A New Look at Exact Input, Output, Productivity, and Technical Change Measurement

by

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WP 96-05
Revised October 1996

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Scientific Article No. _______, Contribution No. _______, from the Maryland Agricultural Experiment Station

Abstract

Superlative input, output, technical change, and productivity measures are derived for quadratic approximations to the directional technology distance function. Input measures can be computed as cost differences using appropriately normalized prices. Output measures can be computed as revenue differences, while technical change and productivity measures can be computed as profit differences.

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1. Introduction

Most economists would probably agree that the best indicator of a firm's performance is some measure of its profitability, be it the cost of producing a fixed bundle of outputs, the revenue generated from a fixed bundle of inputs, or its profit. If true, then changes in profitability would be natural measures of changes in the firm's performance. And, in particular, profitability differences should be a good way to compare inputs, outputs, and productivity over firms or over time.

This paper examines cost, revenue, and profit's role in measuring inputs, outputs, and productivity. It shows that profit differences (not ratios) calculated using appropriately normalized prices can be exact measures of input use, production, technical change and productivity. In particular, measures originally suggested by Bennet (1920) and Bowley (1928) in the context of cost of living and welfare measurement provide exact indexes when the underlying technology is suitably quadratic. The Bennet-Bowley measures correspond to cost, revenue, and profit differences calculated using normalized average prices.

The approach taken in this paper departs from earlier work on indexes by relying on a version of Luenberger's (1992b, 1995) shortage function, the *directional technology distance function*, to characterize technology. The directional technology distance function enables one to construct new representations of differences in input use, output production, and productivity that I refer to as *Luenberger indicators.* These Luenberger indicators are novel because they are based on a translation (not radial) representation of the technology and, thus, are all specified in difference (not ratio) form. Hence, their basic normalization property is that they are translation invariant (i.e., are invariant to the choice of the origin). It is these Luenberger indicators for which the Bennet-Bowley measures are exact when the directional technology distance function is quadratic in inputs and outputs. And because the quadratic is a flexible functional form, the Bennet-Bowley measures are thus superlative in Diewert's (1976) sense.

In what follows, I first introduce my notation and terminology. Then to make my ideas concrete, I consider the measurement of various types of *translation-neutral technical change* using directional distance functions. I show that as long as the appropriate directional distance function is quadratic, appropriately normalized Bennet-Bowley measures of profit differences provide exact measures of input- and output-translation neutral technical change, respectively. In the following sections, I consider, successively: input indicators, output indicators, and productivity measurement. Following Diewert (1976) and Caves, Christensen, and Diewert (1982a, 1982b), I then show that appropriate Bennet-Bowley measures are exact for the Luenberger input indicator, the Luenberger output indicator, and the Luenberger productivity indicator provided that the directional technology distance function is quadratic.

2. Notation, Assumptions, and Definitions

Let $\mathbf{x} \in \Re^n_+$ denote a vector of inputs and $\mathbf{y} \in \Re^m_+$denote a vector of outputs. Superscripts on input and output vectors are typically used to differentiate vectors either across time or across firms. (Exceptions are $\mathbf{0}^k$ and $\mathbf{1}^k$ which denote the k vectors of zeroes and ones, respectively.) For example, $\mathbf{x}^h$ will be interpreted variously as firm h's input use or as input use in period h. The technology is defined in terms of a set $T \subset \Re^n_+ \times \Re^m_+$:

$$T = \{(\mathbf{x} \in \Re^n_+, \mathbf{y} \in \Re^m_+) : \mathbf{x} \text{ can produce } \mathbf{y}\}.$$

T satisfies the following properties:

T.1: T is closed;

T.2: Inputs and outputs are freely disposable, i.e., if $(\mathbf{x}', -\mathbf{y}') \geq (\mathbf{x}, -\mathbf{y})$ then $(\mathbf{x}, \mathbf{y}) \in T \Rightarrow (\mathbf{x}', \mathbf{y}') \in T$;

T.3: Doing nothing is feasible, i.e. $(\mathbf{0}^n, \mathbf{0}^m) \in T$.

Related to T are the *input set*, $V(\mathbf{y}) = \{\mathbf{x} : (\mathbf{x}, \mathbf{y}) \in T\}$, and the *output set*, $Y(\mathbf{x}) = \{\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in T\}$.

Slightly modifying Luenberger's (1992, 1995) shortage function and following Chambers, Chung, and Färe (1996), I define the *directional technology distance function* as:

$$\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) = \max\{\beta \in \Re : (\mathbf{x} - \beta\mathbf{g}_x, \mathbf{y} + \beta\mathbf{g}_y) \in T\}, \mathbf{g}_x \in \Re^n_+, \mathbf{g}_y \in \Re^m_+, (\mathbf{g}_x, \mathbf{g}_y) \neq (\mathbf{0}^n, \mathbf{0}^m),$$

if $(\mathbf{x}-\beta\mathbf{g}_x, \mathbf{y}+\beta\mathbf{g}_y) \in T$ for some β and $d_T(\mathbf{y}, \mathbf{g}_y) = \inf\{\delta \in \Re : \mathbf{y}+\delta\mathbf{g}_y \in \Re^m_+\}$ otherwise[1]. Here $(\mathbf{g}_x, \mathbf{g}_y)$ is a reference vector of inputs and outputs. $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)$ represents the maximal translation of the input and output vector in the direction of $(-\mathbf{g}_x, \mathbf{g}_y)$that keeps the translated input and output vector inside T. When $(-\mathbf{g}_x, \mathbf{g}_y) = (-\mathbf{1}^n, \mathbf{1}^m)$, the directional technology distance function, therefore, corresponds to Blackorby and Donaldson's (1980) *translation function* for T. Figure 1 illustrates $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{1}^n, \mathbf{1}^m)$as the ratio OA/OB for the point (x, y). As Chambers, Chung, and Färe (1996) show, all known distance and directional distance functions can be depicted as special cases of $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)$. In particular, the *directional input distance function* defined by Chambers, Chung, and Färe (1995) is

$$\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) = \max\left\{\beta \in \Re : (\mathbf{x}-\beta\mathbf{g}_x, \mathbf{y}) \in T\right\}, \mathbf{g}_x \in \Re^n_+, \mathbf{g}_x \neq \mathbf{0}^n,$$

if $(\mathbf{x}-\beta\mathbf{g}_x, \mathbf{y}) \in T$ for some β and $-\infty$ otherwise[2], and the *directional output distance function* is $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y)$.

