leq C, \tag{3.1}$$

and that $\vec{D}(C, \mathbf{r}, \mathbf{w}, \mathbf{p}; \mathbf{g})$ is nonincreasing and translatable in $\mathbf{r}$ (i.e., $\vec{D}(C, \mathbf{r}+\alpha\mathbf{g}, \mathbf{w}, \mathbf{p}; \mathbf{g}) = \vec{D}(C, \mathbf{r}, \mathbf{w}, \mathbf{p}; \mathbf{g}) - \alpha$, for $\alpha \in \Re$), nondecreasing in C (Chambers, Chung, and Färe, 1996).

By inspection, we see that:

Lemma 1

$$\begin{aligned} e^c(\mathbf{r},\mathbf{p},\mathbf{w}) &= \vec{D}(C(\mathbf{w},\mathbf{r},\mathbf{p}),\mathbf{0}^S,\mathbf{w},\mathbf{p};\mathbf{1}^S) \\ p(\mathbf{r},\mathbf{p},\mathbf{w}) &= \bar{r} - \vec{D}(C(\mathbf{w},\mathbf{r},\mathbf{p}),\mathbf{0}^S,\mathbf{w},\mathbf{p};\mathbf{1}^S) \\ &= -\vec{D}(C(\mathbf{w},\mathbf{r},\mathbf{p}),\bar{\mathbf{r}},\mathbf{w},\mathbf{p};\mathbf{1}^S). \end{aligned}$$

So, the technology is inherently risky if and only if $\vec{D}(C(\mathbf{w},\mathbf{r},\mathbf{p}),\bar{\mathbf{r}},\mathbf{w},\mathbf{p};\mathbf{1}^S) \leq 0$, because this presumption and (3.1) imply that producing $\bar{\mathbf{r}}$ is more costly than $\mathbf{r}$. Conversely, the technology is not inherently risky if and only if $\vec{D}(c(\mathbf{w},\mathbf{z}),\bar{\mathbf{z}};\mathbf{1}^S) \geq 0$, which by (3.1) implies that $\bar{\mathbf{r}}$ is less costly than $\mathbf{r}$. But these implications are quite intuitive because they mean that removing uncertainty for an inherently risky technology always requires higher costs, while no additional costs are necessary for a technology that is not inherently risky. By using CR.1 and CR.7, it is also easy to establish

Lemma 2 The certainty equivalent revenue satisfies: $e^c(\mathbf{r},\mathbf{p},\mu\mathbf{w}) = e^c(\mathbf{r},\mathbf{p},\mathbf{w})$, $e^c(\mu\mathbf{r},\mu\mathbf{p},\mathbf{w}) = \mu e^c(\mathbf{r},\mathbf{p},\mathbf{w})$, $\mu > 0$. The production risk premium satisfies $p(\mathbf{r},\mathbf{p},\mu\mathbf{w}) = p(\mathbf{r},\mathbf{p},\mathbf{w})$, $p(\mu\mathbf{r},\mu\mathbf{p},\mathbf{w}) = \mu p(\mathbf{r},\mathbf{p},\mathbf{w})$, $\mu > 0$.

The certainty equivalent revenue and the production risk premium are alternative characterizations of the technology. Formally, this can be verified by noting that, by the properties of the directional distance function and Lemma 1, the certainty equivalent revenue is a nondecreasing transformation of revenue-cost. It proves useful to have classes of technologies that are easily characterized in terms of either the production risk premium or the certainty equivalent. We define a state-contingent technology as displaying *constant absolute riskiness* if for all $\mathbf{r}, t \in \Re$:

$$p\left(\mathbf{r}+t\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right) = p\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right).$$

From this definition, properties CR, and Lemmata 1 and 2, it follows almost immediately that:

Result 1 The technology displays constant absolute riskiness if and only if

$$C(\mathbf{w}, \mathbf{r}, \mathbf{p}) = \hat{C}\left(\mathbf{w}, \mathbf{T}\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)$$

where

$$T\left(\mathbf{r} + \delta\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right) = T\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right) + \delta, \quad \delta \in \Re,$$

$$T\left(\lambda\mathbf{r}, \lambda\mathbf{p}, \mathbf{w}\right) = \lambda T\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right), \quad T\left(\mathbf{r}, \mathbf{p}, \lambda\mathbf{w}\right) = T\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right) \quad \lambda > 0,$$

and $\hat{C}(\mathbf{w}, \mathbf{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \mathbf{p})$ is positively linearly homogeneous in input prices, homogeneous of degree zero in $T(\mathbf{r}, \mathbf{p}, \mathbf{w})$ and $\mathbf{p}$, nondecreasing in $T(\mathbf{r}, \mathbf{p}, \mathbf{w})$, and nonincreasing in $\mathbf{p}$. $T(\mathbf{r}, \mathbf{p}, \mathbf{w})$ can be chosen to be nondecreasing and convex in the state-contingent revenues, and $\hat{C}$ can be chosen to be convex in $T(\mathbf{r}, \mathbf{p}, \mathbf{w})$.

Proof See Appendix.

Geometrically, if a revenue-cost function displays constant absolute riskiness, rays parallel to the equal-revenue ray will cut successive isocost contours for the revenue-cost function at points of equal slope.

The production risk premium defined above is an absolute measure of the inherent riskiness of the technology. A measure of the relative riskiness of the technology is given by the *relative production risk premium*

$$r(\mathbf{r}, \mathbf{p}, \mathbf{w}) = \frac{e^c(\mathbf{r}, \mathbf{p}, \mathbf{w})}{\bar{r}}.$$

By analogy with the treatment of constant absolute riskiness, we say a technology displays *constant relative riskiness* if for all $\mathbf{r}, t \in \Re_+$:

$$r(t\mathbf{r}, \mathbf{p}, \mathbf{w}) = r(\mathbf{r}, \mathbf{p}, \mathbf{w}).$$

From this definition, CR, and Lemmata 1 and 2 it follows almost immediately that:

Result 2 : The technology displays constant relative riskiness if and only if

$$C(\mathbf{w}, \mathbf{r}, \mathbf{p}) = \bar{C}\left(\mathbf{w}, \bar{\mathbf{T}}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \mathbf{p}\right)$$

where

$$\bar{T}(\lambda\mathbf{r}, \mathbf{p}, \mathbf{w}) = \lambda\bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \quad \bar{T}(\mathbf{r}, \mathbf{p}, \lambda\mathbf{w}) = \bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}),$$

$$\bar{T}(\mathbf{r}, \lambda\mathbf{p}, \mathbf{w}) = \bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}) \quad \lambda > 0,$$

and $\bar{C}\left(\mathbf{w}, \bar{\mathbf{T}}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \mathbf{p}\right)$ is positively linearly homogeneous in input prices, homogeneous of degree zero in $\bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w})$ and output prices, nondecreasing in $\bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w})$, and nonincreasing in output prices. $\bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w})$ can be chosen to be nondecreasing and convex in the state-contingent revenues, and $\bar{C}$ can be chosen to be convex in $\bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w})$.

4. Objective functions

Quiggin and Chambers (1997) analyze general preferences of the form $W : Y^S \rightarrow \Re$, where $Y \subseteq \Re_+$. Thus the analysis is concerned with preferences over state-contingent income vectors $\mathbf{y} \in \Re_+^S$. It is assumed that preferences are subjectively risk-averse in the sense that there exists a vector $\pi \in \Re^S$, with $\sum_{s=1}^{S} \pi_s = 1$ and

$$W(E_\pi[\mathbf{y}]\mathbf{1}_S) \geq W(\mathbf{y}), \forall \mathbf{y}$$

where $E_\pi[\mathbf{y}] = \sum_{s=1}^{S} \pi_s y_s$, and $E_\pi[\mathbf{y}]\mathbf{1}_S$ is the state-contingent outcome vector with $E_\pi[\mathbf{y}]$ occurring in every state of nature. Thus, the elements of π may be considered as subjective probabilities.

