Input and Output Indicators

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1 Introduction

This paper develops new input and output measures. The approach used to construct these measures relies on earlier work by Chambers (1996) that employed a version of Luenberger's (1992, 1995) shortage function to develop input, output, and productivity measures. The measures developed by Chambers (1996), being based on a translation representation of the technology, are to be contrasted directly with more conventional input and output measures which rely upon radial representations of the technology: namely, input and output distance functions (Caves, Christensen, and Diewert 1982a, 1982b).

In what follows, I first briefly discuss the translation measure upon which I base my new measures of inputs and outputs. Following Chambers, Chung, and Färe (1996), I refer to it as the *directional technology distance function* to emphasize that it represents a complete generalization of Shephard's (1970) input and output distance functions. After briefly developing the properties of the directional technology distance function, I specify two parametric representations of it which are flexible in the sense of Diewert (1976). The first has the attractive property of automatically satisfying the translation properties of directional technology distance functions. I refer to this form as the *logarithmic-transcendental.* The second form is the quadratic, which was studied extensively in Chambers (1996). Next, I briefly discuss previous work on bilateral input, output, and productivity measurement and provide a synopsis of the main results in Chambers (1996) on Bennet-Bowley input and output measures. Then, I define new bilateral measures of input and output aggregates that are particularly appropriate for the logarithmic-transcendental technology and show how they can be calculated directly from observed data on input and output quantities and their prices. My last substantive section derives necessary and sufficient restrictions on the technology that insure that the bilateral input and output indicators defined here and by Chambers (1996) satisfy a form of additive transitivity that Blackorby and Donaldson (1980) refer to as additive circularity. These necessary and sufficient conditions are so restrictive that I then develop new multilateral input and output measures which satisfy additive circularity, but which can be constructed directly from a series of bilateral input and output indicators.

2 Notation, Assumptions, and Definitions

Let $\mathbf{x} \in \Re^n_+$ denote a vector of inputs and $\mathbf{y} \in \Re^m_+$an output vector. Superscripts on input and output vectors are typically used to differentiate vectors either across time or across firms. (Exceptions are $\mathbf{0}^k$ and $\mathbf{1}^k$ which denote the k vectors of zeroes and ones, respectively.) For example, $\mathbf{x}^h$ will be interpreted variously as firm h's input use or as input use in period h. The technology is defined in terms of a set $T \subset \Re^n_+ \times \Re^m_+$:

$$T = \{(\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+) : \mathbf{x} \text{ can produce } \mathbf{y}\}.$$

T satisfies the following properties:
T.1: T is closed;
T.2: Inputs and outputs are freely disposable, i.e., if $(\mathbf{x}^{'}, -\mathbf{y}^{'}) \geq (\mathbf{x}, -\mathbf{y})$ then $(\mathbf{x}, \mathbf{y}) \in T \Rightarrow (\mathbf{x}^{'}, \mathbf{y}^{'}) \in T$;
T.3: T is convex.

Slightly modifying Luenberger's (1992, 1995) shortage function and following Chambers, Chung, and Färe (1996), the *directional technology distance function* is defined by:

$$\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) = \sup\{\beta \in \Re : (\mathbf{x} - \beta\mathbf{g}_x, \mathbf{y} + \beta\mathbf{g}_y) \in T\}, \mathbf{g}_x \in \Re^n_+, \mathbf{g}_y \in \Re^m_+.$$

where $(\mathbf{g}_x, \mathbf{g}_y) \neq (\mathbf{0}^n, \mathbf{0}^m)$. $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)$ represents the maximal translation of the input and output vector in the direction of $(-\mathbf{g}_x, \mathbf{g}_y)$that keeps the translated input and output vector inside T. When $(-\mathbf{g}_x, \mathbf{g}_y) = (-\mathbf{1}^n, \mathbf{1}^m)$, the directional technology distance function, therefore, is analogous to Blackorby and Donaldson's (1980) *translation function* for T. Figure 1 illustrates $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{1}^n, \mathbf{1}^m)$as the ratio OB/OA for the point (x, y). All known distance and directional distance functions can be depicted as special cases of $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)$. In particular, the *directional input distance function* defined by Chambers, Chung, and Färe (1995) is $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m)$, and the *directional output distance function* is $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y)$.

An important property of the directional technology distance function is that it offers a complete function representation of the technology in that:

$$(\mathbf{x}, \mathbf{y}) \in T \Leftrightarrow \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \geq 0 \tag{1}$$

(Luenberger, 1992). Points on the boundary of T are characterized by $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) = 0$. Denote input prices by $\mathbf{w} \in \Re^n_{++}$ [1]and output prices by $\mathbf{p} \in \Re^m_{++}$. From (1),

[1] $\Re^k_{++}$ denotes the strictly positive k-orthant.

it follows immediately (Luenberger, 1992; Chambers, Chung, and Färe (1995, 1996)) that a profit maximizing firm solves:

$$\sup\left\{\mathbf{p}\cdot\left(\mathbf{y}+\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)\,\mathbf{g}_y\right)-\mathbf{w}\left(\mathbf{x}-\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)\,\mathbf{g}_x\right)\right\}. \tag{2}$$

Assuming that the directional technology distance function is differentiable in both inputs and outputs, the first-order conditions for an interior solution to (2) are:

$$\begin{aligned}\mathbf{p}&=-\nabla_y\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)\left(\mathbf{p}\cdot\mathbf{g}_y+\mathbf{w}\cdot\mathbf{g}_x\right)\\ \mathbf{w}&=\nabla_x\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)\left(\mathbf{p}\cdot\mathbf{g}_y+\mathbf{w}\cdot\mathbf{g}_x\right).\end{aligned} \tag{3}$$

In equations (3) the notation ∇_z denotes the gradient of the function with respect to the vector $\mathbf{z}$.

The other properties of $\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)$are summarized by Chambers, Chung, and Färe(1996):

D.1: $\overrightarrow{D_T}(\mathbf{x}-\alpha\mathbf{g}_x,\mathbf{y}+\alpha\mathbf{g}_\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)=\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)-\alpha,\alpha\in\Re$;

D.2: $\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\lambda\mathbf{g}_x,\lambda\mathbf{g}_y)=\frac{1}{\lambda}\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y),\lambda>0$;

D.3: $(\mathbf{x}',-\mathbf{y}')\geq(\mathbf{x},-\mathbf{y})\Rightarrow\overrightarrow{D_T}\left(\mathbf{x}',\mathbf{y}';\mathbf{g}_x,\mathbf{g}_y\right)\geq\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)$, i.e., nondecreasing in inputs and nonincreasing in output;

D.4: $\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)$is concave in $(\mathbf{x},\mathbf{y})$ if T.3 is satisfied.

In what follows, I deal exclusively with a special case of the directional technology distance function (analogous to Blackorby and Donaldson's (1980) translation function), and I shall always deploy the more concise notation:

$$\begin{aligned}T_t(\mathbf{x},\mathbf{y})&\equiv\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{1}^n,\mathbf{1}^m),\\ T_i(\mathbf{x},\mathbf{y})&\equiv\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{1}^n,\mathbf{0}^m),\\ T_o(\mathbf{x},\mathbf{y})&\equiv\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{0}^n,\mathbf{1}^m),\end{aligned}$$

and refer to them, respectively, as the technology, input, and output translation functions.

The first parametric specification of the technology translation function that I consider is the *logarithmic-transcendental.* Firm h's technology translation function is logarithmic-transcendental if it can be written in the form:

$$\begin{aligned}\exp T_t^h(\mathbf{x},\mathbf{y})=&\frac{1}{2}\sum_{j=1}^n\sum_{i=1}^n a_{ij}^h\exp(\frac{x_i}{2})\exp(\frac{x_j}{2})+\frac{1}{2}\sum_{k=1}^m\sum_{j=1}^m b_{jk}^h\exp(\frac{-y_j}{2})\exp(\frac{-y_k}{2})\\ &+\sum_{j=1}^n\sum_{k=1}^m c_{kj}^h\exp(\frac{x_j}{2})\exp(\frac{-y_k}{2}),\end{aligned}$$

where $a_{ij}^h=a_{ji}^h$ for all i and j, $b_{jk}^h=b_{kj}^h$ for j and k, and $c_{kj}^h=c_{jk}^h$ for all j and

k. The logarithmic-transcendental, being a member of the generalized-quadratic class of functions (Blackorby, Primont, and Russell, 1978), is second-order flexible. Also notice that the logarithmic-transcendental form automatically satisfies property D.1 for the technology translation function.