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[1] Diewert (1983) defines a closely related concept in the context of measuring waste in a production sector.

[2] Note that because $\mathbf{y}+\delta\mathbf{0}^m \in \Re^m_+$ for all real δ we set $d_T(\mathbf{y}, \mathbf{0}^m) = -\infty$.

The directional technology distance function satisfies[3]:

D.1: $\overrightarrow{D_T}\left(\mathbf{x}-\alpha\mathbf{g}_x, \mathbf{y}+\alpha\mathbf{g}_\mathbf{y}; \mathbf{g}_x, \mathbf{g}_y\right) = \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) - \alpha, \alpha \in \Re$;

D.2: $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)$ is upper semi-continuous in $\mathbf{x}$ and $\mathbf{y}$ (jointly);

D.3: $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \lambda\mathbf{g}_x, \lambda\mathbf{g}_y) = \frac{1}{\lambda}\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y), \lambda > 0$;

D.4: $(\mathbf{x}', -\mathbf{y}') \geq (\mathbf{x}, -\mathbf{y}) \Rightarrow \overrightarrow{D_T}\left(\mathbf{x}', \mathbf{y}'; \mathbf{g}_x, \mathbf{g}_y\right) \geq \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)$, i.e., nondecreasing in inputs and nonincreasing in output;

D.5 $\overrightarrow{D_T}(\mathbf{0}^n, \mathbf{0}^m; \mathbf{g}_x, \mathbf{g}_y) \geq 0$;

D.6: $(\mathbf{x}, \mathbf{y}) \in T \Leftrightarrow \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \geq 0$;

D.7: If T is convex, $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)$ is concave in $(\mathbf{x}, \mathbf{y})$ (jointly);

D.8: If $V(\mathbf{y})$ is convex, $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m)$ is concave in $\mathbf{x}$,

D.9: If $Y(\mathbf{x})$ is convex, $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y)$ is concave in $\mathbf{y}$.

D.6 is particularly important because it implies that the directional technology distance function is a complete function representation of the technology. Firms are said *to operate efficiently if* $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) = 0$. One can establish that:

Lemma 1. *If* $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)$ *satisfies D.1 through D.6, then* $\hat{T} = \{(\mathbf{x}, \mathbf{y}) : \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \geq 0\}$ *satisfies T.1-T.3.*

Denote input prices by $\mathbf{w} \in \Re^n_{++}$[4] and output prices by $\mathbf{p} \in \Re^m_{++}$. By D.6, the

[3]These properties are established in a number of places including Luenberger (1992a, 1992b, 1995, 1996) and Chambers, Chung, and Färe (1995, 1996). Most are straightforward.

[4]$\Re^n_{++}$ denotes the strictly positive n-orthant.

firm's cost minimization problem can be written:

$$\min_{x} \left\{ \mathbf{w} \cdot \mathbf{x} : \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) \geq 0 \right\}$$

if $V(\mathbf{y})$ is nonempty. However, a convenient consequence of D.6 is that the firm's cost minimization problem can be rewritten as the unconstrained minimization problem:

$$\min_{x} \left\{ \mathbf{w} \cdot \left(\mathbf{x} - \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) \, \mathbf{g}_x \right) \right\}. \tag{2.1}$$

so long as $V(\mathbf{y})$ is nonempty. This result is easily demonstrated. Let

$$\mathbf{x}' \in \arg\min_{x} \left\{ \mathbf{w} \cdot \mathbf{x} : \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) \geq 0 \right\}.$$

We first establish that cost minimizers operate efficiently, i.e., $\overrightarrow{D_T}(\mathbf{x}', \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) = 0$. Suppose instead that $\overrightarrow{D_T}(\mathbf{x}', \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) > 0$. If so, there must exist a strictly cheaper input bundle than $\mathbf{x}'$ that is technically feasible thus violating the definition of $\mathbf{x}'$ as cost minimizing. Next note that all technically feasible input bundles satisfy $\mathbf{x}-\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,0^m)\,\mathbf{g}_x \in V(\mathbf{y})$ which, in turn, implies $\mathbf{w}\cdot\mathbf{x}' \leq \mathbf{w}\cdot\left(\mathbf{x}-\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,0^m)\,\mathbf{g}_x\right)$so that the cost minimizing solution provides a lower bound to the minimand in (2.1). But this lower bound is also achieved because $\mathbf{w}\cdot\mathbf{x}' = \mathbf{w}\cdot\left(\mathbf{x}'-\overrightarrow{D_T}(\mathbf{x}',\mathbf{y};\mathbf{g}_x,0^m)\,\mathbf{g}_x\right)$. Similar arguments establish that the revenue maximization and profit maximization problems can be rewritten as the following unconstrained maximization problems:

$$\max_{y}\left\{\mathbf{p}\cdot\left(\mathbf{y}+\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{0}^n,\mathbf{g}_y)\,\mathbf{g}_y\right)\right\} \tag{2.2}$$

and[5]

$$\max_{y,x}\left\{\mathbf{p}\cdot\left(\mathbf{y}+\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)\,\mathbf{g}_y\right)-\mathbf{w}\cdot\left(\mathbf{x}-\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{g}_y)\,\mathbf{g}_x\right)\right\}. \tag{2.3}$$

Assuming that the directional technology distance function is differentiable, the first-order conditions for an interior solution to (2.1) are:

$$\mathbf{w} = \nabla_x\overrightarrow{D_T}(\mathbf{x},\mathbf{y};\mathbf{g}_x,0^m)\,(\mathbf{w}\cdot\mathbf{g}_x) \tag{2.4}$$

[5] See, for example, Luenberger (1992b,1995).

while the first-order conditions for an interior solution to (2.2) are:

$$\mathbf{p} = -\nabla_y \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y)\left(\mathbf{p} \cdot \mathbf{g}_y\right), \tag{2.5}$$

and the first-order conditions for an interior solution to (2.3) are:

$$\begin{aligned} \mathbf{p} &= -\nabla_y \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)\left(\mathbf{p} \cdot \mathbf{g}_y + \mathbf{w} \cdot \mathbf{g}_x\right) \\ \mathbf{w} &= \nabla_x \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y)\left(\mathbf{p} \cdot \mathbf{g}_y + \mathbf{w} \cdot \mathbf{g}_x\right). \end{aligned} \tag{2.6}$$

In equations (2.4) through (2.6) the notation ∇_z denotes the gradient of the function with respect to the vector $\mathbf{z}$.