We focus on the case when $\mathbf{y}$ is a vector of net returns. Net returns for state s are given by

$$\begin{aligned} y_s &= \mathbf{p}^s \bullet \mathbf{z}^s - \mathbf{w} \bullet \mathbf{x} \\ &= r_s - C(\mathbf{w}, \mathbf{r}, \mathbf{p}). \end{aligned}$$

Hence

$$\mathbf{y} = \mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S.$$

Using this notation, then the producer's objective function can be expressed as

$$W(\mathbf{y}) = W(\mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S).$$

5. Risk neutrality

The simplest optimisation problem for the firm under uncertainty arises when preferences are characterised by risk neutrality. Under risk neutrality, the producer chooses state-contingent outputs to maximize her expected return from production. Formally, therefore, her optimization problem can be written in terms of the effort-cost function as:

$$\max_{\mathbf{z}} \sum_s \pi_s \sum_m p_{ms} z_{ms} - c(\mathbf{w}, \mathbf{z}),$$

which can be conveniently reduced to the following S-dimensional problem by using the revenue-cost function:

$$\max_{\mathbf{r}} \sum_{s} \pi_s r_s - C(\mathbf{w}, \mathbf{r}, \mathbf{p})$$

The first-order conditions on $\mathbf{r}$ may be written in the notation of complementary slackness as

$$\pi_s - C_s(\mathbf{w}, \mathbf{r}, \mathbf{p}) \leq 0, \quad r_s \geq 0, \quad s \in \Omega$$

where

$$C_s(\mathbf{w}, \mathbf{r}, \mathbf{p}) = \frac{\partial C(\mathbf{w}, \mathbf{r}, \mathbf{p})}{\partial r_s}.$$

That is, the marginal cost of increasing revenue in any state is at least equal to the subjective probability of that state. Pictorially, therefore, we represent the producer equilibrium in a manner reminiscent of the representation of production equilibrium in the non-stochastic multi-product case by a hyperplane being tangent to an isocost curve of the producer. Figure 1 illustrates. Here the slope of the hyperplane is determined by the ratio of the producer's subjective probabilities, the fair-odds line, and the isocost curve is determined by the equilibrium level of revenue-cost. Instead of determining an optimal mix of outputs as in the non-stochastic multi-product case, the producer equilibrium now determines the optimal mix of state-contingent revenues. This analogy naturally suggests interpreting the producer's subjective probabilities as the producer's subjective prices of the state-contingent revenues.

Summing these first-order conditions yields an *arbitrage condition*

$$\sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) \geq \sum_{s\in\Omega} \pi_s = 1 \tag{5.1}$$

To see why we refer to (5.1) as an arbitrage condition, notice that the far left-hand side of the expression represents the directional derivative of the cost function in the direction of the equal-revenue ray (the bisector in Figure 1), i.e.,

$$\sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) = \frac{\partial C(\mathbf{w},\mathbf{r}+\gamma\mathbf{1}^S,\mathbf{p})}{\partial\gamma}\,|_{\gamma=\mathbf{0}}\,.$$

So, intuitively speaking, $\sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p})$ is the marginal cost of increasing all state-contingent revenues by the same small amount, and (5.1) simply requires that this cost be at least as large as the expected return. If it were not, it would obviously be profitable for the decisionmaker to continue increasing each state-contingent revenue. For an interior solution, (5.1) must hold as an equality.

We shall refer to the set of revenue vectors $\mathbf{r}$ satisfying (5.1) for given $\mathbf{w}$,$\mathbf{p}$ as the *efficient set* , denoted $\Xi(\mathbf{w},\mathbf{p})$,

$$\Xi(\mathbf{w},\mathbf{p}) = \left\{\mathbf{r} : \sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) \geq 1\right\}.$$

We call the boundary of $\Xi(\mathbf{w},\mathbf{p})$ the *efficient frontier* and note that its elements are given by:

$$\bar{\Xi}(\mathbf{w},\mathbf{p}) = \left\{\mathbf{r} : \sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) = 1\right\}.$$

Because the revenue-cost function is positively linearly homogeneous in input prices (CR.1) and homogeneous of degree zero in $(\mathbf{r},\mathbf{p})$(CR.7), it is trivially linearly homogeneous in $(\mathbf{w},\mathbf{r},\mathbf{p})$. Therefore differentiating both sides of

$$C(\theta\mathbf{w},\theta\mathbf{r},\theta\mathbf{p}) = \theta C(\mathbf{w},\mathbf{r},\mathbf{p}), \quad \theta > 0$$

with respect to r_s gives:

$$\theta C_s(\theta\mathbf{w},\theta\mathbf{r},\theta\mathbf{p}) = \theta C_s(\mathbf{w},\mathbf{r},\mathbf{p})$$

which implies that

$$C_s(\theta\mathbf{w},\theta\mathbf{r},\theta\mathbf{p}) = C_s(\mathbf{w},\mathbf{r},\mathbf{p}).$$

This homogeneity property of marginal cost allows us to establish the following property of the efficient set:

Lemma 3 $\Xi(\theta\mathbf{w}, \theta\mathbf{p}) = \theta\Xi(\mathbf{w}, \mathbf{p})$ and $\bar{\Xi}(\theta\mathbf{w}, \theta\mathbf{p}) = \theta\bar{\Xi}(\mathbf{w}, \mathbf{p})$, $\theta > 0$ the efficient set and the efficient frontier are positively linearly homogeneous in input and output prices.

We establish this result for the efficient set, and the leave the obvious extension to the efficient frontier to the reader. By the definition of the efficient set:

$$\begin{aligned}
\Xi(\theta\mathbf{w}, \theta\mathbf{p}) &= \left\{\mathbf{r} : \sum_{s=1}^{S} C_s(\theta\mathbf{w}, \mathbf{r}, \theta\mathbf{p}) \geq 1\right\} \\
&= \left\{\mathbf{r} : \sum_{s=1}^{S} C_s(\mathbf{w}, \frac{\mathbf{r}}{\theta}, \mathbf{p}) \geq 1\right\} \\
&= \theta\left\{\frac{\mathbf{r}}{\theta} : \sum_{s=1}^{S} C_s(\mathbf{w}, \frac{\mathbf{r}}{\theta}, \mathbf{p}) \geq 1\right\} \\
&= \theta\Xi(\mathbf{w}, \mathbf{p}).
\end{aligned}$$

Here the second equality follows by the fact that $C_s(\mathbf{w}, \mathbf{r}, \mathbf{p})$ is homogeneous of degree zero in all prices and revenues. The obvious implication is that expanding all state-contingent output prices and input prices radially expands the efficient set by the same proportion. Consequently, expanding input and output prices proportionately leads a risk-neutral individual to expand optimal revenues by the same proportion.

Different risk neutral decision-makers may hold different subjective probabilities. However, a revenue vector $\mathbf{r}$ is potentially optimal for some risk-neutral decision-maker only if (5.1) holds. If (5.1) holds for an arbitrary revenue vector, $\hat{\mathbf{r}}$ say, then one can say that that revenue vector is consistent with expected profit maximizing behavior for an individual with the subjective probabilities $\hat{\pi}_s = C_s(\mathbf{w}, \hat{\mathbf{r}}, \mathbf{p})$. The correspondence of the producer's subjective probabilities with these state-contingent marginal costs then determines the optimal point on the efficient set.