The quadratic[2] k translation function (k= i,o) for firm h is:

$$T_k^h(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} a_i^h x_i + \sum_{k=1}^{m} b_k^h y_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}^h x_i x_j + \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m} \beta_{kl}^h y_k y_l + \sum_{i=1}^{n}\sum_{k=1}^{m} \gamma_{ik}^h x_i y_k,$$

with $\alpha_{ij}^h = \alpha_{ji}^h, \beta_{kl}^h = \beta_{lk}^h$. If this form is interpreted as an input-translation function, the following parametric restrictions insure that D.1 is satisfied:

$$\sum_{i=1}^{n} a_i^h = 1; \sum_{j=1}^{n} \alpha_{ij}^h = 0, i = 1, ..., n; \sum_{i=1}^{n} \gamma_{ik}^h = 0, k = 1, ..., m.$$

And finally if the quadratic is interpreted as an output translation function then the following insure that D.1 is satisfied:

$$\sum_{k=1}^{m} b_k^h = -1; \sum_{l=1}^{m} \beta_{kl}^h = 0, k = 1, ..., m; \sum_{k=1}^{m} \gamma_{ik}^h = 0, i = 1, .., n.$$

3 Previous Work on Input and Output Indexes

Input and output indexes are summary measures of two things: multiple input use or multiple output production and input and output differences either over time, place, or firms. For the purposes of our present discussion, suppose that one is considering how a firm's use of a single input, which we denote as x, changes over time. There are at least two ways to measure this change: The first and perhaps the most obvious is simply the difference between input use in period one, x^1, and input use in the base period, x^0, i.e., $x^1 - x^0$. This approach brings with it the advantage that changes in the origin from which these two numbers are measured has no effect on the measure of input change, but it also has the disadvantage that the input-change measure is not unit free. For example, if we go from measuring the input in terms of hours to measuring in terms of hundreds of hours, the input measure changes. A second approach, which remedies the unit problem, is to consider the ratio of input use in period one to input use in the base period, i.e., x^1/x^0. However, this ratio-approach has the disadvantage that changes in the origin from which these two numbers are measured does have an effect on the resulting measure of input change. So, for example, if

[2] Notice that the logarithmic-transcendental has fewer parameters than the quadratic. However, the logarithmic-transcendental automatically satisfies D.1, while the quadratic does not. To see how the quadratic must be further restricted, differentiate D.1, $T_t(\mathbf{x}-\alpha\mathbf{1}^n, \mathbf{y}+\alpha\mathbf{1}^m) = T_t(\mathbf{x},\mathbf{y}) - \alpha$ with respect to α to obtain $-\nabla_{\mathbf{x}} T_t(\mathbf{x}-\alpha\mathbf{1}^n, \mathbf{y}+\alpha\mathbf{1}^m)\cdot\mathbf{1}^n + \nabla_{\mathbf{y}} T_t(\mathbf{x}-\alpha\mathbf{1}^n, \mathbf{y}+\alpha\mathbf{1}^m)\cdot\mathbf{1}^m = -1$. Now differentiate this expression with respect to α and evaluate both expressions at $\alpha = 1$.

we are originally measuring input committal in hours worked and then move to measuring input committal in terms of hours over 5 hours worked, the ratio measure must typically change, and in some instances the measure will not even be well defined. To see this clearly, consider the case where x^0 was originally 5 hours worked, and the translation of the origin described above takes place. The new index is not well-defined.

In fact, one of the most common empirical problems that occurs in applying ratio-based indexes is what to do with zero observations, as ratio-based indexes are typically not well defined at the origin. Difference based indexes typically will be well-defined at the origin precisely because they are invariant to changes in the origin.

Index numbers have almost exclusively been calculated using the ratio approach. All the traditionally familiar indexes (Laspeyres, Paasche, and Fisher's ideal) are ratio-based measures. Moreover, these more traditional indexes were all calculated using a "test" or axiomatic approach to index-number construction. That is, tests reflecting reasonable properties that an index should possess were specified, and indexes meeting these tests were then derived. A more recent development has been the derivation of indexes using what Samuelson and Swamy (1974) have referred to as the economic approach to index numbers. In the economic approach to index numbers, indexes are constructed from primal representations of preferences and technology under the presumption that individual agents are economic optimizers. Following the original work of Konüs (1939) and Malmquist (1953), virtually all of these indexes have been calculated using a ratio approach. Here the basic idea is relatively simple: Take a radial measure of the technology, a distance function, and then express input and output indexes in terms of ratios of these measures of the technology. Because radial functional representations of technology are positively linearly homogeneous, these economic measures are invariant to the units in which quantities are calculated.

Perhaps the most influential works in this area have been the papers by Caves, Christensen, and Diewert (1982a, 1982b) which show that the Törnqvist approximation to the Divisia index is an exact index that can be derived by taking geometric averages of Malmquist indexes for transcendental logarithmic input and output distance functions. Because the transcendental-logarithmic function is second-order flexible, these results imply that the Törnqvist index is 'superlative' in Diewert's (1976) sense.

The only studies, to my knowledge, which have pursued economic indexes of input and outputs using the difference approach are Diewert (1992), Diewert (1993) and Chambers (1996). Diewert (1993) briefly discusses Bennet's (1920) index as a possible way of accounting for changes in inputs or outputs but does not develop any such indexes in detail. Although raised in a different context, the early works of Bennet (1920) on measuring the cost of living and Bowley (1928) on welfare evaluation are direct predecessors of this difference approach. In fact, Chambers (1996a) shows that the Bennet-Bowley index is an exact measure of Allais' (1943) disposable surplus when the consumer's benefit function is quadratic.

I now briefly survey the results by Chambers (1996) on input and output measurement. As a starting point, it is convenient to define two measures of profitability. The *cost-based measure* for input prices $\mathbf{w}$ and input levels $\mathbf{x}^1$ and $\mathbf{x}^0$ is:

$$C\left(\mathbf{w};\mathbf{x}^1,\mathbf{x}^0\right)=\mathbf{w}\cdot\left(\mathbf{x}^1-\mathbf{x}^0\right).$$

The *revenue-based measure* for output prices $\mathbf{p}$ and output bundles $\mathbf{y}^1$ and $\mathbf{y}^0$ is:

$$R\left(\mathbf{p};\mathbf{y}^1,\mathbf{y}^0\right)=\mathbf{p}\cdot(\mathbf{y}^1-\mathbf{y}^0).$$

Depending upon where input prices are evaluated, $C\left(\mathbf{w};\mathbf{x}^1,\mathbf{x}^0\right)$ and $R\left(\mathbf{p};\mathbf{y}^1,\mathbf{y}^0\right)$are the analogues in difference form of the Laspeyres or Paasche input and output indexes.

The *Bennet-Bowley cost-based measure* is the average of the Laspeyres and Paasche cost-based measures:

$$BC(\mathbf{w}^1,\mathbf{w}^0;\mathbf{x}^1,\mathbf{x}^0)=\frac{1}{2}\left(C\left(\mathbf{w}^1;\mathbf{x}^1,\mathbf{x}^0\right)+C\left(\mathbf{w}^0;\mathbf{x}^1,\mathbf{x}^0\right)\right).$$

The *Bennet-Bowley revenue-based measure* is:

$$BR(\mathbf{p}^1,\mathbf{p}^0;\mathbf{y}^1,\mathbf{y}^0)=\frac{1}{2}\left(R\left(\mathbf{p}^1;\mathbf{y}^1,\mathbf{y}^0\right)+R\left(\mathbf{p}^0;\mathbf{y}^1,\mathbf{y}^0\right)\right).$$

The Bennet-Bowley measures, of course, are the difference analogues of the appropriate Fisher ideal indexes. Notice, however, that they also have the attractive intuitive property that they can be interpreted as cost differences, revenue differences, and profit differences evaluated at average prices (e.g., the Bennet-Bowley cost-based measure measures the difference in costliness of the two input bundles $\mathbf{x}^1$ and $\mathbf{x}^0$ at average input prices$\frac{1}{2}\left(\mathbf{w}^0+\mathbf{w}^1\right)$).

Chambers (1996) defines the *1-technology Luenberger input indicator* for $\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^1\right)$by:

$$X^1\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^1\right)=T_i^1\left(\mathbf{x}^0,\mathbf{y}^1\right)-T_i^1\left(\mathbf{x}^1,\mathbf{y}^1\right),$$

and the 0-technology Luenberger input indicator for $\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^1\right)$ by:

$$X^0\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^0\right)=T_i^0\left(\mathbf{x}^0,\mathbf{y}^0\right)-T_i^0\left(\mathbf{x}^1,\mathbf{y}^0\right).$$

Figure 2 illustrates $X^1\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^1\right)$ in the case where firms operate inefficiently as the difference between the amounts that one can translate $\mathbf{x}^0$ and $\mathbf{x}^1$in the direction of the bisector and still keep both input bundles in the input set for technology 1. In the case illustrated, $X^1(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1) > 0$, suggesting that $\mathbf{x}^0$ is *larger* than $\mathbf{x}^1$when the difference in the input bundles is measured relative to the ability to produce $\mathbf{y}^1$using technology 1. On the other hand, $X^0(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0)$ compares the two input bundles' ability to produce $\mathbf{y}^0$ relative to technology 0. It would be desirable to have an indicator that is invariant to the technology chosen to make the comparison. A natural compromise is to take the average of these two indicators.