Properties D.1 to D.6 let us establish that, although they are alternative characterizations of T and both special cases of the directional technology distance function, the directional input distance function and the directional output distance function generally provide different evaluations of a given bundle of inputs and outputs.

Lemma 2. $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) = \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y)$ *for all* $\mathbf{x} \in \Re^n_+$ *and* $\mathbf{y} \in \Re^m_+$ *if and only if* $(\mathbf{x}, \mathbf{y}) \in T \Rightarrow (\mathbf{x}+\gamma\mathbf{g}_x, \mathbf{y}+\gamma\mathbf{g}_y) \in T, \gamma \in \Re, (\mathbf{x}+\gamma\mathbf{g}_x, \mathbf{y}+\gamma\mathbf{g}_y) \geq (\mathbf{0}^n, \mathbf{0}^m)$.

Proof. *Suppose that* $\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) = \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y)$. *Evaluating this expression at* $(\mathbf{x}+\gamma\mathbf{g}_x, \mathbf{y}+\gamma\mathbf{g}_y) \geq (\mathbf{0}^n, \mathbf{0}^m)$ *yields:*

$$\begin{aligned}
\overrightarrow{D_T}(\mathbf{x}+\gamma\mathbf{g}_x, \mathbf{y}+\gamma\mathbf{g}_y; \mathbf{g}_x, \mathbf{0}^m) &= \overrightarrow{D_T}(\mathbf{x}+\gamma\mathbf{g}_x, \mathbf{y}+\gamma\mathbf{g}_y; \mathbf{0}^n, \mathbf{g}_y) \\
&= \overrightarrow{D_T}(\mathbf{x}+\gamma\mathbf{g}_x, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y) - \gamma \\
&= \overrightarrow{D_T}(\mathbf{x}+\gamma\mathbf{g}_x, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) - \gamma \\
&= \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m).
\end{aligned}$$

The second equality follows by D.1, the third by the supposed identity, and the fourth by D.1 again. This establishes necessity. To prove sufficiency, notice that if the required property holds:

$$\begin{aligned}
\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) &= \sup\{\beta : (\mathbf{x}-\beta\mathbf{g}_x, \mathbf{y}) \in T\} \\
&= \sup\{\beta : (\mathbf{x}, \mathbf{y} + \beta\mathbf{g}_\mathbf{y}) \in T\} \\
&= \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y).
\end{aligned}$$

Hence, the directional input and directional output distance functions only provide the same evaluation of an input-output bundle if translating a technically feasible input-output bundle in the direction of $(\mathbf{g}_x, \mathbf{g}_y)$ always stays within T.

I define several different measures of change in profitability. The *cost-based measure* for input prices $\mathbf{w}$ and input levels $\mathbf{x}^1$ and $\mathbf{x}^0$ is:

$$C\left(\mathbf{w}; \mathbf{x}^1, \mathbf{x}^0\right) = \mathbf{w} \cdot \left(\mathbf{x}^1 - \mathbf{x}^0\right).$$

The *revenue-based measure* for output prices $\mathbf{p}$ and output bundles $\mathbf{y}^1$ and $\mathbf{y}^0$ is:

$$R\left(\mathbf{p}; \mathbf{y}^1, \mathbf{y}^0\right) = \mathbf{p} \cdot (\mathbf{y}^1 - \mathbf{y}^0).$$

Depending upon where input prices are evaluated, $C\left(\mathbf{w}; \mathbf{x}^1, \mathbf{x}^0\right)$ and $R\left(\mathbf{p}; \mathbf{y}^1, \mathbf{y}^0\right)$are the analogues in difference form of the Laspeyres or Paasche input and output indexes. The *profit-based measure* is:

$$P\left(\mathbf{p}, \mathbf{w}; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0\right) = R\left(\mathbf{p}; \mathbf{y}^1, \mathbf{y}^0\right) - C\left(\mathbf{w}; \mathbf{x}^1, \mathbf{x}^0\right).$$

The *Bennet-Bowley cost-based measure* is the average of the Laspeyres and Paasche cost-based measures:

$$BC(\mathbf{w}^1, \mathbf{w}^0; \mathbf{x}^1, \mathbf{x}^0) = \frac{1}{2}\left(C\left(\mathbf{w}^1; \mathbf{x}^1, \mathbf{x}^0\right) + C\left(\mathbf{w}^0; \mathbf{x}^1, \mathbf{x}^0\right)\right).$$

The *Bennet-Bowley revenue-based measure* and the *Bennet-Bowley profit-based measure* are defined, respectively, as:

$$BR(\mathbf{p}^1, \mathbf{p}^0; \mathbf{y}^1, \mathbf{y}^0) = \frac{1}{2}\left(R\left(\mathbf{p}^1; \mathbf{y}^1, \mathbf{y}^0\right) + R\left(\mathbf{p}^0; \mathbf{y}^1, \mathbf{y}^0\right)\right),$$

and

$$BP(\mathbf{p}^1, \mathbf{w}^1, \mathbf{p}^0, \mathbf{w}^0; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0) = \frac{1}{2}\left(P\left(\mathbf{p}^1, \mathbf{w}^1; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0\right) + P\left(\mathbf{p}^0, \mathbf{w}^0; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0\right)\right).$$

The Bennet-Bowley measures, of course, are the difference analogues of the appropriate Fisher ideal indexes. Notice, however, that they also have the attractive intuitive property that they can be interpreted as cost differences, revenue differences, and profit differences evaluated at average prices (e.g., $\frac{1}{2}(\mathbf{w}^0 + \mathbf{w}^1)$)[6].