The efficient frontier is easily characterized under the presumption that the revenue cost function displays constant absolute riskiness. By Result 1, revenue cost in this case can be written

$$C(\mathbf{w}, \mathbf{r}, \mathbf{p}) = \hat{C}(\mathbf{w}, \mathbf{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \mathbf{p})$$

where $T(\mathbf{r}, \mathbf{p}, \mathbf{w})$ is now interpretable as a *revenue aggregate*, which satisfies

$$T\left(\mathbf{r} + \delta \mathbf{1}^S, \mathbf{p}, \mathbf{w}\right) = T(\mathbf{r}, \mathbf{p}, \mathbf{w}) + \delta.$$

Differentiating this last expression with respect to δ and evaluating the expression at $\delta = 0$ gives:

$$\sum_{s \in \Omega} T_s(\mathbf{r}, \mathbf{p}, \mathbf{w}) = 1, \tag{5.2}$$

while differentiating with respect to r_s gives:

$$T_s\left(\mathbf{r} + \delta \mathbf{1}^S, \mathbf{p}, \mathbf{w}\right) = T_s\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right), \quad \forall \delta. \tag{5.3}$$

Substituting these results into the first-order conditions for an interior solution yields:

$$\frac{\pi_s}{\pi_k} = \frac{T_s\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right)}{T_k\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right)} = \frac{T_s\left(\mathbf{r} + \delta \mathbf{1}^S, \mathbf{p}, \mathbf{w}\right)}{T_k\left(\mathbf{r} + \delta \mathbf{1}^S, \mathbf{p}, \mathbf{w}\right)}, \quad \delta \in \Re, \forall k, s \in \Omega, \tag{5.4}$$

from which we conclude, by using the homogeneity properties of the revenue aggregate cited in Result 1, that the expansion path is homogeneous of degree zero in input prices and parallel to the equal-revenue vector. Moreover, substituting (5.2) into the definition of the efficient set gives:

$$\begin{aligned} \Xi(\mathbf{w}, \mathbf{p}) &= \left\{\mathbf{r} : \frac{\partial \hat{C}\left(\mathbf{w}, \mathbf{T}\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial T} \sum_{s \in \Omega} T_s(\mathbf{r}, \mathbf{p}, \mathbf{w}) \geq 1\right\} \\ &= \left\{\mathbf{r} : \frac{\partial \hat{C}\left(\mathbf{w}, \mathbf{T}\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial T} \geq 1\right\}. \end{aligned}$$

This last expression tells us that the efficient frontier uniquely determines the level of the revenue aggregate, and thereby the revenue-cost level, if the revenue-cost function exhibits constant absolute riskiness. Summarizing results, we have:

Result 3 If the revenue-cost function exhibits constant absolute riskiness, the expansion path is homogeneous of degree zero in input prices, parallel to the equal-revenue vector, and all elements of the efficient frontier are equally costly.

Hence we are led to conclude that when there are interior solutions to the expected profit maximizing problem, then for fixed input and output prices all individuals possessing a technology exhibiting constant absolute riskiness will incur the same level of revenue-cost regardless of their (risk-neutral) preferences towards state-contingent outcomes. Put another way, the optimal level of revenue-cost is independent of a risk-neutral decisionmaker's subjective probabilities when the technology exhibits constant absolute riskiness.

The reason that this happens is transparent. When the revenue-cost function exhibits constant absolute riskiness, it can be written in terms of a single revenue aggregate $T(\mathbf{r}, \mathbf{p}, \mathbf{w})$. This revenue aggregate is positively linearly homogeneous in revenues and state-contingent prices, homogeneous of degree zero in input prices, and has the desirable property that when all revenues increase by the same amount the aggregate goes up by that same amount. Since the arbitrage condition that determines the efficient set requires that the marginal cost of increasing all revenues by the same amount equal one, then for such a technology, the arbitrage condition boils down to requiring that the marginal cost of the aggregate equal one.

We now consider what constant absolute riskiness implies about the producer's optimal response to re-scaling input prices. For an interior solution, upon rescaling input prices, (5.1) becomes:

$$\frac{\partial \hat{C}\left(\mu \mathbf{w}, T\left(\mathbf{r}, \mathbf{p}, \mu \mathbf{w}\right), \mathbf{p}\right)}{\partial T} = \frac{\partial \hat{C}\left(\mu \mathbf{w}, T\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial T} = 1$$

where $\mu > 0$ is the common factor by which all input prices are multiplied, and the first equality follows by the fact that $T\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right)$ is homogeneous of degree zero in input prices. Because the revenue-cost function is positively linearly homogeneous in input prices, this condition can be rewritten:

$$\frac{\partial \hat{C}\left(\mathbf{w}, \mathbf{T}\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial T} = \frac{1}{\mu}.$$

Therefore, we conclude that if $\mu < 1$, the convexity of $\hat{C}$ in $\mathbf{T}(\mathbf{r}, \mathbf{p}, \mathbf{w})$ implies that the optimal $\mathbf{T}(\mathbf{r}, \mathbf{p}, \mathbf{w})$ increases as a result of a re-scaling of input prices. Conversely, when $\mu > 1$, a re-scaling of input prices leads to a decline in the revenue aggregate. Now recall that from (5.4) we know that the expansion path is homogeneous of degree zero in $\mathbf{w}$ and parallels the equal-revenue ray. The fact that the revenue-aggregate increases as input prices are proportionately decreased when combined with this fact implies that all state-contingent revenues go up or down by the same amount in response to a rescaling of input prices. Stated in more geometric terms, a proportional decrease in all input prices leads the optimal state-contingent revenue vector to expand in a direction parallel to the equal-revenue vector.

Now consider what happens when output prices expand or contract radially. Even when the revenue-cost function exhibits constant absolute riskiness, a proportional change in output prices will generally affect the expansion path. Hence, we cannot show that all state-contingent revenues expand equally as they do when input prices change proportionately. However, it is possible to show that effort must increase as a result of a radial expansion of output prices. When output prices are changed proportionately, the first-order condition determining the revenue aggregate (under the presumption of constant absolute riskiness) can be rewritten as:

$$\frac{\partial \hat{C}\left(\mathbf{w}, T\left(\mathbf{r}, \mu \mathbf{p}, \mathbf{w}\right), \mu \mathbf{p}\right)}{\partial T} = 1.$$

Using the homogeneity properties of $\hat{C}$ and T, this condition can be expressed

$$\frac{\partial \hat{C}\left(\mathbf{w}, T\left(\frac{\mathbf{r}}{\mu}, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial T} = \mu.$$

Evaluating the left-hand side of this expression at the state-contingent revenues that were optimal before the price re-scaling shows that it must be smaller than the right-hand side if $\mu > 1$. Consequently, the optimal level of T and revenue cost must increase as a result of the re-scaling of output prices.

Result 4 If the revenue-cost function is characterized by constant absolute riskiness, then re-scaling input prices leads a risk-neutral producer to adjust all state-contingent revenues by the same amount. Increasing (decreasing) output prices proportionately leads a risk-neutral producer to increase (decrease) the revenue aggregate and revenue cost.

When the revenue-cost function exhibits constant absolute riskiness, output price is non-stochastic, and there is only a single stochastic output, the second part of Result 4 can be considerably strengthened. In this case because there is a single state-contingent output whose price is non-stochastic we can write the revenue aggregate and the revenue-cost function as, respectively,

$$T(\mathbf{r}, \mathbf{p}, \mathbf{w}) = T\left(p\mathbf{z}, p\mathbf{1}^S, \mathbf{w}\right) = pT\left(\mathbf{z}, \mathbf{1}^S, \mathbf{w}\right)$$

and

$$\hat{C}\left(\mathbf{w}, pT\left(\mathbf{z}, \mathbf{1}^{\mathbf{S}}, \mathbf{w}\right), p\mathbf{1}^S\right) = \hat{C}\left(\mathbf{w}, T\left(\mathbf{z}, \mathbf{1}^{\mathbf{S}}, \mathbf{w}\right), \mathbf{1}^S\right)$$

where we have exploited the homogeneity properties of the revenue aggregate and the revenue-cost function. Using these results, we now see that a risk-neutral entrepreneur facing such a technology chooses the state-contingent output vector $\mathbf{z} \in \Re_+^S$ to

$$\max\left\{p\sum_{s\in\Omega}\pi_s z_s - \hat{C}\left(\mathbf{w}, T\left(\mathbf{z}, \mathbf{1}^{\mathbf{S}}, \mathbf{w}\right), \mathbf{1}^S\right)\right\}.$$