The *Luenberger input indicator,* denoted $X(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1)$, is defined:

$$X\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = \frac{1}{2}\left(X^1\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1\right) + X^0\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0\right)\right).$$

An obvious consequence of these definitions and D.1 (the translation property) is

Theorem 1 *(Chambers, 1996)*$X^k\left(\mathbf{x}^0 - \alpha\mathbf{1}^n, \mathbf{x}^1 - \alpha\mathbf{1}^n, \mathbf{y}^k\right) = X^k\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^k\right)$, $k = 0, 1$.

Corollary 2 *(Chambers, 1996)* $X(\mathbf{x}^0 - \alpha\mathbf{1}^n, \mathbf{x}^1 - \alpha\mathbf{1}^n, \mathbf{y}^1, \mathbf{y}^0) = X(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1)$.

Put in words, the theorem and the corollary say that all the Luenberger input indicators are *translation invariant in inputs.* This should be contrasted directly with Malmquist input indexes' homogeneity of degree zero in inputs. In the case of Malmquist indexes, zero degree homogeneity emerges from the linear homogeneity of input distance functions in inputs. Here, translation invariance follows from D.1. Effectively, it makes the difference between input aggregates independent of the choice of the origin. Chambers (1996) main result on Luenberger input indicators is:

Theorem 3 *(Chambers, 1996) If the firm minimizes cost, the input-translation function is quadratic with $\alpha^0_{ij} = \alpha^1_{ij}$ for all i and j, then:*

$$X\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = -BC(\bar{\mathbf{w}}^1, \bar{\mathbf{w}}^0; \mathbf{x}^1, \mathbf{x}^0).$$

where $\bar{\mathbf{w}}^k = \frac{\mathbf{w}^k}{\mathbf{w}^k \cdot \mathbf{1}^n}$.

This theorem is important because it implies that the Bennet-Bowley cost-based measure is an exact indicator for a second-order flexible technology. (Notice, as Caves, Christensen, and Diewert (1982) point out, that the restriction $\alpha^0_{ij} = \alpha^1_{ij}$ restricts the flexibility of the technology.) Hence, the Bennet-Bowley cost-based measure might be thought of as a superlative indicator of input differences.

Parallel to the definition of the input indicators, Chambers (1996) defines the 1-*technology Luenberger output indicator* for $(\mathbf{x}^1, \mathbf{y}^1, \mathbf{y}^0)$by:

$$Y^1\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^1\right) = T^1_o\left(\mathbf{x}^1, \mathbf{y}^0\right) - T^1_o\left(\mathbf{x}^1, \mathbf{y}^1\right),$$

and the 0-technology Luenberger output indicator for $(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0)$ by:

$$Y^0\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0\right) = T_o^0\left(\mathbf{x}^0, \mathbf{y}^0\right) - T_o^0\left(\mathbf{x}^0, \mathbf{y}^1\right).$$

$Y^k\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^k\right)$ thus measures the difference between the amounts $\mathbf{y}^0$ and $\mathbf{y}^1$ can be projected in the direction of the bisector and still keep both of them in the $\mathbf{x}^k$ output set for technology k . The *Luenberger output indicator* is the average of $Y^1(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^1)$ and $Y^0(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0)$:

$$Y\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0, \mathbf{x}^1\right) = \frac{1}{2}\left(Y^1\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^1\right) + Y^0\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0\right)\right).$$

An obvious consequence of these definitions and D.1 is

Theorem 4 *(Chambers, 1996)*$Y^k\left(\mathbf{y}^0 + \alpha\mathbf{1}^m, \mathbf{y}^1 + \alpha\mathbf{1}^m, \mathbf{x}^k\right) = Y^k\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^k\right), k = 0, 1.$

Corollary 5 *(Chambers, 1996)*$Y\left(\mathbf{y}^0 + \alpha\mathbf{1}^m, \mathbf{y}^1 + \alpha\mathbf{1}^m, \mathbf{x}^0, \mathbf{x}^1\right) = Y\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0, \mathbf{x}^1\right).$

Chambers (1996) also shows that the Bennet-Bowley revenue based measure is an exact measure (and thus superlative) of the Luenberger output indicator under appropriate assumptions on the technology:

Theorem 6 *(Chambers, 1996)If the firm maximizes revenue, the output translation function is quadratic with $\beta_{ij}^0 = \beta_{ij}^1$ for all i and j, then*

$$Y\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0, \mathbf{x}^1\right) = BR(\bar{\mathbf{p}}^1, \bar{\mathbf{p}}^0; \mathbf{y}^1, \mathbf{y}^0),$$

where $\bar{\mathbf{p}}^k = \frac{\mathbf{p}^k}{\mathbf{p}^k \cdot \mathbf{1}^m}$.

4 Exponential Input and Output Indicators

Where Chambers (1996) defines indicators in terms of input and output translation function, here I want to define input and output indicators in terms of the logarithmic-transcendental technology translation function. To that end, I define the *1-technology exponential input indicator* for $(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1)$by:

$$EX^1\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1\right) = \exp T_t^1\left(\mathbf{x}^0, \mathbf{y}^1\right) - \exp T_t^1\left(\mathbf{x}^1, \mathbf{y}^1\right),$$

and the 0-technology exponential input indicator for $(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1)$ by:

$$EX^0\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0\right) = \exp T_t^0\left(\mathbf{x}^0, \mathbf{y}^0\right) - \exp T_t^0\left(\mathbf{x}^1, \mathbf{y}^0\right).$$

The primary difference between the exponential input indicators introduced here and the indicators studied in Chambers (1996) is that these indicators are specified in terms of differences of exponentials of technology translation functions, while those in Chambers (1996) are specified in terms of differences of input translation functions. However, as with the Luenberger indicators defined by Chambers (1996), both compare the ability of the two input bundles to produce different output bundles relative to a different technology.

The *Luenberger exponential input indicator* is the simple average of the 1-technology and 0-technology exponential input indicators, i.e.,

$$EX\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^0,\mathbf{y}^1\right)=\frac{1}{2}\left(EX^1\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^1\right)+EX^0\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^0\right)\right).$$

An obvious consequence of these definitions and D.1 (the translation property) is

Theorem 7 $EX^k\left(\mathbf{x}^0-\alpha\mathbf{1}^n,\mathbf{x}^1-\alpha\mathbf{1}^n,\mathbf{y}^k+\alpha\mathbf{1}^m\right)=\exp\left(-\alpha\right)EX^k\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^k\right),$ $k=0,1,\alpha\in\Re.$

Corollary 8 $EX\left(\mathbf{x}^0-\alpha\mathbf{1}^n,\mathbf{x}^1-\alpha\mathbf{1}^n,\mathbf{y}^0+\alpha\mathbf{1}^m,\mathbf{y}^1+\alpha\mathbf{1}^m\right)=\exp\left(-\alpha\right)EX\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^0,\mathbf{y}^1\right),$ $\alpha\in\Re.$

I define the *1-technology exponential output indicator* for $(\mathbf{x}^1,\mathbf{y}^1,\mathbf{y}^0)$by:

$$EY^1\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^1\right)=\exp T_t^1\left(\mathbf{x}^1,\mathbf{y}^0\right)-\exp T_t^1\left(\mathbf{x}^1,\mathbf{y}^1\right),$$

and the 0-technology exponential output indicator for $(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0)$ by:

$$EY^0\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0\right)=\exp T_t^0\left(\mathbf{x}^0,\mathbf{y}^0\right)-\exp T_t^0\left(\mathbf{x}^0,\mathbf{y}^1\right).$$

$EY^k\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^k\right)$thus measures the difference between the amounts $\mathbf{y}^0$and $\mathbf{y}^1$can be projected in the direction of the bisector and still keep both of them in the $\mathbf{x}^k$ output set for technology k . The *Luenberger exponential output indicator* is the average of $EY^1\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^1\right)$ and $EY^0\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0\right)$:

$$EY\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1\right)=\frac{1}{2}\left(EY^1\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^1\right)+EY^0\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0\right)\right)$$

An obvious consequence of these definitions and D.1 is

Theorem 9 $EY^k\left(\mathbf{y}^0+\alpha\mathbf{1}^m,\mathbf{y}^1+\alpha\mathbf{1}^m,\mathbf{x}^k-\alpha\mathbf{1}^n\right)=\exp(-\alpha)EY^k\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^k\right),$ $k=0,1.$

Corollary 10 $EY\left(\mathbf{y}^0+\alpha\mathbf{1}^m,\mathbf{y}^1+\alpha\mathbf{1}^m,\mathbf{x}^0-\alpha\mathbf{1}^n,\mathbf{x}^1-\alpha\mathbf{1}^n\right)=\exp(-\alpha)EY\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1\right).$

5 Exact Measures of the Luenberger Exponential Input and Output Indicators

This section derives exact measures of the exponential input and output indicators introduced in the previous section that can be calculated without econometric estimation. My first result is:

Theorem 11 *If the technology translation function is logarithmic-transcendental with $a_{ij}^0 = a_{ij}^1$, for all i and j, and firms maximize profit, then:*

$$EX\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^0,\mathbf{y}^1\right) = \sum_{k=1}^{n}\left(\exp\left(\frac{-x_k^0}{2}\right)\bar{w}_k^0 + \exp\left(\frac{-x_k^1}{2}\right)\bar{w}_k^1\right)\left(\exp\left(\frac{x_k^0}{2}\right) - \exp\left(\frac{x_k^1}{2}\right)\right),$$

where $\bar{w}_k^h = \frac{w_k^h}{\mathbf{w}^h\cdot\mathbf{1}^n+\mathbf{p}^h\cdot\mathbf{1}^m}$, $h = 0, 1$.