The quadratic directional technology distance function for firm h is:

$$\overrightarrow{D_T^h}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) = a_0^h + \sum_{i=1}^{n} a_i^h x_i + \sum_{k=1}^{m} b_k^h y_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}^h x_i x_j + \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m} \beta_{kl}^h y_k y_l + \sum_{i=1}^{n}\sum_{k=1}^{m} \gamma_{ik}^h x_i y_k,$$

with

$$\begin{aligned} \alpha_{ij}^h &= \alpha_{ji}^h, \beta_{kl}^h = \beta_{lk}^h, \sum_{k=1}^{m} b_k^h g_{yk} - \sum_{i=1}^{n} a_i^h g_{xi} = -1; \\ \sum_{k=1}^{m} \gamma_{ik}^h g_{yk} - \sum_{j=1}^{n} \alpha_{ij}^h g_{xj} &= 0, i = 1, ..., n; \sum_{l=1}^{m} \beta_{kl}^h g_{yl} - \sum_{i=1}^{n} \gamma_{ik}^h g_{xi} = 0, k = 1, ..., m. \end{aligned}$$

This form can be interpreted as a quadratic directional input distance function when $g_{yk} = 0, k = 1, ..., m$ and as a quadratic directional output distance function when $g_{xj} = 0, j = 1, ..., n$. The following lemma will prove useful in later developments:

Lemma 3. *The quadratic directional technology (input, output) distance function can provide a second-order approximation in* $(\mathbf{x}, \mathbf{y})$ *to any twice-continuously*

[6] In a consumer context, the Bennet-Bowley cost measures correspond to Hicks' many-market consumer surplus measure and to Harberger's welfare indicator.

differentiable technology directional technology (input, output) distance function with the same reference vectors.

Proof. *The result is demonstrated for the directional technology distance function. The extension to the directional input and output distance functions is straightforward. By D.1, any directional distance function must satisfy:* $\overrightarrow{D_T}\left(\mathbf{x}-\alpha\mathbf{g}_x, \mathbf{y}+\alpha\mathbf{g}_\mathbf{y}; \mathbf{g}_x, \mathbf{g}_y\right) = \overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) - \alpha$. *Differentiation of this expression with respect to* α *at* $\alpha = 0$ *gives:*

$$\nabla_\mathbf{y}\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \cdot g_\mathbf{y} - \nabla_\mathbf{x}\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \cdot g_\mathbf{x} = -1. \tag{2.7}$$

Differentiation of (2.7) with respect to $\mathbf{x}$ *gives:*

$$\nabla_\mathbf{yx}\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \cdot g_\mathbf{y} - \nabla_\mathbf{xx}\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \cdot g_\mathbf{x} = 0, \tag{2.8}$$

while differentiation with respect to $\mathbf{y}$ *gives:*

$$\nabla_\mathbf{yy}\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \cdot g_\mathbf{y} - \nabla_\mathbf{xy}\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{g}_y) \cdot g_\mathbf{x} = 0. \tag{2.9}$$

Therefore, any twice continuously differentiable distance function must satisfy (2.7)-(2.9)[7]*.Differentiation verifies that the quadratic directional technology distance function satisfies (2.7)-(2.9). To complete the demonstration of the result, we need to show that there exists a set of parameters for the quadratic directional distance function for which the quadratic function value, and first and second-order partial derivatives coincide with those of the original directional distance function at a particular point. Let the point in question be* $(\mathbf{x}^*, \mathbf{y}^*)$. *We start with the Hessians: Set*

$$\alpha_{ij}^{h} = \frac{\partial \overrightarrow{D_T}\left(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y\right)}{\partial x_i \partial x_j}$$

for $j = 1, 2, ..., n-1$ *and all* i,

$$\gamma_{ik}^{h} = \frac{\partial \overrightarrow{D_T}\left(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y\right)}{\partial x_i \partial y_k}$$

[7] *Luenberger (1996) obtains (2.8) and (2.9) by using a duality argument.*

for all i and k,

$$g_{xn}\alpha_{in}^{h} = \sum_{k=1}^{m} \frac{\partial \overrightarrow{D_T}(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y)}{\partial x_i \partial y_k} g_{yk} - \sum_{j=1}^{n-1} \frac{\partial \overrightarrow{D_T}(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y)}{\partial x_i \partial x_j} g_{xj},$$

for all i,

$$\beta_{kl}^{h} = \frac{\partial \overrightarrow{D_T}(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y)}{\partial y_k \partial y_l},$$

for $l = 1, 2, ..., m-1$ and all k, and

$$-\beta_{km}^{h} g_{ym} = \sum_{l=1}^{m-1} \frac{\partial \overrightarrow{D_T}(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y)}{\partial y_k \partial y_l} g_{yl} - \sum_{i=1}^{n} \frac{\partial \overrightarrow{D_T}(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y)}{\partial x_i \partial y_k} g_{xi}$$

for all k. At these parameter values direct computation of the Hessians for the two directional distance functions shows that they coincide and jointly satisfy (2.8) and (2.9). Now using the values of the second-order parameters defined above set

$$b_k^{h} = \frac{\partial \overrightarrow{D_T}(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y)}{\partial y_k} - \sum_{i=1}^{m} \beta_{ki}^{h} y_i^* - \sum_{i=1}^{n} \gamma_{ik}^{h} x_i^*,$$

$k = 1, 2, ..., m,$

$$a_i^h = \frac{\partial \overrightarrow{D_T}\left(\mathbf{x}^*, \mathbf{y}^*; \mathbf{g}_x, \mathbf{g}_y\right)}{\partial x_i} - \sum_{k=1}^{n} \alpha_{ik}^h x_k^* - \sum_{k=1}^{m} \gamma_{ik}^h y_k^*,$$

for $i = 1, 2, ..., n-1$, *and using these parameter values set:*

$$a_n^h g_{xn} = \sum_{k=1}^{m} b_k^h g_{yk} - \sum_{i=1}^{n-1} a_i^h g_{xi} + 1.$$

Direct computation using these parameter values of the gradients of both functions reveals that they are equal and satisfy (2.7). The constant term of the quadratic directional technology distance function, a_0^h, *can now be chosen to insure that the function values coincide.*

3. Profit as an Exact Indicator of Technical Change

I first consider the case where the index h is an indicator of the state of the technology for the same firm. Technical change is *input-translation neutral* if:

$$\overrightarrow{D_T^h}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m) = \overrightarrow{D_T}(\mathbf{x}+A(h)\mathbf{g}_x, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m).$$

Input-translation neutral technical change is illustrated for the case $\mathbf{g}_x = (1, 1)$ in Figure 2 where it is seen to project the frontier of the input set (the isoquant) to the southwest in the direction of $(-1, -1)$. Intuitively, it is quite similar to quasi-input homotheticity in the sense that cost minimizing input ratios for all states of the technology (i.e., for all h) can be found by finding the cost minimizing input ratio for the reference technology ($\overrightarrow{D_T}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m)$) and then projecting that input ratio inwards (for progressive technical change) or outwards (for regressive technical change) along the reference vector. Notice, in particular, that input-translation neutrality generalizes the more familiar concept of Harrod neutral technical change. (If there are two inputs, capital and labor, Harrod neutrality is the special case of input-translation neutrality where $g_x = (0, 1)$.) When technical change is input-translation neutral, the cost function for the $\overrightarrow{D_T^h}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m)$

technology can always be written:

$$c^h(\mathbf{w},\mathbf{y}) = c(\mathbf{w},\mathbf{y}) - A(h)\mathbf{w}\cdot\mathbf{g}_x$$

where $c(\mathbf{w},\mathbf{y})$ is the cost function for the reference technology. Hence, input-translation neutrality can always be interpreted as shifting the intercepts of the cost minimizing derived demands downward.