The associated first-order conditions are:

$$p\pi_s - \hat{C}_T\left(\mathbf{w}, T\left(\mathbf{z}, \mathbf{1}^{\mathbf{S}}, \mathbf{w}\right), \mathbf{1}^S\right) T_s\left(\mathbf{z}, \mathbf{1}^{\mathbf{S}}, \mathbf{w}\right) \leq 0, \quad z_s \geq 0$$

in the notation of complementary slackness. Note first that using (5.3) with these conditions imply that the expansion path for an interior solution obeys:

$$\frac{\pi_s}{\pi_k} = \frac{T_s\left(\mathbf{z}, \mathbf{1^S}, \mathbf{w}\right)}{T_k\left(\mathbf{z}, \mathbf{1^S}, \mathbf{w}\right)} = \frac{T_s\left(\mathbf{z} + \delta \mathbf{1}^S, \mathbf{1}^S, \mathbf{w}\right)}{T_k\left(\mathbf{z} + \delta \mathbf{1}^S, \mathbf{1}^S, \mathbf{w}\right)}$$

for all δ.

Accordingly, the expansion path is parallel to the equal-revenue vector and independent of the level of the non-stochastic output price. Where exactly the individual is on the expansion path can be determined by summing these first-order conditions over all states while using (5.2) to obtain the first-order condition (the arbitrage condition) for the output-aggregate:

$$p - \hat{C}_T\left(\mathbf{w}, T\left(\mathbf{z}, \mathbf{1^S}, \mathbf{w}\right), \mathbf{1}^S\right) \leq 0.$$

Or, perhaps in more intuitive terms, the level of the output aggregate is chosen to equate its marginal cost to its price. Increasing the non-stochastic output price then naturally leads to an increase in T. This gives us the conclusion that we were looking for: Increasing the non-stochastic output price leads to an expansion of the state-contingent output vector along a ray that parallels the equal-outcome vector.

Result 5 If the revenue-cost function exhibits constant absolute riskiness and $\mathbf{z} \in \Re_{+}^{S}$, $\mathbf{p} = p\mathbf{1}^{S}$, with $p \in \Re_{++}$, the state-contingent output vector expands parallel to $\mathbf{1}^{S}$ in response to a change in the non-stochastic output price.

A similar analysis may be undertaken for constant relative riskiness. Using Result 2, the risk-neutral individual's objective function in that case is:

$$\max_{\mathbf{r}} \sum_{s} \pi_s r_s - \bar{C}\left(\mathbf{w}, \bar{\mathbf{T}}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \mathbf{p}\right)$$

where $\bar{\mathbf{T}}(\mathbf{r}, \mathbf{p}, \mathbf{w})$ satisfies the properties detailed in Result 2. The associated first-order conditions for this problem are:

$$\pi_s - \frac{\partial \bar{C}\left(\mathbf{w}, \bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \mathbf{p}\right)}{\partial \bar{T}} \bar{T}_s(\mathbf{r}, \mathbf{p}, \mathbf{w}) \leq 0, \quad r_s \geq 0, \quad s \in \Omega$$

in the notation of complementary slackness. For an interior solution it follows immediately that

$$\frac{\pi_s}{\pi_k} = \frac{\bar{T}_s(\mathbf{r}, \mathbf{p}, \mathbf{w})}{\bar{T}_k(\mathbf{r}, \mathbf{p}, \mathbf{w})}, \forall s, k.$$

By Result 2, the right-hand side of this expression is unaffected by proportional changes in either input prices or output prices and is homogeneous of degree zero in state-contingent revenues. Hence, so long as input prices or output prices move proportionally the optimal mixture of state-contingent revenues (i.e., relative state-contingent revenues) is unchanging.

Now that we have shown that the optimal mixture of state-contingent revenues is invariant to proportional changes in prices all we need to do to complete our examination is to show that a proportional reduction in input prices leads to an increase in the revenue aggregate $\bar{T}(\mathbf{r},\mathbf{p},\mathbf{w})$. By complementary slackness and the positive linear homogeneity of this aggregate in state-contingent revenues, we have from the first-order conditions that:

$$\sum_s \pi_s r_s = \frac{\partial \bar{C}\left(\mathbf{w}, \bar{T}(\mathbf{r},\mathbf{p},\mathbf{w}), \mathbf{p}\right)}{\partial \bar{T}} \bar{T}(\mathbf{r},\mathbf{p},\mathbf{w}). \tag{5.5}$$

The right-hand side of this expression is an increasing function of $\bar{T}(\mathbf{r},\mathbf{p},\mathbf{w})$ and a decreasing function of $\mathbf{w}$. Factoring in a proportional reduction in input prices obviously leads the right-hand side to fall implying that $\bar{T}(\mathbf{r},\mathbf{p},\mathbf{w})$ must adjust upward. Thus,

Result 6 If the revenue-cost function displays constant relative riskiness, a proportional reduction (increase) in all input prices leads to a proportional increase (reduction) in all state-contingent revenues.

In this case, the statement of the equivalent result in terms of output prices is straightforward. The revenue aggregate is now homogeneous of degree zero in output prices (Result 2) so that a proportional change in output prices changes (5.5) to

$$\sum_s \pi_s r_s = \frac{\partial \bar{C}\left(\mathbf{w}, \bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \lambda \mathbf{p}\right)}{\partial \bar{T}} \bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w})$$

which using the homogeneity properties of $\bar{C}$ becomes

$$\sum_s \pi_s r_s = \frac{\partial \bar{C}\left(\frac{\mathbf{w}}{\lambda}, \frac{\bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w})}{\lambda}, \mathbf{p}\right)}{\partial \bar{T}} \bar{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}).$$

Hence,

Corollary 6.1 If the revenue-cost function displays constant relative riskiness, a proportional increase (reduction) in all state-contingent output prices leads to a proportional increase (reduction) in all state-contingent revenues.

6. Maximin preferences

Risk neutrality is the polar case of net returns corresponding to the absence of aversion to risk. For purposes of comparison and to illustrate the generality of our approach, we start our analysis of risk-averse decisionmaking by considering the most extreme form of risk aversion as typified by the maximin objective function.

Maximin is particularly convenient for two reasons: It corresponds to infinite risk aversion (by our definition, it is risk-averse for all possible probability distributions) and thus offers a convenient polar case with which to compare the results from risk neutrality. It also corresponds to a case, where producers are strictly risk averse, but where preferences exhibit both constant absolute risk aversion and constant relative risk aversion[5]. Finally, maximin preferences are not consistent with the expected-utility hypothesis.

In the maximin case, the producer's problem is:

$$\begin{aligned} & \max_{r} \{\min\{r_1 - C(\mathbf{w},\mathbf{r},\mathbf{p}), ..., r_S - C(\mathbf{w},\mathbf{r},\mathbf{p})\}\} \\ = \; & \max_{r} \{\min\{r_1, ..., r_S\} - C(\mathbf{w},\mathbf{r},\mathbf{p})\}. \end{aligned}$$

Because the objective function here is not smoothly differentiable, we cannot generally rely upon first-order conditions and the Kuhn-Tucker theorem to guide identification of an optimum. Even so, the results one expects to emerge are transparent intuitively. We expect the producer to produce at a point where her

[5] Chambers and Quiggin (1998) give definitions of constant absolute risk aversion and constant relative risk aversion for general preferences, analogous to the corresponding definitions for production technology given in this paper. Only risk-neutral preferences are consistent with constant absolute risk aversion and constant relative risk aversion and the expected-utility hypothesis.

indifference curve just 'sits' on one of her isocost curves. Under maximin preferences, indifference curves are 'L-shaped' around the equal-revenue ray. Therefore, we expect the producer's L-shaped indifference curve to sit on an isocost curve at a point on the equal-revenue ray. In other words, the producer chooses a non-stochastic production pattern. (Figure 2 illustrates.)