Proof *By Diewert's (1976) quadratic identity:*

$$\exp T_t^1\left(\mathbf{x}^0,\mathbf{y}^1\right) - \exp T_t^1\left(\mathbf{x}^1,\mathbf{y}^1\right) = \frac{1}{2}\left[\nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^1\left(\mathbf{x}^0,\mathbf{y}^1\right) + \nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^1\left(\mathbf{x}^1,\mathbf{y}^1\right)\right] \cdot\left[\exp\left(\frac{\mathbf{x}^0}{2}\right) - \exp\left(\frac{\mathbf{x}^1}{2}\right)\right].$$

Similarly,

$$\exp T_t^0\left(\mathbf{x}^0,\mathbf{y}^0\right) - \exp T_t^0\left(\mathbf{x}^1,\mathbf{y}^0\right) = \frac{1}{2}\left[\nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^0\left(\mathbf{x}^0,\mathbf{y}^0\right) + \nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^0\left(\mathbf{x}^1,\mathbf{y}^0\right)\right] \cdot\left[\exp\left(\frac{\mathbf{x}^0}{2}\right) - \exp\left(\frac{\mathbf{x}^1}{2}\right)\right]$$

Adding these two equalities gives:

$$\left[\nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^0\left(\mathbf{x}^0,\mathbf{y}^0\right) + \nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^1\left(\mathbf{x}^1,\mathbf{y}^1\right)\right]\cdot\left[\exp\left(\frac{\mathbf{x}^0}{2}\right) - \exp\left(\frac{\mathbf{x}^1}{2}\right)\right] +$$
$$\frac{1}{2}\begin{bmatrix}\nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^1\left(\mathbf{x}^0,\mathbf{y}^1\right) + \nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^0\left(\mathbf{x}^1,\mathbf{y}^0\right)\\ -\nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^0\left(\mathbf{x}^0,\mathbf{y}^0\right) - \nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^1\left(\mathbf{x}^1,\mathbf{y}^1\right)\end{bmatrix}\cdot\left[\exp\left(\frac{\mathbf{x}^0}{2}\right) - \exp\left(\frac{\mathbf{x}^1}{2}\right)\right]$$

Under the assumption that $a_{ij}^0 = a_{ij}^1$ for all i and j the second term in this expression is zero. Hence,

$$EX\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^0,\mathbf{y}^1\right) = \frac{1}{2}\left[\nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^0\left(\mathbf{x}^0,\mathbf{y}^0\right) + \nabla_{\exp\left(\frac{\mathbf{x}}{2}\right)}\exp T_t^1\left(\mathbf{x}^1,\mathbf{y}^1\right)\right] \cdot\left[\exp\left(\frac{\mathbf{x}^0}{2}\right) - \exp\left(\frac{\mathbf{x}^1}{2}\right)\right] \qquad (4)$$

Differentiation establishes that:

$$\exp T_t(\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{x}}T_t(\mathbf{x},\mathbf{y})=\frac{1}{2}\exp(\frac{\mathbf{x}}{2})\circ\nabla_{\exp(\frac{\mathbf{x}}{2})}\exp T_t(\mathbf{x},\mathbf{y})$$

where $\mathbf{z}\circ\mathbf{v}$ *denotes the vector consisting of the component-wise products of the vectors* $\mathbf{z}$ *and* $\mathbf{v}$. *When the firm maximizes profit, this last expression upon using (3) reduces to:*

$$\exp(-\frac{\mathbf{x}}{2})\circ\bar{\mathbf{w}}=\frac{1}{2}\nabla_{\exp(\frac{\mathbf{x}}{2})}\exp T_t(\mathbf{x},\mathbf{y}).$$

This last expression when substituted into (4) establishes the theorem.

A parallel argument establishes:

Theorem 12 *If the technology translation function is logarithmic-transcendental with* $b^0_{ij}=b^1_{ij}$ *for all* i *and* j *and firms maximize profit, then:*

$$EY\left(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1\right)=\sum_{k=1}^{m}\left[\exp\left(\frac{y^0_k}{2}\right)\bar{p}^0_k+\exp\left(\frac{y^1_k}{2}\right)\bar{p}^1_k\right]\left[\exp\left(\frac{-y^0_k}{2}\right)-\exp\left(\frac{-y^1_k}{2}\right)\right].$$

where $\bar{p}^h_k=\frac{p^h_k}{\mathbf{p}^h\cdot\mathbf{1}^m+\mathbf{w}^h\cdot\mathbf{1}^n}$, $h=0,1$.

Proof *By Diewert's (1976) identity:*

$$\exp T^1_t\left(\mathbf{x}^1,\mathbf{y}^0\right)-\exp T^1_t\left(\mathbf{x}^1,\mathbf{y}^1\right)=\frac{1}{2}\left[\nabla_{\exp(\frac{-\mathbf{y}}{2})}\exp T^1_t\left(\mathbf{x}^1,\mathbf{y}^0\right)+\nabla_{\exp(\frac{-\mathbf{y}}{2})}\exp T^1_t\left(\mathbf{x}^1,\mathbf{y}^1\right)\right]\cdot\left[\exp\left(\frac{-\mathbf{y}^0}{2}\right)-\exp\left(\frac{-\mathbf{y}^1}{2}\right)\right].$$

Similarly,

$$\exp T^0_t\left(\mathbf{x}^0,\mathbf{y}^0\right)-\exp T^0_t\left(\mathbf{x}^0,\mathbf{y}^1\right)=\frac{1}{2}\left[\nabla_{\exp(\frac{-\mathbf{y}}{2})}\exp T^0_t\left(\mathbf{x}^0,\mathbf{y}^0\right)+\nabla_{\exp(\frac{-\mathbf{y}}{2})}\exp T^0_t\left(\mathbf{x}^0,\mathbf{y}^1\right)\right]\cdot\left[\exp\left(\frac{-\mathbf{y}^0}{2}\right)-\exp\left(\frac{-\mathbf{y}^1}{2}\right)\right].$$

Adding these two expressions together and rearranging under the assumption that $b^0_{ij}=b^1_{ij}$ *gives:*

$$\left[\nabla_{\exp(\frac{-\mathbf{y}}{2})}\exp T^0_t\left(\mathbf{x}^0,\mathbf{y}^0\right)+\nabla_{\exp(\frac{-\mathbf{y}}{2})}\exp T^1_t\left(\mathbf{x}^1,\mathbf{y}^1\right)\right]\cdot\left[\exp\left(\frac{-\mathbf{y}^0}{2}\right)-\exp\left(\frac{-\mathbf{y}^1}{2}\right)\right]$$

Differentiation establishes:

$$\exp T_t(\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{y}}T_t(\mathbf{x},\mathbf{y})=-\frac{1}{2}\exp\left(-\frac{\mathbf{y}}{2}\right)\circ\nabla_{\exp(\frac{-\mathbf{y}}{2})}\exp T_t(\mathbf{x},\mathbf{y}).$$

Together with (3), this last result establishes for a profit maximizing firm that:

$$\frac{1}{2}\nabla_{\exp\left(\frac{-\mathbf{y}}{2}\right)} \exp T_t(\mathbf{x}, \mathbf{y}) = \bar{\mathbf{p}} \circ \exp\left(\frac{\mathbf{y}}{2}\right).$$

The result follows immediately.