Technical change is *output-translation neutral* if

$$\overrightarrow{D_T^h}(\mathbf{x},\mathbf{y};\mathbf{0}^n,\mathbf{g}_y) = \overrightarrow{D_T}(\mathbf{x},\mathbf{y}-A(h)\mathbf{g}_y;\mathbf{0}^n,\mathbf{g}_y)$$

which can be visualized (although it is not drawn) as projecting the frontier of the output set to the northeast in the direction of $\mathbf{g}_y$. Intuitively, it is quite similar to the notion of quasi-output homotheticity. Economically, it requires that the intercepts of the revenue maximizing supplies be shifted upwards, as reflected by the fact that the revenue function for the $\overrightarrow{D_T^h}(\mathbf{x},\mathbf{y};\mathbf{0}^n,\mathbf{g}_y)$ technology can be written:

$$r^h(\mathbf{p},\mathbf{x}) = r(\mathbf{p},\mathbf{x}) + A(h)\mathbf{p}\cdot\mathbf{g}_y$$

where $r(\mathbf{p},\mathbf{x})$ is the revenue function for the reference technology. Note that the $A(h)$ functions in the input and output directional distance functions will not, in general, coincide.

Our input-based technical-change indicator is the shift in the directional input distance function that can be associated solely with changes in the index of the state of the technology:

$$t_i(\mathbf{x},\mathbf{y},1,0) = \overrightarrow{D_T^1}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{0}^m) - \overrightarrow{D_T^0}(\mathbf{x},\mathbf{y};\mathbf{g}_x,\mathbf{0}^m)$$

The output-based technical-change indicator is the shift in the output directional distance function associated solely with changes in the index of the state of the technology:

$$t_o(\mathbf{x},\mathbf{y},1,0) = \overrightarrow{D_T^1}(\mathbf{x},\mathbf{y};\mathbf{0}^n,\mathbf{g}_y) - \overrightarrow{D_T^0}(\mathbf{x},\mathbf{y};\mathbf{0}^n,\mathbf{g}_y)$$

So when technical change is either input- or output-translation neutral, D.1 implies that the respective technical-change indicator is independent of the level of input and outputs and is expressible simply as: $A(1) - A(0)$.

I am now ready to state my results on measurement of translation neutral technical change:

Theorem 1. *If the firm maximizes profit, technical change is input-translation neutral, and* $\overrightarrow{D_T^h}(\mathbf{x}, \mathbf{y}; \mathbf{g}_x, \mathbf{0}^m)$ *is quadratic then*

$$t_i(\mathbf{x}, \mathbf{y}, 1, 0) = BP(\bar{\mathbf{p}}^1, \bar{\mathbf{w}}^1, \bar{\mathbf{p}}^0, \bar{\mathbf{w}}^0; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0),$$

with $\bar{\mathbf{p}}^k = \frac{(\mathbf{p}^k)}{\mathbf{w}^k \cdot \mathbf{g}_x}, \bar{\mathbf{w}}^k = \frac{(\mathbf{w}^k)}{\mathbf{w}^k \cdot \mathbf{g}_x}$.

Proof: *The proof follows Diewert (1976). If the technology is input-translation neutral and the firm is a profit maximizer then:*

$$\overrightarrow{D_T^h}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{g}_x, \mathbf{0}^m\right) = \overrightarrow{D_T}\left(\mathbf{x}^1 + A(1)\mathbf{g}_x, \mathbf{y}^1; \mathbf{g}_x, \mathbf{0}^m\right) = A(1) + \overrightarrow{D_T}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{g}_x, \mathbf{0}^m\right) = 0$$

where the second equality follows by D.1 and similarly

$$\overrightarrow{D_T^h}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{g}_x,\mathbf{0}^m\right)=A(0)+\overrightarrow{D_T}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{g}_x,\mathbf{0}^m\right)=0,$$

whence

$$t_i(\mathbf{x},\mathbf{y},1,0)=A(1)-A(0)=\overrightarrow{D_T}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{g}_x,\mathbf{0}^m\right)-\overrightarrow{D_T}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{g}_x,\mathbf{0}^m\right).$$

Applying Diewert's (1976) quadratic lemma to this expression establishes:

$$\begin{aligned} t_i(\mathbf{x},\mathbf{y},1,0) &= \frac{1}{2}\left(\nabla_x\overrightarrow{D_T}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{g}_x,\mathbf{0}^m\right)+\nabla_x\overrightarrow{D_T}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{g}_x,\mathbf{0}^m\right)\right)\cdot\left(\mathbf{x}^0-\mathbf{x}^1\right) \\ &\quad +\frac{1}{2}\left(\nabla_y\overrightarrow{D_T}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{g}_x,\mathbf{0}^m\right)+\nabla_y\overrightarrow{D_T}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{g}_x,\mathbf{0}^m\right)\right)\cdot\left(\mathbf{y}^0-\mathbf{y}^1\right) \end{aligned} \tag{3.1}$$

Using (2.6) in (3.1) with $g_y=0^m$ establishes the theorem.

Having established this result, applying an exactly parallel argument in the case of output-translation neutral technical change leads to an analogous result. Hence, I give it without proof.