This is quite easy to show. In fact, we can conclude even more: The producer not only chooses a non-stochastic production pattern, but she chooses to produce where the efficient frontier intersects the equal-revenue ray. Let $\mathbf{r}^*$ denote the producer's optimal state-contingent revenue vector. Now suppose, contrary to our assertion, that $\mathbf{r}^*$ does not lie on the equal-revenue vector, and consider perturbing any single element of $\mathbf{r}^*$, say r_s, by the small amount δr_s. The associated variation in the producer's objective function is

$$\left(\delta^{\min} - C_s(\mathbf{w}, \mathbf{r}, \mathbf{p})\right) \delta r_s$$

where $\delta^{\min} = 1$ if $r_s \in \min\{r_1, ..., r_S\}$ and 0 otherwise. So if $r_s \notin \min\{r_1, ..., r_S\}$, the variation in the producer's objective function is

$$-C_s(\mathbf{w}, \mathbf{r}, \mathbf{p})\delta r_s$$

which implies that the producer's welfare can be increased by decreasing this state-contingent revenue towards the equal-revenue vector. Hence, the optimal state-contingent revenue vector must involve no revenue uncertainty.

Because the optimal production pattern can involve no revenue uncertainty, the decisionmaker's problem then reduces to

$$\max_{r}\left\{r - C\left(\mathbf{w}, r\mathbf{1}^{s}, \mathbf{p}\right)\right\}$$

with the associated first-order condition:

$$1 - \sum_{s\in\Omega} C_s(\mathbf{w}, r\mathbf{1}^S, \mathbf{p}), \quad r \geq 0$$

in the notation of complementary slackness. This last condition tells us that the revenue choice must meet (5.1) and hence be on the efficient frontier if r is strictly positive. Putting these arguments together, while using Lemma 3, lets us conclude that:

Result 7 A producer with maximin preferences completely stabilizes revenue and for an interior solution produces where the efficient frontier intersects the equal-revenue vector. A proportional increase in input and output prices leads to a proportional increase in the optimal non-stochastic revenue.

An immediate corollary of Result 7 is obtained by assuming that the revenue-cost function exhibits constant absolute riskiness and applying Result 3 to obtain:

Corollary 7.1 If the producer has maximin preferences and uses a technology exhibiting constant absolute riskiness, then for an interior solution the producer incurs the same level of cost as a risk-neutral producer.

This latter case is also illustrated in Figure 2. There we have drawn the isocost curve associated with the efficient frontier under the presumption of constant absolute riskiness. The risk-neutral producer with fair-odds as depicted produces at the point of tangency between the fair-odds line and the isocost curve. The completely risk-averse producer facing the same technology produces on the bisector.

Figure 2 illustrates the essential relationship between the solutions for a risk taker and a risk averter that emerge when we consider more general risk-averse preferences. The risk averter sacrifices expected returns in order to self-insure by arranging (in her view) a less risky production pattern than would be chosen by a risk-neutral individual with the same subjective probabilities. Higher expected returns are sacrificed for more stable returns.

When the technology exhibits constant absolute riskiness, this producer's response to a radial contraction of input prices or a radial expansion of output prices parallels that of a risk-neutral individual. In the case of a radial change in input prices, the producer's first-order condition for an interior solution requires:

$$\frac{\partial \hat{C}\left(\mu\mathbf{w}, T\left(r1^S, \mathbf{p}, \mu\mathbf{w}\right), \mathbf{p}\right)}{\partial T} = 1,$$

which by the homogeneity properties established in Result 1 reduces to:

$$\frac{\partial \hat{C}\left(\mathbf{w}, T\left(r1^S, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial T} = \frac{1}{\mu},$$

so that $T\left(r1^S, \mathbf{p}, \mathbf{w}\right)$ must increase if input prices decrease proportionately. By Result 1, this also means that the maximin individual's optimal certain revenue must increase as a result of a proportional decrease in input prices. Similar arguments establish that a proportional change in output prices transforms the producer's first-order conditions to

$$\frac{\partial \hat{C}\left(\mathbf{w}, T\left(\frac{r}{\mu}1^S, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial T} = \mu,$$

implying that $T\left(r1^S, \mathbf{p}, \mathbf{w}\right)$ and r must increase as a result of a proportional increase in state-contingent output prices.

Corollary 7.2 If the producer has maximin preferences and uses a technology exhibiting constant absolute riskiness, then for an interior solution a proportional increase (decrease) in state-contingent output prices or a proportional decrease (increase) in input prices leads to an increase (decrease) in the optimal non-stochastic revenue chosen by the producer and an increase (decrease) in her revenue cost.

When a producer with maximin preferences uses a technology displaying constant relative riskiness, his or her level of revenue cost relative to that of a risk-neutral individual depends critically upon the level of the certain revenue that he or she produces. In this case, the first-order conditions and complementary slackness require:

$$r = \frac{\partial \bar{C}\left(\mathbf{w}, \bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial \bar{T}} \bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right).$$

The right-hand side of this expression, by Result 2, is a nondecreasing function of the revenue aggregate, and therefore by recalling (5.5), one sees that the revenue aggregate here can be higher than that used by a risk-neutral individual only if the non-stochastic revenue produced here exceeds mean revenue produced by the risk-neutral individual. Conversely, the revenue aggregate here will be lower than that for a risk-neutral individual only if the certain revenue is lower than the risk-neutral individual's mean revenue. And from the fact that $\bar{C}$ is non-decreasing in the revenue aggregate, we have:

Corollary 7.3 If the producer has maximin preferences and uses a technology exhibiting constant relative riskiness, then the producer incurs a greater level of revenue cost than a risk-neutral producer if and only if her certain revenue exceeds the risk-neutral individual's mean revenue.

Now consider how a re-scaling of input prices affects the producer's optimal choice of the non-stochastic revenue. In that instance, Result 2 implies that the complementary slackness conditions become:

$$r = \frac{\partial \bar{C}\left(\mu \mathbf{w}, \bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial \bar{T}} \bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right),$$

which can now be rewritten as:

$$\frac{r}{\mu} = \frac{\partial \bar{C}\left(\mathbf{w}, \bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right), \mathbf{p}\right)}{\partial \bar{T}} \bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right),$$

so that a radial decrease in state-contingent prices must lead to an increase in

$\bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right)$ and in the non-stochastic revenue as well.

Similarly, a radial expansion of the state-contingent output prices transforms the first-order condition to:

$$r = \frac{\partial \bar{C}\left(\mathbf{w}, \bar{T}\left(r\mathbf{1}^S, \mu\mathbf{p}, \mathbf{w}\right), \mu\mathbf{p}\right)}{\partial \bar{T}} \bar{T}\left(r\mathbf{1}^S, \mu\mathbf{p}, \mathbf{w}\right).$$

By the homogeneity properties of $\bar{C}$ and $\bar{T}$, this expression can be rewritten as:

$$\mu r = \frac{\partial \bar{C}\left(\mathbf{w}, \frac{\bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right)}{\mu}, \mathbf{p}\right)}{\partial \bar{T}} \bar{T}\left(r\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right),$$

from which we conclude that $\bar{T}$, and thus r, must rise if $\mu > 1$.

Corollary 7.4 If the producer has maximin preferences and uses a technology exhibiting constant relative riskiness, then a radial expansion (contraction) of output prices or a radial contraction (expansion) of input prices leads to an expansion (contraction) of both revenue cost and the optimal non-stochastic revenue.