6 Transitivity of Input and Output Indicators

So far, all the indicators studied only make bilateral comparisons either across firms or time. More generally, however, one will want to make multilateral comparisons. For example, one potential application of the input and output indicators is for the construction of a time series of an aggregate input from multiple time series of single inputs. A property that is usually deemed desirable in the construction of multilateral indexes is Frisch circularity. Here the analogue of Frisch circularity corresponds to what Blackorby and Donaldson (1980) have referred to as *additive circularity.* A multilateral indicator $G(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k)$ satisfies additive circularity if:

$$G(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k) = G(\mathbf{x}^h, \mathbf{x}^j, \mathbf{y}^h, \mathbf{y}^j) + G(\mathbf{x}^j, \mathbf{x}^k, \mathbf{y}^j, \mathbf{y}^k)$$

for all $h, j,$ and k. From this definition, it is apparent why additive circularity might be a desirable property for an indicator to possess. Consider using a multilateral indicator $G(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k)$ to construct a series of observations on aggregate inputs across firms. Let the observation on the base firm's (denoted now by a superscript 1) aggregate input be normalized to G^1. Notice, for example, that there are at least two ways to construct the aggregate input for firm 3: First, one can construct it directly by using the formula: $G^3 = G^1 + G(\mathbf{x}^3, \mathbf{x}^1, \mathbf{y}^3, \mathbf{y}^1)$. Or, one could construct it more indirectly by first constructing G^2, and then computing $G^2 + G(\mathbf{x}^3, \mathbf{x}^2, \mathbf{y}^3, \mathbf{y}^2)$. Unless the indicator satisfies additive circularity, there is no reason for the result of both computations to be the same.

The reason, of course, that this happens is that the choice of a base observation here is essentially arbitrary. Generally speaking, when looking at comparisons across firms (or across countries for that matter), there is no natural way of ranking firms. However, in some applications, there is a natural ordering, and when there is, additive circularity becomes a less compelling property to possess. A clear example of this is in the construction of a time series of aggregate inputs. In that case, the ordering is clear, and one can usefully construct meaningful aggregates by making successive bilateral comparisons from the base period.[3]

When additive circularity is important it comes at a severe cost in terms of restricting the technologies to which it implies. My first result of this section is that:

[3] In the literature on time-series aggregates, this is referred to as chain linking.

Theorem 13 *The Luenberger input indicator satisfies additive circularity for all* $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$ *if and only if:*

$$T_i^h(\mathbf{x}, \mathbf{y}) = a^h(\mathbf{y}) + b(\mathbf{x}),$$

where $b(\mathbf{x}+\alpha\mathbf{1}^n) = b(\mathbf{x}) + \alpha$.

Proof *By Lemma 1 in Blackorby and Donaldson (1980), the Luenberger input indicator satisfies additive circularity if and only if it can be expressed as:*

$$X\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = v\left(\mathbf{x}^0, \mathbf{y}^0\right) - v\left(\mathbf{x}^1, \mathbf{y}^1\right)$$

where $v\left(\mathbf{x}^0+\alpha\mathbf{1}^n, \mathbf{y}^0\right) = v\left(\mathbf{x}^0, \mathbf{y}^0\right) + k\alpha$. *Substituting into the definition of the Luenberger input indicator and setting* $\mathbf{x}^1 = \mathbf{0}^n$ *and* $\mathbf{y}^1 = \mathbf{0}^m$ *gives:*

$$T_i^0\left(\mathbf{x}^0, \mathbf{y}^0\right) = T_i^0\left(\mathbf{0}^n, \mathbf{y}^0\right) + 2v\left(\mathbf{x}^0, \mathbf{y}^0\right) - 2v\left(\mathbf{0}^n, \mathbf{0}^m\right) + T_i^1\left(\mathbf{x}^0, \mathbf{0}^m\right) - T_i^1\left(\mathbf{0}^n, \mathbf{0}^m\right).$$

Performing a similar operation for $X\left(\mathbf{x}^0, \mathbf{x}^h, \mathbf{y}^0, \mathbf{y}^h\right)$ *establishes that:*

$$T_i^1\left(\mathbf{x}^0, \mathbf{0}^m\right) - T_i^1\left(\mathbf{0}^n, \mathbf{0}^m\right) = T_i^h\left(\mathbf{x}^0, \mathbf{0}^m\right) - T_i^h\left(\mathbf{0}^n, \mathbf{0}^m\right),$$

so that we can rewrite the above in the obvious renormalization:

$$T_i^0\left(\mathbf{x}^0, \mathbf{y}^0\right) = t_i^0\left(\mathbf{y}^0\right) + m\left(\mathbf{x}^0, \mathbf{y}^0\right).$$

Because this same argument can be applied for arbitrary h *and* k *it follows immediately that:*

$$T_i^h(\mathbf{x}, \mathbf{y}) = t_i^h(\mathbf{y}) + m(\mathbf{x}, \mathbf{y}). \tag{5}$$

where $m(\mathbf{x}, \mathbf{y})$ *must satisfy D.1 in* $\mathbf{x}$. *Substituting this result into the additive circularity condition and simplifying yields:*

$$-m\left(\mathbf{x}^1, \mathbf{y}^0\right) + m\left(\mathbf{x}^0, \mathbf{y}^1\right) - m\left(\mathbf{x}^2, \mathbf{y}^1\right) + m\left(\mathbf{x}^1, \mathbf{y}^2\right) = -m\left(\mathbf{x}^2, \mathbf{y}^0\right) + m\left(\mathbf{x}^0, \mathbf{y}^2\right).$$

Now set $\mathbf{x}^1 = \mathbf{x}^2 = \mathbf{0}^n, \mathbf{y}^1 = \mathbf{y}^0 = \mathbf{0}^m$ *to obtain:*

$$m\left(\mathbf{x}^0, \mathbf{0}^m\right) - m\left(\mathbf{0}^n, \mathbf{0}^m\right) + m\left(\mathbf{0}^n, \mathbf{y}^2\right) = m\left(\mathbf{x}^0, \mathbf{y}^2\right)$$

so that:

$$m(\mathbf{x}, \mathbf{y}) = b(\mathbf{x}) + n(\mathbf{y})$$

in an obvious renormalization. This result along with (5) establishes necessity. Sufficiency is obvious.

Corollary 14 If the Luenberger input indicator satisfies additive circularity for all $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$, then it is independent of output.

Hence, transitivity in the form of additive circularity places severe restrictions on the classes of technology which will permit one to construct meaningful Luenberger input indicators: All firms must possess a technology whose input translation function is additively separable in inputs and outputs, and where only the function dealing with outputs can be specific to firms. Some intuitive insight into the form that this technology assumes can be had by noticing that applying D.1 to the form in the theorem yields:

$$T_i^h(\mathbf{x}, \mathbf{y}) = a^h(\mathbf{y}) + b(\mathbf{x}) = b(\mathbf{x}+a^h(\mathbf{y}) \cdot \mathbf{1}^n).$$

So transitive Luenberger input indicators are only available if the technology differences across firms can be summarized by a common input-translation function that is independent of the level of output, $b(\mathbf{x})$, and actual differences emerge as a result of a firm-specific translation of inputs along the unit vector, where the magnitude of the translation depends upon the output choice. In a sense, the differences across firms are restricted to changing the efficiency with which a given vector of inputs is utilized. Figure 3 illustrates: The base technology (h=1) is represented by the input set characterized by the isoquant labelled V^1. The technology for h=2 is found by translating every element of the base isoquant by $a^2(\mathbf{y}) - a^1(\mathbf{y})$along the unit vector.

Turning to the quadratic input translation function, it follows immediately that:

Corollary 15 *If the input-translation function is quadratic, it satisfies additive circularity for all $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$ if and only if $a_i^h = a_i^k$ for all h and k and $i = 1, 2, ..., n, \alpha_{ij}^h = \alpha_{ij}^k$ for all h and k and $i, j = 1, 2, ..., n$, and $\gamma_{ik}^h = 0$ for all h, i, k.*

An exactly parallel argument establishes:

Theorem 16 *The Luenberger output indicator satisfies additive circularity for all $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$ if and only if:*

$$T_o^h(\mathbf{x}, \mathbf{y}) = a(\mathbf{y}) + b^h(\mathbf{x}),$$

where $a(\mathbf{y}+\beta\mathbf{1}^m) = a(\mathbf{y}) - \beta$ for all h.

Corollary 17 If the Luenberger output indicator satisfies additive circularity for all $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$, it is independent of inputs.

Corollary 18 *If the output-translation function is quadratic, it satisfies additive circularity for all* $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$ *if and only if* $b^h_k = b^j_k$ *for all* h *and* j, $k = 1, 2, ..m, \beta^h_{kl} = \beta^j_{kl}$ *for all* h *and* j, $k, l = 1, 2, ..., m$, *and* $\gamma^h_{ik} = 0$ *for all* h, i, k.

So for Luenberger output indicators to be transitive it must be true that the technology can be described as though there exists a single reference output set common across firms with the only effect that inputs have on production to be in terms of translating outputs, by a firm-specific amount that depends on the input mix, along the unit vector.