Theorem 2. *If the firm maximizes profit, technical change is output-translation neutral, and* $\overrightarrow{D_T^h}(\mathbf{x}, \mathbf{y}; \mathbf{0}^n, \mathbf{g}_y)$ *is quadratic then*

$$t_o(\mathbf{x}, \mathbf{y}, 1, 0) = BP(\vec{\mathbf{p}}^1, \vec{\mathbf{w}}^1, \vec{\mathbf{p}}^0, \vec{\mathbf{w}}^0; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0),$$

with $\vec{\mathbf{p}}^k = \frac{(\mathbf{p}^k)}{\mathbf{p}^k \cdot \mathbf{g}_y}, \vec{\mathbf{w}}^k = \frac{(\mathbf{w}^k)}{\mathbf{p}^k \cdot \mathbf{g}_y}$

These theorems establish that the difference in profitability between $(\mathbf{x}^1, \mathbf{y}^1)$ and $(\mathbf{x}^0, \mathbf{y}^0)$calculated using the average of normalized prices for 1 and 0 is an exact measure of the technical-change indicator if technical change is either input- or output-translation neutral and if the directional distance function is quadratic in inputs and outputs. Hence, the Bennet-Bowley profit measures provide superlative measures of translation-neutral technical change, and rather intuitively, technical change is progressive when $(\mathbf{x}^1, \mathbf{y}^1)$ is more profitable than $(\mathbf{x}^0, \mathbf{y}^0)$. The only difference that emerges in the computation of the technical change indicator in the two cases is in the normalization used in the case of input and output neutrality. (As we shall see later, this difference reflects how translation of the input-output vector in the direction of $(\mathbf{g}_x, \mathbf{g}_y)$ affects production feasibility.) The main restriction embodied in these theorems is that embodied in most superlative indexes of technical change: Firms must operate efficiently and maximize profit. Hence, there is an implicit presumption that the technology does not manifest increasing returns to scale, i.e., T is convex. However, there is no need to assume that inputs or outputs are separable, or that separate aggregators exist for inputs and outputs. Finally, these two theorems should be recognized as the natural extension of Diewert's (1976) results on superlative measures of technical change (especially his equation (3.8)) to the case where the technical change indicator is specified in difference and not ratio form.

4. Luenberger Input Indicators and the Exactness of Cost

The basic idea behind the construction of input-index or output-index numbers is to create a summary measure of inputs or outputs that can be used to assess how these quantities vary over time or over place. In the case of a scalar input, for example, there are at least two natural ways to create an index of input utilization by firm 1 relative to input use by firm 0: The first, and by far more common, is to use the ratio $\frac{\mathbf{x}^1}{\mathbf{x}^0}$. A natural advantage of this approach is that the index is independent of any rescaling of the inputs. A second approach is to use the difference $\mathbf{x}^1 - \mathbf{x}^0$. This has the advantage of being independent of any translations, i.e. changes in origin, of both inputs, but it is not invariant to any rescaling of the inputs (it is however linearly homogeneous in any rescaling of the inputs). And because economists routinely prefer to work in terms of quantities which are unit free, as a rule, most practical indexes have used the ratio approach.[8] And when indexes for multiple inputs and outputs were developed, a natural progression, therefore, was to define these indexes in terms of ratios of linearly homogeneous aggregator functions. In recent years, Malmquist indexes have become very popular because a number of commonly computed indexes including Fisher's ideal index and Törnqvist's index can be rationalized as exact measures of ratios of radial representations of the technology.

Notice, however, that the ratio-based measure is not independent of changes in origin: Suppose for example that one was originally measuring input committal in terms of hours worked and then moved to measuring input committal in terms of hours over 5 hours worked. In this case, the ratio measure must typically change, and in some instances the new ratio measure may not even be well defined. This is most clearly illustrated, for example, by supposing that $\mathbf{x}^0$ was originally 5 hours. The new ratio is not well defined. In fact, one of the most common practical problems with ratio-based indexes is what to do with zero observations, as ratio-based indexes are frequently not well defined in the neighborhood of the

[8] Diewert (1993) is an exception. He briefly considers indexes expressed in difference form.

origin. (For example, Törnqvist's index is not well defined if some input or output quantities have zero values for some observations.)

Although directional distance functions can always be interpreted as radial measures of the technology (relative to a translated origin), they are more naturally thought of as translation measures. Hence, when working with a translation representation of the technology, it seems more natural to specify measures of input use in difference form. In all that follows, I pursue this approach, and following Diewert (1993) I refer to them as *indicators* to distinguish them from the more familiar radial measures.

I define the *1-technology Luenberger input indicator* for $(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1)$by:

$$X^1\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1\right) = \overrightarrow{D_T^1}\left(\mathbf{x}^0, \mathbf{y}^1; \mathbf{g}_x, \mathbf{0}^m\right) - \overrightarrow{D_T^1}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{g}_x, \mathbf{0}^m\right),$$

and the 0-technology Luenberger input indicator for $(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1)$ by:

$$X^0\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0\right) = \overrightarrow{D_T^0}\left(\mathbf{x}^0, \mathbf{y}^0; \mathbf{g}_x, \mathbf{0}^m\right) - \overrightarrow{D_T^0}\left(\mathbf{x}^1, \mathbf{y}^0; \mathbf{g}_x, \mathbf{0}^m\right). \tag{4.1}$$

Figure 3 illustrates $X^1(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1)$for $\mathbf{g}_x = \mathbf{1}^n$ as the difference between the amounts that one can translate $\mathbf{x}^0$ and $\mathbf{x}^1$in the direction of the bisector and still keep both input bundles in the input set for technology 1. In the case illustrated, $X^1(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1) > 0$, suggesting that $\mathbf{x}^0$ is *larger* than $\mathbf{x}^1$.

The *Luenberger input indicator,* denoted $X(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1)$, is the average of $X^1(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1)$ and $X^0(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0)$, i.e.,

$$X\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = \frac{1}{2}\left(X^1\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1\right) + X^0\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0\right)\right).$$

An obvious consequence of these definitions and D.1 (the translation property) is

Theorem 1. $X^k\left(\mathbf{x}^0 - \alpha\mathbf{g}_x, \mathbf{x}^1 - \alpha\mathbf{g}_x, \mathbf{y}^k\right) = X^k\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^k\right), k = 0, 1.$

Corollary 2. $X(\mathbf{x}^0 - \alpha\mathbf{g}_x, \mathbf{x}^1 - \alpha\mathbf{g}_x, \mathbf{y}^1, \mathbf{y}^0) = X(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1).$

Put in words, the theorem and the corollary say that all the input indicators are *translation invariant in inputs.* This should be contrasted directly with Malmquist input indexes' homogeneity of degree zero in inputs. In the case of Malmquist indexes, zero degree homogeneity emerges from the linear homogeneity of input distance functions in inputs. Here, translation invariance follows from D.1 and plays the same role for these indicators as homogeneity plays for Malmquist indexes. With these definitions, I am now ready to state my next result:

Theorem 3. *If the firm minimizes cost, the directional input distance function is quadratic with $\alpha_{ij}^0 = \alpha_{ij}^1$ for all i and j, then:*

$$X\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = -BC(\bar{\mathbf{w}}^1, \bar{\mathbf{w}}^0; \mathbf{x}^1, \mathbf{x}^0).$$

where $\bar{\mathbf{w}}^k = \frac{\mathbf{w}^k}{\mathbf{w}^k \cdot \mathbf{g}_x}$.