7. Generalized Schur-concave preferences

We now turn attention to the general case where the objective function is of the net returns form

$$W(\mathbf{y}) = W(\mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S).$$

and the function W represents risk-averse preferences for some given probability vector $\boldsymbol{\pi}$. To make the idea of risk-aversion more precise we introduce the notion of generalized Schur-concavity. As in Rothschild and Stiglitz (1970), we say that $\mathbf{y}'$ is a mean-preserving spread of $\mathbf{y}$ (denoted notationally as $\mathbf{y} \preceq_\pi \mathbf{y}'$) if for all y

$$\int_{-\infty}^{y} F_{\mathbf{y}}(t)dF_{\mathbf{y}}(t) \geq \int_{-\infty}^{y} F_{\mathbf{y}'}(t)dF_{\mathbf{y}'}(t)$$

where $F_{\mathbf{y}}(t) = Pr\{\mathbf{y} \leq t\}$, and, following Chambers and Quiggin (1997b), define a preference function W to be *generalized Schur-concave* for π if and only if W: $\Re^S \rightarrow \Re$ satisfies:

$$\mathbf{y} \preceq_\pi \mathbf{y}' \Rightarrow W(\mathbf{y}) \geq W(\mathbf{y}').$$

Generalized Schur concavity thus encompasses all forms of preferences (including expected utility) which are risk-averse in our sense for the probabilities $\boldsymbol{\pi}$. The crucial property of generalized Schur-concavity, which is proven in Chambers and Quiggin (1997), for our purposes is:

Lemma 4 A smoothly differentiable preference function W is generalized Schur-concave for π only if

$$y_s \geq y_{s'} \Leftrightarrow W_s(\mathbf{y})/\pi_s \leq W_{s'}(\mathbf{y})/\pi_{s'} \quad \forall \mathbf{y}, s, s'$$

Assuming W is generalized Schur-concave for π, the producer's maximisation problem is well-behaved. The first-order condition on r_s becomes:

$$W_s(\mathbf{y}) - C_s(\mathbf{w}, \mathbf{r}, \mathbf{p}) \sum_{t \in \Omega} W_t(\mathbf{y}) \leq 0, \quad r_s \geq 0,$$

with complementary slackness. The arbitrage condition derived from summing these first-order conditions is

$$\sum_{s \in \Omega} C_s(\mathbf{w}, \mathbf{r}, \mathbf{p}) \geq 1 \tag{7.1}$$

just as in the case of expected profit maximization. We conclude from (7.1) that a producer maximizing a generalized Schur-concave function of net returns will choose a revenue vector that is in the efficient set.

Observe, as we have illustrated with the maximin case above, that condition (7.1) holds with equality for an interior solution even in the absence of differentiability of W, since, for any $\mathbf{r}$ that does not satisfy the condition, there exists an $\mathbf{r}'$ such that $\mathbf{y}' - \mathbf{y} \geq \mathbf{0}$ with strict inequality in at least one state. Pictorially, the production equilibrium is illustrated by a tangency between one of the producer's indifference curve and one of her isocost curves.

An immediate implication of (7.1) and Result 3 is that an individual with generalized Schur concave preferences and a technology exhibiting constant absolute riskiness will incur the same level of costs as a risk-neutral producer (as well as one with completely risk-averse preferences). Hence, we have:

Result 8 If the producer has generalized Schur-concave preferences and uses a technology exhibiting constant absolute riskiness in state-contingent revenues, then for an interior solution the producer incurs the same level of cost as a risk-neutral producer and a producer with maximin preferences.

An immediate implication of this result is a 'separation' result:

Corollary 8.1 If the producer has generalized Schur-concave preferences and uses a technology exhibiting constant absolute riskiness, her optimal cost level is independent of her risk preferences.

Result 8 can also be illustrated with the use of Figure 2. A person with generalized Schur-concave preferences and a technology with constant absolute riskiness will produce on the isocost curve between the bisector and the point of risk-neutral production. In particular, since a risk-neutral producer maximizes expected profits at given $\boldsymbol{\pi}$, any other producer operating on the efficient set with the same level of cost must have lower expected profit and, therefore, lower expected revenue. Therefore, under these conditions, the expected-profit maximizing vector of net returns must be riskier (that is, have a higher risk premium in terms of the risk averter's preferences) than the vector of net returns chosen by the risk-averse producer. To confirm this statement, let the optimal state-contingent revenue vector for the risk-neutral individual be denoted $\mathbf{r}^N$ and the optimal state-contingent revenue vector for the risk-averse individuals be denoted $\mathbf{r}^A$. The risk premium associated with $\mathbf{r}^N$ from the risk-averter's perspective is:

$$\sum_{s\in\Omega}\pi_s\left(r_s^N - C\left(\mathbf{w},\mathbf{r}^A,\mathbf{p}\right)\right) - e\left(\mathbf{y}^A\right) = \sum_{s\in\Omega}\pi_s\left(r_s^N - r_S^A\right) + \sum_{s\in\Omega}\pi_s r_S^A - e\left(\mathbf{y}^A\right).$$

The term on the right-hand side of the equality represents the difference between the risk-neutral individual's expected revenue and that for the risk-averse individual plus the risk-averse individual's risk premium for his state-contingent revenue vector. In terms of Figure 2, the difference between the two risk premiums can be visualized as the difference (not drawn) between where the fair-odds line through the risk-neutral individual's choice intersects the bisector and where the fair-odds line through the risk-averter's choice intersects the bisector. Summarizing, we have:

Corollary 8.2 If the producer has generalized Schur-concave preferences and uses a technology exhibiting constant absolute riskiness, then a risk-neutral individual using the same technology adopts a riskier state-contingent revenue vector (from the risk-averter's perspective) than the risk averter.

We have seen that an individual with generalized Schur-concave preferences using a technology exhibiting constant absolute riskiness will adopt a state-contingent revenue vector that is more risky than that for an individual with maximin preferences but less risky than of a risk-neutral individual.Combining Lemma 4 with the first-order conditions for an interior solution shows that an optimally chosen state-contingent revenue vector must be *risk-aversely efficient* (in the sense of Peleg and Yaari, 1978) with respect to π:

$$r_s \geq r_t \Leftrightarrow C_s(\mathbf{w}, \mathbf{r},\ \mathbf{p})/\pi_s \leq C_t(\mathbf{w}, \mathbf{r}, \mathbf{p})/\pi_t,$$

or

$$\left(\frac{C_s(\mathbf{w},\mathbf{r},\mathbf{p})}{\pi_s} - \frac{C_t(\mathbf{w},\mathbf{r},\mathbf{p})}{\pi_t}\right)(r_s - r_t) \leq 0. \tag{7.2}$$

The notion of risk-averse efficiency is due to Peleg and Yaari and can be heuristically identified with the notion that for any state-contingent revenue vector satisfying it there will be some risk-averse individual who would optimally adopt that vector if she incurred the same level of revenue-cost.

Once again, the result may be extended to the case where W is not differentiable but still generalized Schur concave by using the observation that for any $\mathbf{r}$ that does not satisfy the condition, there exists $\mathbf{r}'$ such that $\mathbf{y}'\preceq_{\pi}\mathbf{y}$. We define the *risk-aversely efficient set for* π as consisting of those elements of the efficient set satisfying (7.2).

By complementary slackness, so long as the preference function is differentiable, we have:

$$\frac{\sum_{s\in\Omega} W_s(\mathbf{y})\, r_s}{\sum\limits_{s\in\Omega} W_s(\mathbf{y})} = \sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) r_s. \tag{7.3}$$

From Lemma 4, generalized Schur-concave preferences W satisfy:

$$\sum_{s\in\Omega} W_s(\mathbf{y})\left(y_s - \sum_{s\in\Omega}\pi_s y_s\right) \leq 0.$$

Substituting $y_s = r_s - C(\mathbf{w}, \mathbf{r}, \mathbf{p})$, this last inequality implies:

$$\sum_{s\in\Omega} \pi_s r_s \geq \frac{\sum_{s\in\Omega} W_s(\mathbf{y})\, r_s}{\sum_{s\in\Omega} W_s(\mathbf{y})} \tag{7.4}$$

which when combined with (7.4) establishes that:

$$\sum_{s\in\Omega} \pi_s r_s - \sum_{s\in\Omega} C_s(\mathbf{w},\ \mathbf{r}, \mathbf{p}) r_s \geq 0.$$

Direct calculation establishes that this last expression equals the marginal change in expected profit associated with a small radial expansion of the revenue vector. Because it is non-negative, we have:

Result 9 If $\mathbf{r}$ is a risk-aversely efficient revenue vector, a small radial expansion in $\mathbf{r}$ leads to an increase in expected profits.