Similar results apply for the Luenberger exponential indicators:

Theorem 19 *The Luenberger exponential input indicator satisfies additive circularity for all* $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$ *if and only if:*

$$\exp T^h_t(\mathbf{x}, \mathbf{y}) = a^h(\mathbf{y}) + b(\mathbf{x}),$$

where $a^h(\mathbf{y}+\beta \mathbf{1}^m) + b(\mathbf{x}-\beta \mathbf{1}^n) = \exp(-\beta) \left(a^h(\mathbf{y}) + b(\mathbf{x}) \right)$ *for all* h.

Corollary 20 *If the Luenberger exponential input indicator satisfies additive circularity for all* $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$, *it is independent of output.*

Corollary 21 *If the technology translation function is logarithmic-transcendental, it satisfies additive circularity of the Luenberger exponential input indicator if and only if:* $a^h_{ij} = a^t_{ij}$ *all* h *and* t, $i, j = 1, 2, ..., n$, *and* $c^h_{kj} = 0$ *for all* h.

Theorem 22 *The Luenberger exponential output indicator satisfies additive circularity for all* $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$ *if and only if:*

$$\exp T^h_t(\mathbf{x}, \mathbf{y}) = a(\mathbf{y}) + b^h(\mathbf{x}),$$

where $a(\mathbf{y}+\beta \mathbf{1}^m) + b^h(\mathbf{x}-\beta \mathbf{1}^n) = \exp(-\beta) \left(a(\mathbf{y}) + b^h(\mathbf{x}) \right)$ *for all* h.

Corollary 23 *If the Luenberger exponential output indicator satisfies additive circularity for all* $\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+$ *it is independent of inputs.*

Corollary 24 *If the technology translation function is logarithmic-transcendental, it satisfies additive circularity of the Luenberger exponential output indicator if and only if:* $b^h_{ij} = v^t_{ij}$ *all* h *and* t, $i, j = 1, 2, ..., m$, *and* $c^h_{kj} = 0$ *for all* h.

The primary implication of the preceding theorems and their corollaries is that it is highly unlikely that any technology will satisfy the restrictions required for additive circularity of the bilateral indicators to hold. Hence, using the bilateral approach that we have developed so far will not be sufficient to ensure the existence of multilateral input and output indicators that satisfy this form of additive transitivity. However, by suitably redefining the indicators previously developed, it is possible to generate both multilateral input and output indicators that satisfy additive circularity. Following Caves, Christensen, and Diewert (1982a, 1982b), we consider two separate approaches: In terms of constructing input indicators, the first approach is to take an arbitrary observation on inputs and outputs, call it $(\mathbf{x}^*, \mathbf{y}^*)$, and to construct all bilateral input indicators relative to this observation, i.e., create as the case warrants either $X\left(\mathbf{x}^h, \mathbf{x}^*, \mathbf{y}^h, \mathbf{y}^*\right)$ or $EX\left(\mathbf{x}^h, \mathbf{x}^*, \mathbf{y}^h, \mathbf{y}^*\right)$. Once these indicators relative to a common base are constructed, new indicators expressed relative to this common base can be defined as:

$$\vec{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right) = X\left(\mathbf{x}^h, \mathbf{x}^*, \mathbf{y}^h, \mathbf{y}^*\right) - X\left(\mathbf{x}^k, \mathbf{x}^*, \mathbf{y}^k, \mathbf{y}^*\right),$$

and

$$E\vec{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right) = EX\left(\mathbf{x}^h, \mathbf{x}^*, \mathbf{y}^h, \mathbf{y}^*\right) - EX\left(\mathbf{x}^k, \mathbf{x}^*, \mathbf{y}^k, \mathbf{y}^*\right).$$

It is easy to verify that these new indicators are, in fact, transitive and satisfy the additive circularity property.

Another alternative is to make all input comparisons relative to the average technology by taking the average of either $X\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right)$ or $EX\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right)$, as appropriate, over all possible k, i.e., define new indicators:

$$\bar{X}(\mathbf{x}^h, \mathbf{y}^h) = \frac{1}{N}\sum_{k=1}^{N} X\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right),$$

and

$$E\bar{X}(\mathbf{x}^h, \mathbf{y}^h) = \frac{1}{N}\sum_{i=1}^{N} EX\left(\mathbf{x}^h, \mathbf{x}^i, \mathbf{y}^h, \mathbf{y}^i\right).$$

$\bar{X}(\mathbf{x}^h, \mathbf{y}^h)$ and $E\bar{X}(\mathbf{x}^h, \mathbf{y}^h)$ can be thought of as indicators of input usage for firm h relative to the average of input usage by the firms considered. Once these average indicators are constructed new bilateral indicators can then be derived as the difference between these average indicators. That is, as

$$\hat{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right) = \bar{X}(\mathbf{x}^h, \mathbf{y}^h) - \bar{X}(\mathbf{x}^k, \mathbf{y}^k),$$

and

$$E\hat{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right) = E\bar{X}(\mathbf{x}^h, \mathbf{y}^h) - E\bar{X}(\mathbf{x}^k, \mathbf{y}^k).$$

Both $\hat{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right)$ and $E\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right)$ satisfy additive circularity, and are thus transitive.

We thus have,

Theorem 25 *If the firm minimizes cost, the input-translation function is quadratic with $\alpha_{ij}^h = \alpha_{ij}^k$ for all h and k, then:*

$$\vec{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right) = BC(\bar{\mathbf{w}}^k, \bar{\mathbf{w}}^*; \mathbf{x}^k, \mathbf{x}^*) - BC(\bar{\mathbf{w}}^*, \bar{\mathbf{w}}^h; \mathbf{x}^*, \mathbf{x}^h),$$

and

$$\hat{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right) = -\frac{1}{N}\sum_{i=1}^{N}\left(BC(\bar{\mathbf{w}}^i, \bar{\mathbf{w}}^h; \mathbf{x}^i, \mathbf{x}^h) - BC(\bar{\mathbf{w}}^i, \bar{\mathbf{w}}^k; \mathbf{x}^i, \mathbf{x}^k)\right)$$

where $\bar{\mathbf{w}}^k = \frac{\mathbf{w}^k}{\mathbf{w}^k \cdot \mathbf{1}^n}$.

Similarly,

Theorem 26 *If the technology translation function is logarithmic-transcendental with $a_{ij}^h = a_{ij}^k$ for all h and k, and firms maximize profit, then:*

$$\begin{aligned} E\vec{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right) = & \sum_{k=1}^{n}\left(\exp\left(\frac{-x_k^h}{2}\right)\bar{w}_k^h + \exp\left(\frac{-x_k^*}{2}\right)\bar{w}_k^*\right)\left(\exp\left(\frac{x_k^h}{2}\right) - \exp\left(\frac{x_k^*}{2}\right)\right) \\ & + \sum_{k=1}^{n}\left(\exp\left(\frac{-x_k^*}{2}\right)\bar{w}_k^* + \exp\left(\frac{-x_k^k}{2}\right)\bar{w}_k^k\right)\left(\exp\left(\frac{x_k^*}{2}\right) - \exp\left(\frac{x_k^k}{2}\right)\right), \end{aligned}$$

and

$$\begin{aligned} E\hat{X}\left(\mathbf{x}^h, \mathbf{x}^k, \mathbf{y}^h, \mathbf{y}^k\right) = & \frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{n}\left(\exp\left(\frac{-x_k^h}{2}\right)\bar{w}_k^h + \exp\left(\frac{-x_k^i}{2}\right)\bar{w}_k^i\right)\left(\exp\left(\frac{x_k^h}{2}\right) - \exp\left(\frac{x_k^i}{2}\right)\right) \\ & -\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{n}\left(\exp\left(\frac{-x_k^k}{2}\right)\bar{w}_k^k + \exp\left(\frac{-x_k^i}{2}\right)\bar{w}_k^i\right)\left(\exp\left(\frac{x_k^k}{2}\right) - \exp\left(\frac{x_k^i}{2}\right)\right) \end{aligned}$$

where $\bar{w}_k^h = \frac{w_k^h}{\mathbf{w}^h \cdot \mathbf{1}^n + \mathbf{p}^h \cdot \mathbf{1}^m}$.