Proof. *The proof is inspired by the method of proof in Caves, Christensen, and Diewert (1982a, 1982b). By Diewert's (1976) quadratic lemma:*

$$X^1\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^1\right) = \frac{1}{2}\left(\nabla_x \overrightarrow{D_T^1}\left(\mathbf{x}^0, \mathbf{y}^1; \mathbf{g}_x, \mathbf{0}^m\right) + \nabla_x \overrightarrow{D_T^1}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{g}_x, \mathbf{0}^m\right)\right) \cdot \left(\mathbf{x}^0 - \mathbf{x}^1\right),$$

and

$$X^0\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0\right) = \frac{1}{2}\left(\nabla_x \overrightarrow{D_T^0}\left(\mathbf{x}^0, \mathbf{y}^0; \mathbf{g}_x, \mathbf{0}^m\right) + \nabla_x \overrightarrow{D_T^0}\left(\mathbf{x}^1, \mathbf{y}^0; \mathbf{g}_x, \mathbf{0}^m\right)\right) \cdot \left(\mathbf{x}^0 - \mathbf{x}^1\right).$$

Adding these two expressions together and rearranging using the assump-

tions on the parameters gives:

$$X\left(\mathbf{x}^0,\mathbf{x}^1,\mathbf{y}^0,\mathbf{y}^1\right)=\frac{1}{2}\left(\nabla_x\overrightarrow{D_T^1}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{g}_x,\mathbf{0}^m\right)+\nabla_x\overrightarrow{D_T^0}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{g}_x,\mathbf{0}^m\right)\right)\cdot\left(\mathbf{x}^0-\mathbf{x}^1\right).$$

Applying (2.4) yields the result.

Several comments should be made about this result: Most importantly, Lemma 3 and this theorem imply that the Bennet-Bowley cost measure is a superlative input indicator. Second, the Bennet-Bowley cost measure calculated using normalized input prices is an exact input indicator regardless of whether the technology exhibits constant returns to scale and regardless of whether the entities involved choose outputs optimally. Also, because this cost difference represents the natural analogue of Fisher's ideal index for ratio measures, this result corresponds to the result that Fisher's ideal index is exact for a quantity aggregator that is the square root of a quadratic function. Next, although the cost-based measures themselves are positively linearly homogeneous in prices, the measures here defined have the attractive property of being homogeneous of degree zero in observed prices because they are specified in normalized form (the normalizing factor is the value of the reference input bundle). Because these measures are linearly homogeneous in normalized prices, however, they are not invariant to the choice of the reference vectors. Therefore, for practical application it will usually be important to pick the reference vectors with some care. Obvious candidates for g_x include either $\mathbf{x}^1$ or $\mathbf{x}^0$, or some average of the two. Finally, if the 0 and 1 refer to two distinct firms operating in the same market at the same time, then it is reasonable to presume that they face the same price, and it is immediate that:

Corollary 4. *If the firm minimizes cost, the input-directional distance function is quadratic with $\alpha_{ij}^0 = \alpha_{ij}^1$ for all i and j then:*

$$X\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = -C\left(\bar{\mathbf{w}}^1; \mathbf{x}^1, \mathbf{x}^0\right).$$

These results on input-change indicators also have immediate applications outside of production economics. Suppose that we now take both $\mathbf{y}^0$ and $\mathbf{y}^1$ to be scalar welfare indicators, the inputs to represent commodities consumed, and T to be a preference set instead of a technology set. Then it follows from the standard analogy of producer and consumer theory that our input indicators are also interpretable as commodity indicators without any change in the mathematical analysis. Hence, under the assumptions that we have laid out so far, perfectly legitimate indicators of consumption can be constructed by using the cost, now expenditure, formulae developed above. And, in this context, it is particularly interesting to note the similarity of $BC(\bar{\mathbf{w}}^1, \bar{\mathbf{w}}^0; \mathbf{x}^1, \mathbf{x}^0)$ to Bowley's (1928) utility-change indicator for a quadratic utility function. Although the two indicators are virtually identical in form, there is an important difference[9]: $BC(\bar{\mathbf{w}}^1, \bar{\mathbf{w}}^0; \mathbf{x}^1, \mathbf{x}^0)$ can be computed using only observed data on prices and quantities. Bowley's indicator requires observed data on prices and quantities *plus* the marginal utility of income. So even with the assumption that marginal utility of income is constant, Bowley's measure only provides a measure of welfare change that is accurate up to a factor of proportionality, while our measure would be exact.

5. Luenberger Output Indicators and the Exactness of Revenue

I define the 1-*technology Luenberger output indicator* for $(\mathbf{x}^1, \mathbf{y}^1, \mathbf{y}^0)$by:

$$Y^1\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^1\right) = \overrightarrow{D_T^1}\left(\mathbf{x}^1, \mathbf{y}^0; \mathbf{0}^n, \mathbf{g}_y\right) - \overrightarrow{D_T^1}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{0}^n, \mathbf{g}_y\right),$$

[9]Diewert (1976) obtains Bowley's measure as a corollary to his quadratic lemma which was used in the proof of theorem 1.

and the 0-technology Luenberger output indicator for $(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0)$ by:

$$Y^0\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0\right) = \overrightarrow{D_T^0}\left(\mathbf{x}^0, \mathbf{y}^0; \mathbf{0}^n, \mathbf{g}_y\right) - \overrightarrow{D_T^0}\left(\mathbf{x}^0, \mathbf{y}^1; \mathbf{0}^n, \mathbf{g}_y\right).$$