An early analogue of Result 6 was first proved by Sandmo (1971) for the expected-utility model with a non-stochastic technology and stochastic prices. Much later, we generalized his result for the expected-utility model with state-contingent production in Chambers and Quiggin (1997).

A comparison of Result 9 with Result 8, and especially with Corollaries 8.1 and 8.2, shows a crucial difference between the analysis of a general state-contingent production technology and the special case of a stochastic production function. Result 9, when applied to the case of a stochastic production function with a scalar input (effort), implies that a risk-averse producer will always commit less effort than a risk-neutral producer. Similarly, in the Sandmo model of non-stochastic technology and stochastic prices, price stabilisation or price insurance always generate an increase in output and costs. For a general state-contingent production technology a risk-averse producer will typically produce less, and, therefore, incur smaller production costs, than a risk-neutral producer constrained to choose a point on the same output ray. However, a risk-averse producer will allocate resources to reducing risk at the expense of a reduction in expected net returns. Result 8 shows that for technology displaying constant absolute riskiness these effects will cancel out as far as costs are concerned, so that the level of costs is determined solely by the arbitrage condition.

Now consider what happens when an individual uses a technology displaying constant relative riskiness. By the first-order conditions, complementary slackness, and Result 2, we have that

$$\frac{\sum_{s\in\Omega} W_s(\mathbf{y})\, r_s}{\sum\limits_{s\in\Omega} W_s(\mathbf{y})} = \frac{\partial \bar{C}\left(\mathbf{w}, \bar{T}(\mathbf{r},\mathbf{p},\mathbf{w}), \mathbf{p}\right)}{\partial \bar{T}} \bar{T}(\mathbf{r},\mathbf{p},\mathbf{w}). \tag{7.5}$$

Expression (5.5), on the other hand, indicated that a risk-neutral individual using the same technology would choose expected revenue so that:

$$\frac{\partial \bar{C}\left(\mathbf{w},\bar{T}\left(\mathbf{r},\mathbf{p},\mathbf{w}\right),\mathbf{p}\right)}{\partial \bar{T}}\bar{T}\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)=\sum_{s}\pi_{s}r_{s}. \tag{7.6}$$

Because $\bar{C}\left(\mathbf{w},\bar{T}\left(\mathbf{r},\mathbf{p},\mathbf{w}\right),\mathbf{p}\right)$ is convex in $\bar{T}$, a risk-averter will incur more effort cost than risk-neutral individual if and only if their directional derivative of $\bar{C}$ in the direction of $\bar{T}$ (the right-hand side of (7.5)) is greater than the left-hand side of (7.6). When this fact is used in conjunction with (7.4), we see that:

Result 10 If the producer has generalized Schur concave preferences and uses a technology exhibiting constant relative riskiness in state-contingent revenues, then the producer incurs a greater level of revenue cost than a risk-neutral producer only if her mean revenue exceeds the risk-neutral individual's mean revenue.

8. Concluding comments

The analysis of the problem of the firm under uncertainty presented in this paper represents a synthesis of the modern theory of production, based on the exploitation of duality, with the idea of state-contingent production sets pioneered by

Arrow and Debreu. This approach is sufficiently flexible to encompass the results of the earlier literature based on the concept of the stochastic production function and to deal with a wide range of relationships between effort-cost and the riskiness of returns.

As has already been observed, the concept of production under uncertainty is central to a wide range of recent developments in economics. Recent papers including Chambers and Quiggin (1996, 1997) have addressed issues including futures markets, pollution control and agrarian exploitation. An application to the problem of moral hazard is presented by Quiggin and Chambers (1998). But the range of application of the approach outlined above is essentially unlimited.

9. Appendix

Proof of Lemma 2: From the definition of the certainty equivalent revenue

$$\begin{aligned} e^c(\mathbf{r},\mathbf{p},\lambda\mathbf{w}) &= \sup\{e : C(\lambda\mathbf{w}, e\mathbf{1}^S, \mathbf{p}) \leq C(\lambda\mathbf{w}, \mathbf{r}, \mathbf{p})\} \\ &= \sup\{e : \lambda C(\mathbf{w}, e\mathbf{1}^S, \mathbf{p}) \leq \lambda C(\mathbf{w}, \mathbf{r}, \mathbf{p})\} \\ &= \sup\{e : C(\mathbf{w}, e\mathbf{1}^S, \mathbf{p}) \leq C(\mathbf{w}, \mathbf{r}, \mathbf{p})\} \end{aligned}$$

where the second equality follows by the positive linear homogeneity, CR.1, of the revenue-cost function. Also by CR. 7:

$$\begin{aligned} e^c(\lambda\mathbf{r},\lambda\mathbf{p},\mathbf{w}) &= \sup\{e : C(\mathbf{w}, e\mathbf{1}^S,\lambda\mathbf{p}) \leq C(\mathbf{w},\lambda\mathbf{r},\lambda\mathbf{p})\} \\ &= \sup\{e : C(\mathbf{w}, \frac{e}{\lambda}\mathbf{1}^S,\mathbf{p}) \leq C(\mathbf{w},\mathbf{r},\mathbf{p})\} \\ &= \lambda\sup\{\frac{e}{\lambda} : C(\mathbf{w}, \frac{e}{\lambda}\mathbf{1}^S,\mathbf{p}) \leq C(\mathbf{w},\mathbf{r},\mathbf{p})\} \\ &= \lambda e^c(\mathbf{r},\mathbf{p},\mathbf{w}). \end{aligned}$$

The properties of the production risk premium follow straightforwardly.

Proof of Result 1: By constant absolute riskiness,

$$p\left(\mathbf{r}+t\mathbf{1}^S,\mathbf{p},\mathbf{w}\right) = p(\mathbf{r},\mathbf{p},\mathbf{w})$$

which by the definition of the risk premium requires

$$\bar{r} + t - e^c\left(\mathbf{r}+t\mathbf{1}^{\mathbf{S}},\mathbf{p},\mathbf{w}\right) = \bar{r} - e^c(\mathbf{r},\mathbf{p},\mathbf{w})$$

whence

$$e^c\left(\mathbf{r}+t\mathbf{1}^{\mathbf{S}},\mathbf{p},\mathbf{w}\right) = e^c(\mathbf{r},\mathbf{p},\mathbf{w}) + t.$$

Applying Lemma 1 establishes that the revenue-cost function must be a monotonic transformation of the certainty-equivalent revenue having this last property. Letting $T(\mathbf{r}, \mathbf{p}, \mathbf{w}) = e^c(\mathbf{r}, \mathbf{p}, \mathbf{w})$ and using Lemma 2 establishes the homogeneity properties of T. The convexity and monotonicity propertries follow from the properties of the revenue-cost function CR and the properties of the directional distance function.

Proof of Result 2: Directly parallels the proof of Result 1 upon noting that constant relative riskiness requires

$$e^c(\lambda\mathbf{r}, \mathbf{p}, \mathbf{w}) = \lambda e^c(\mathbf{r}, \mathbf{p}, \mathbf{w})$$

which together with Lemma 2 establishes that

$$e^c(\lambda\mathbf{r}, \mathbf{p}, \mathbf{w}) = e^c(\lambda\mathbf{r}, \lambda\mathbf{p}, \mathbf{w})$$

implying that the certainty-equivalent revenue is homogeneous of degree zero in the stochastic output prices.