Construction of output indicators satisfying additive circularity follows a similar procedure: Either define a common base against which all outputs are compared, or use the average output indicator as the common base. Therefore, we have:

$$\vec{Y}\left(\mathbf{y}^h, \mathbf{y}^k, \mathbf{x}^h, \mathbf{x}^k\right) = Y\left(\mathbf{y}^h, \mathbf{y}^*, \mathbf{x}^h, \mathbf{x}^*\right) - Y\left(\mathbf{y}^k, \mathbf{y}^*, \mathbf{x}^k, \mathbf{x}^*\right),$$

$$E\vec{Y}\left(\mathbf{y}^h, \mathbf{y}^k, \mathbf{x}^h, \mathbf{x}^k\right) = EY\left(\mathbf{y}^h, \mathbf{y}^*, \mathbf{x}^h, \mathbf{x}^*\right) - EY\left(\mathbf{y}^k, \mathbf{y}^*, \mathbf{x}^k, \mathbf{x}^*\right),$$

$$\hat{Y}\left(\mathbf{y}^h, \mathbf{y}^k, \mathbf{x}^h, \mathbf{x}^k\right) = \frac{1}{N}\sum_{j=1}^{N} Y\left(\mathbf{y}^h, \mathbf{y}^j, \mathbf{x}^h, \mathbf{x}^j\right) - \frac{1}{N}\sum_{j=1}^{N} Y\left(\mathbf{y}^k, \mathbf{y}^j, \mathbf{x}^k, \mathbf{x}^j\right),$$

and

$$E\hat{Y}\left(\mathbf{y}^h,\mathbf{y}^k,\mathbf{x}^h,\mathbf{x}^k\right)=\frac{1}{N}\sum_{j=1}^{N}EY\left(\mathbf{y}^h,\mathbf{y}^j,\mathbf{x}^h,\mathbf{x}^j\right)-\frac{1}{N}\sum_{j=1}^{N}EY\left(\mathbf{y}^k,\mathbf{y}^j,\mathbf{x}^k,\mathbf{x}^j\right).$$

Each of these new indicators, which we shall refer to as *Luenberger multilateral output indicators and Luenberger exponential multilateral output indicators*, respectively, satisfy additive circularity, and we obtain as before:

Theorem 27 *If firms maximize revenue, the output translation function is quadratic with $\beta_{ij}^m=\beta_{ij}^n$ for all m and n, then*

$$\vec{Y}\left(\mathbf{y}^h,\mathbf{y}^k,\mathbf{x}^h,\mathbf{x}^k\right)=BR(\bar{\mathbf{p}}^*,\bar{\mathbf{p}}^h;\mathbf{y}^h,\mathbf{y}^*)-BR(\bar{\mathbf{p}}^*,\bar{\mathbf{p}}^k;\mathbf{y}^k,\mathbf{y}^*),$$

and

$$\hat{Y}\left(\mathbf{y}^h,\mathbf{y}^k,\mathbf{x}^h,\mathbf{x}^k\right)=\frac{1}{N}\sum_{j=1}^{N}BR(\bar{\mathbf{p}}^h,\bar{\mathbf{p}}^j;\mathbf{y}^j,\mathbf{y}^h)-\frac{1}{N}\sum_{j=1}^{N}BR(\bar{\mathbf{p}}^j,\bar{\mathbf{p}}^k;\mathbf{y}^j,\mathbf{y}^k),$$

where $\bar{\mathbf{p}}^k=\frac{\mathbf{p}^k}{\mathbf{p}^k\cdot\mathbf{1}^m}$.

Theorem 28 *If the technology translation function is logarithmic-transcendental with $b_{ij}^m=b_{ij}^n$ for all m and n, and firms maximize profit, then:*

$$\begin{aligned}E\vec{Y}\left(\mathbf{y}^h,\mathbf{y}^k,\mathbf{x}^h,\mathbf{x}^k\right)=&\sum_{k=1}^{m}\left[\exp\left(\frac{y_k^h}{2}\right)\bar{p}_k^h+\exp\left(\frac{y_k^*}{2}\right)\bar{p}_k^*\right]\left[\exp\left(\frac{-y_k^h}{2}\right)-\exp\left(\frac{-y_k^*}{2}\right)\right]\\&-.\sum_{k=1}^{m}\left[\exp\left(\frac{y_k^k}{2}\right)\bar{p}_k^1+\exp\left(\frac{y_k^*}{2}\right)\bar{p}_k^*\right]\left[\exp\left(\frac{-y_k^k}{2}\right)-\exp\left(\frac{-y_k^*}{2}\right)\right],\end{aligned}$$

and

$$\begin{aligned}E\vec{Y}\left(\mathbf{y}^h,\mathbf{y}^k,\mathbf{x}^h,\mathbf{x}^k\right)=&\sum_{j=1}^{N}\sum_{k=1}^{m}\left[\exp\left(\frac{y_k^h}{2}\right)\bar{p}_k^h+\exp\left(\frac{y_k^j}{2}\right)\bar{p}_k^i\right]\left[\exp\left(\frac{-y_k^h}{2}\right)-\exp\left(\frac{-y_k^j}{2}\right)\right]\\&-.\sum_{j=1}^{N}\sum_{k=1}^{m}\left[\exp\left(\frac{y_k^k}{2}\right)\bar{p}_k^k+\exp\left(\frac{y_k^j}{2}\right)\bar{p}_k^j\right]\left[\exp\left(\frac{-y_k^k}{2}\right)-\exp\left(\frac{-y_k^j}{2}\right)\right]\end{aligned}$$

where $\bar{p}_k^h=\frac{p_k^h}{\mathbf{p}^h\cdot\mathbf{1}^m+\mathbf{w}^h\cdot\mathbf{1}^n}$.

These last theorems show that it is possible to define multilateral Luenberger indicators which satisfy the additive circularity property, and for which there exist superlative measures which can be calculated without the need for econometric estimation.

7 Conclusion

This paper has studied the construction of new input and output indicators along the lines suggested by Chambers (1996). Bilateral indicators analogous to those developed in Chambers (1996) have been defined and shown to be calculable under suitable behavioral assumptions directly from observable market data. However, these bilateral Luenberger input and output indicators will only satisfy additive circularity under extreme restrictions on the technology. Consequently, multilateral Luenberger indicators, which satisfy additive circularity and which can be calculated directly from the bilateral indicators, have been defined and shown to be calculable using only data on market prices and quantities.

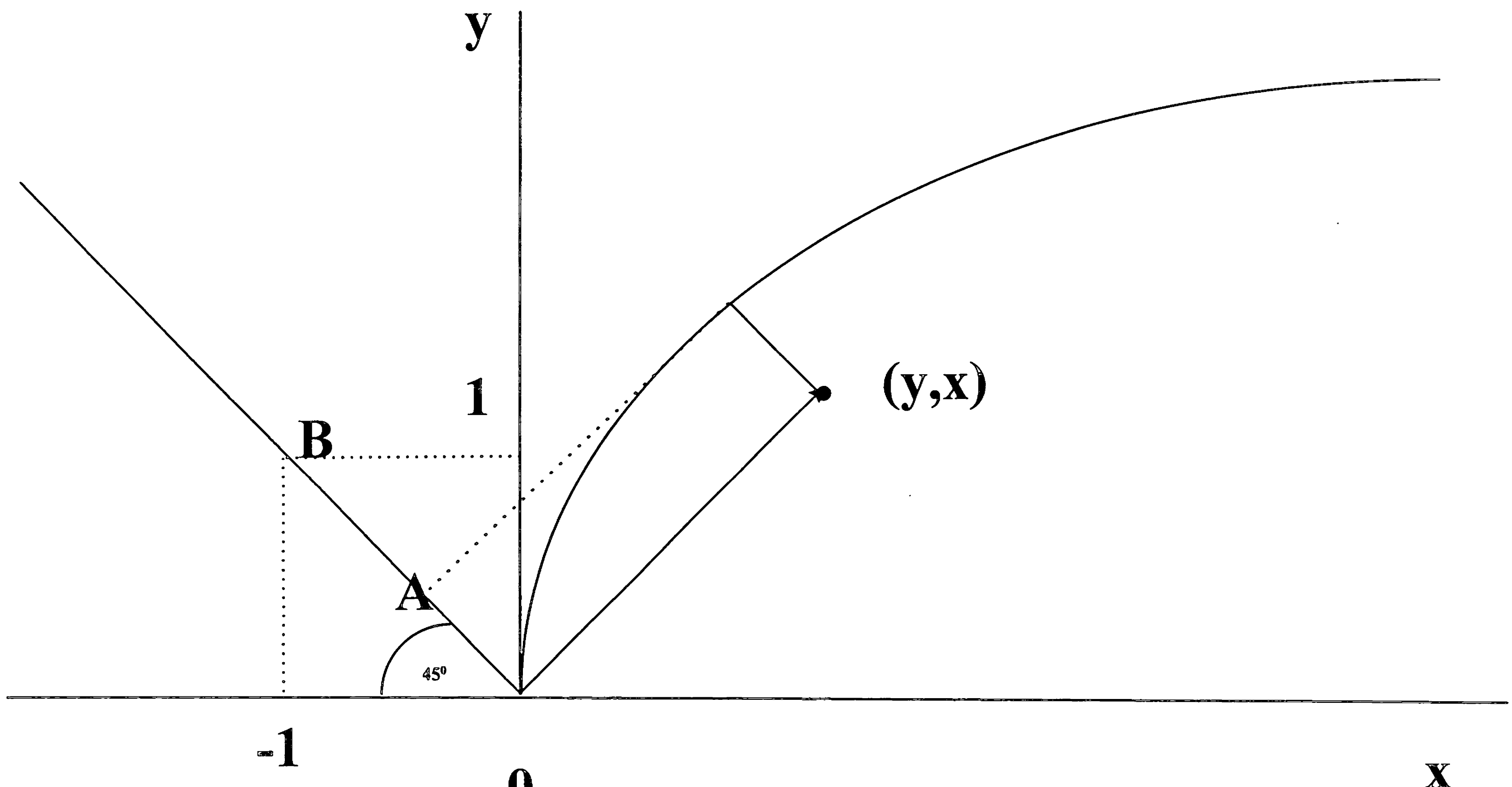

Figure 1: The Directional Technology Distance Function

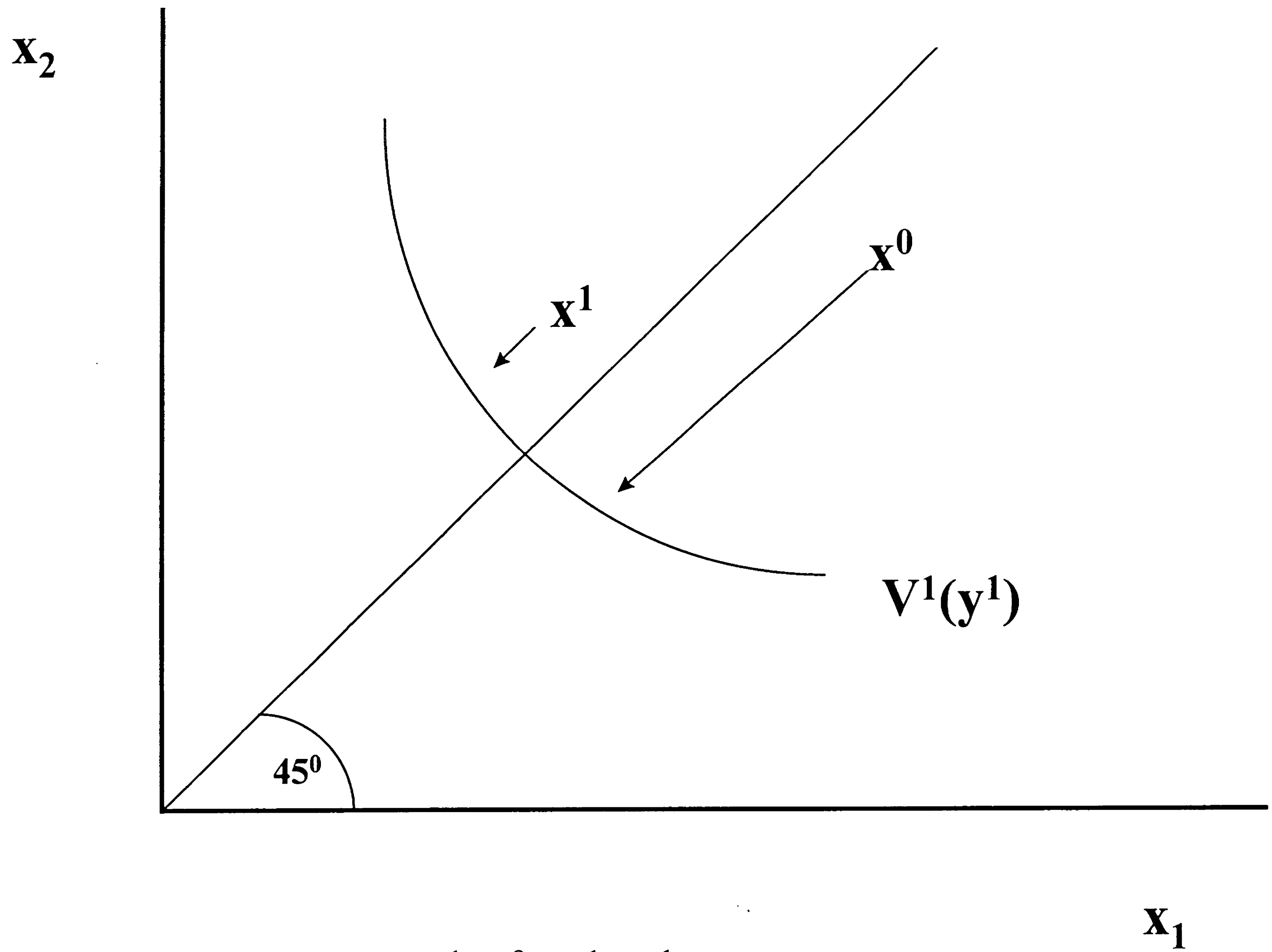

Figure 2: $X^1(x^0, x^1, y^1)$

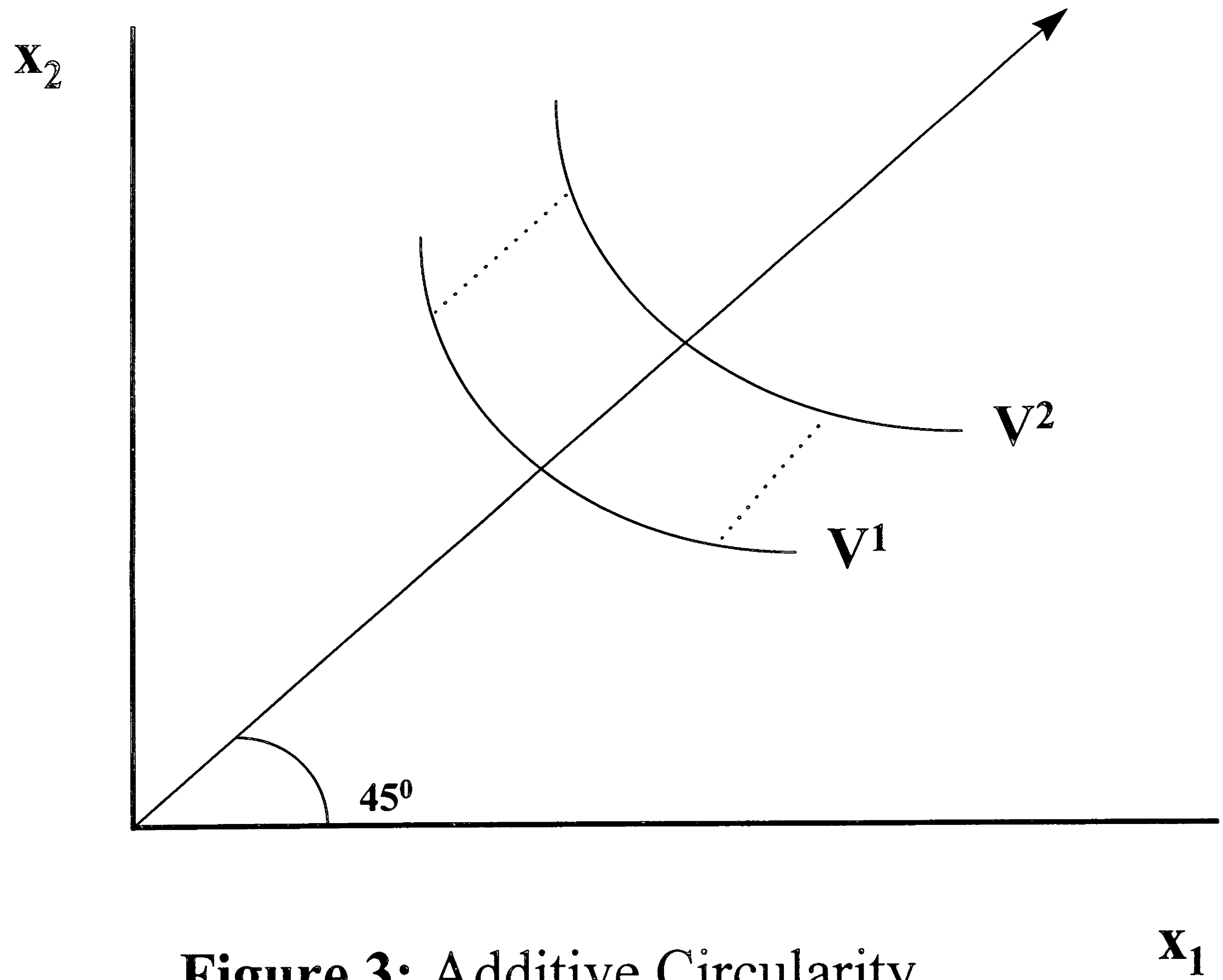

Figure 3: Additive Circularity

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