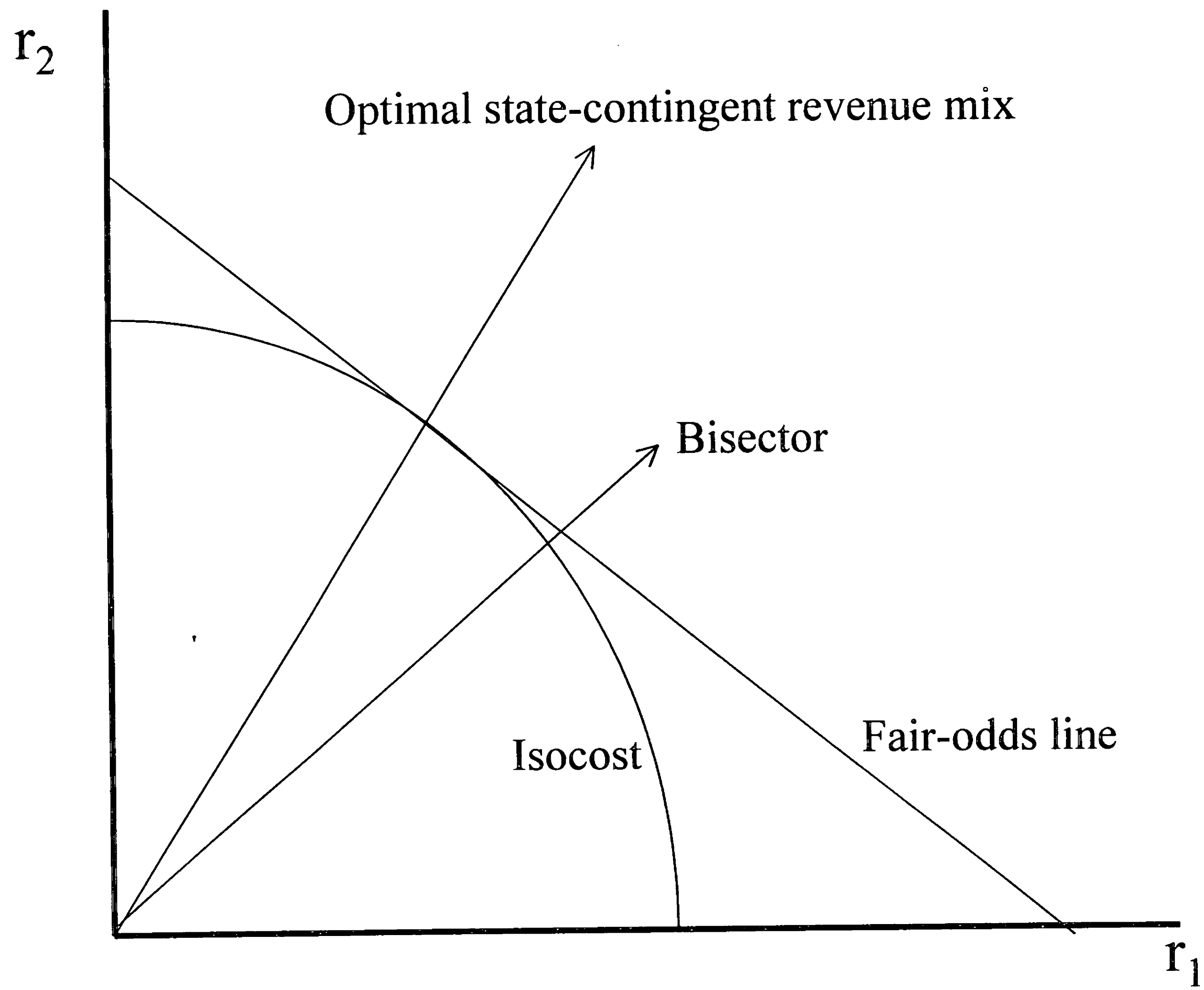

Figure 1: Risk-neutral production equilibrium

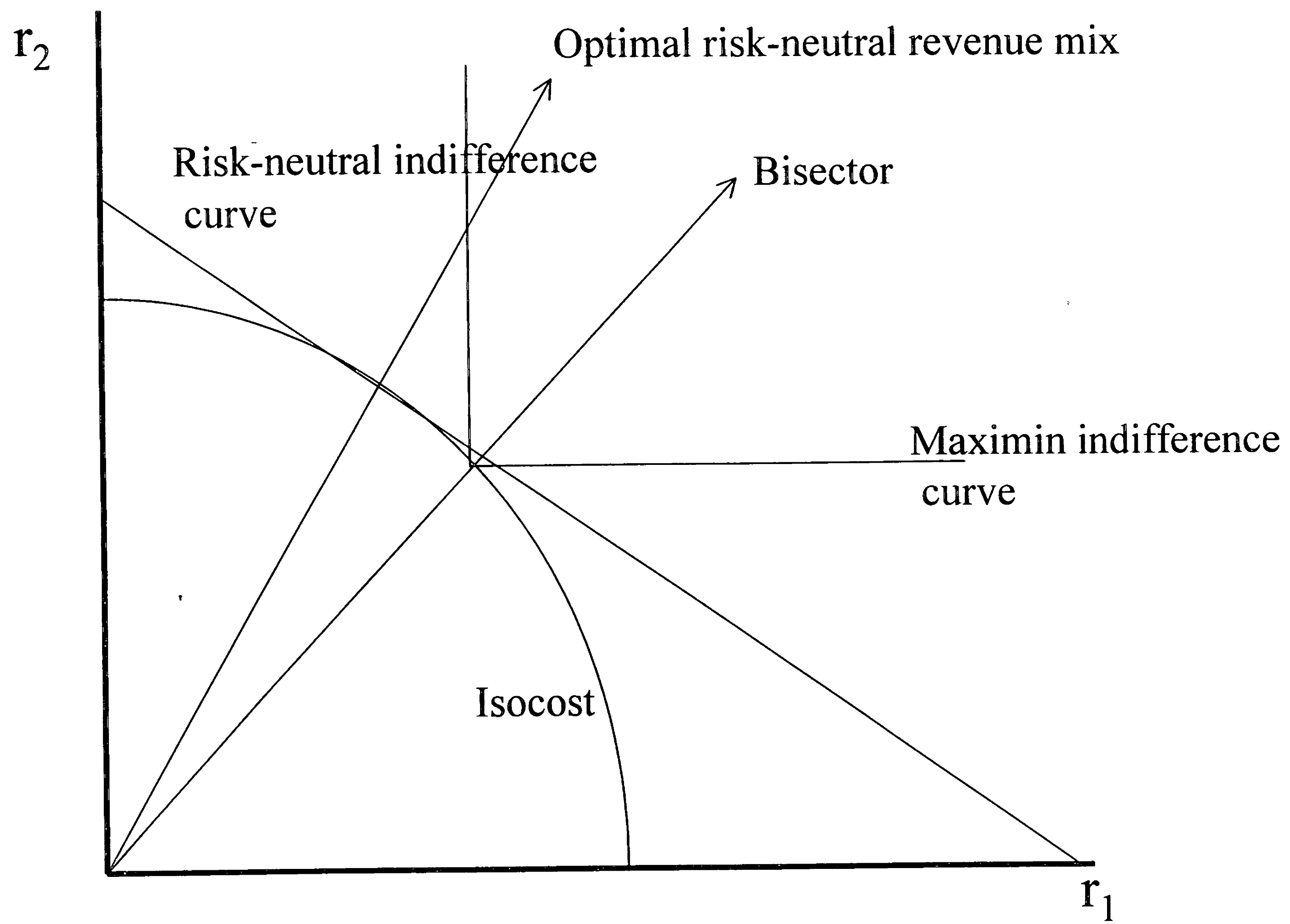

Figure 2: Risk-neutral and maximin production equilibria with constant absolute riskiness