$Y^k\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^k\right)$ thus measures the difference between the amounts $\mathbf{y}^0$ and $\mathbf{y}^1$ can be projected in the direction of the the reference vector and still keep both of them in the $\mathbf{x}^k$ output set for technology k . The *Luenberger output indicator* is the average of $Y^1(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^1)$ and $Y^0(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0)$:

$$Y\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0, \mathbf{x}^1\right) = \frac{1}{2}\left(Y^1\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^1\right) + Y^0\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0\right)\right).$$

An obvious consequence of these definitions and D.1 is

Theorem 1. $Y^k\left(\mathbf{y}^0 + \alpha\mathbf{g}_y, \mathbf{y}^1 + \alpha\mathbf{g}_y, \mathbf{x}^k\right) = Y^k\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^k\right), k = 0, 1.$

Corollary 2. $Y(\mathbf{y}^0 + \alpha\mathbf{g}_y, \mathbf{y}^1 + \alpha\mathbf{g}_y, \mathbf{x}^0, \mathbf{x}^1) = Y(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0, \mathbf{x}^1).$

The derivation of an exact indicator for $Y(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0, \mathbf{x}^1)$ now follows exactly the same steps used to establish that cost differences were an exact indicator for

$X\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right)$except that $Y^1\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^1\right)$ and $Y^0\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0\right)$and (2.2) and (2.5) are used. Thus,

Theorem 3. *If the firm maximizes revenue, the output directional distance function is quadratic with $\beta_{ij}^0 = \beta_{ij}^1$ for all i and j, then*

$$Y\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0, \mathbf{x}^1\right) = BR(\bar{\mathbf{p}}^1, \bar{\mathbf{p}}^0; \mathbf{y}^1, \mathbf{y}^0),$$

where $\bar{\mathbf{p}}^k = \frac{\mathbf{p}^k}{\mathbf{p}^k \cdot \mathbf{g}_y}$.

Corollary 4. *If the firm maximizes revenue, the output directional distance function is quadratic with $\beta_{ij}^0 = \beta_{ij}^1$ for all i and j and $\mathbf{p}^0 = \mu\ \mathbf{p}^1$ for $\mu > 0$,*

$$Y\left(\mathbf{y}^0, \mathbf{y}^1, \mathbf{x}^0, \mathbf{x}^1\right) = R\left(\bar{\mathbf{p}}^1; \mathbf{y}^1, \mathbf{y}^0\right) = R\left(\bar{\mathbf{p}}^0; \mathbf{y}^1, \mathbf{y}^0\right).$$

As with the results on the input-change indicators, these results apply regardless of whether the technology exhibits constant returns to scale and regardless of whether, in this case, inputs are optimally chosen.

6. Productivity Indicators

One obvious approach to take in constructing a productivity indicator is to follow Caves, Christensen, and Diewert (1982a, 1982b) and define both output-based and input-based indicators using the output- and input-directional distance functions, respectively. Computable formulae for the resulting Luenberger output-based productivity indicator (assuming firms operate efficiently)

$$\frac{1}{2}\left(\overrightarrow{D_T^1}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{0}^n,\mathbf{g}_y\right)-\overrightarrow{D_T^0}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{0}^n,\mathbf{g}_y\right)\right)$$

and the input-based productivity indicator,

$$\frac{1}{2}\left(\overrightarrow{D_T^1}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{g}_x,\mathbf{0}^m\right)-\overrightarrow{D_T^0}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{g}_x,\mathbf{0}^m\right)\right)$$

then correspond to the Bennet-Bowley profit-based measures $BP(\vec{\mathbf{p}}^1,\vec{\mathbf{w}}^1,\vec{\mathbf{p}}^0,\vec{\mathbf{w}}^0;\mathbf{y}^1,\mathbf{y}^0,\mathbf{x}^1,\mathbf{x}^0)$ and $BP(\bar{\mathbf{p}}^1,\bar{\mathbf{w}}^1,\bar{\mathbf{p}}^0,\bar{\mathbf{w}}^0;\mathbf{y}^1,\mathbf{y}^0,\mathbf{x}^1,\mathbf{x}^0)$, respectively, under the assumption that the second-order terms of the respective quadratic directional distance functions are the same across technologies 0 and 1. These measures, which I shall refer to as the *Bennet-Bowley input- and output-based productivity measures*, correspond to the technical-change indicators derived in Theorems 2 and 3.[10] Hence, under the assumption of profit maximization and the appropriate form of neutral technical change, these productivity measures exactly measure technical change.

However, another approach, for which there is no analogue in the Caves, Christensen, and Diewert framework, is pursued in this paper: Construct the productivity indicator directly from the technology directional distance function.

The *technology-1 Luenberger productivity indicator* for $(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1)$ is defined by:

$$L^1\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = \overrightarrow{D_T^1}\left(\mathbf{x}^0, \mathbf{y}^0; \mathbf{g}_x, \mathbf{g}_y\right) - \overrightarrow{D_T^1}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{g}_x, \mathbf{g}_y\right),$$

while the *technology-0 Luenberger productivity indicator* is

$$L^0\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = \overrightarrow{D_T^0}\left(\mathbf{x}^0, \mathbf{y}^0; \mathbf{g}_x, \mathbf{g}_y\right) - \overrightarrow{D_T^0}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{g}_x, \mathbf{g}_y\right).$$

[10]The Bennet-Bowley input- and output-based productivity measures, of course, are quite similar, but they are not identical. Below it is shown that their differences can be characterized in terms of how the technology responds to a translation of the input and outputs along the reference vectors.

And finally the *Luenberger productivity indicator* is the average of $L^1(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1)$ and $L^0(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1)$:

$$L\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = \frac{1}{2}\left(L^1\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) + L^0\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right)\right).$$

Naturally, these indicators are translation invariant.

Theorem 1. $L^k(\mathbf{x}^0 - \alpha\mathbf{g}_x, \mathbf{x}^1 - \alpha\mathbf{g}_x, \mathbf{y}^0 + \alpha\mathbf{g}_y, \mathbf{y}^1 + \alpha\mathbf{g}_y) = L^k(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1), k = 0, 1.$

Corollary 2. $L(\mathbf{x}^0 - \alpha\mathbf{g}_x, \mathbf{x}^1 - \alpha\mathbf{g}_x, \mathbf{y}^0 + \alpha\mathbf{g}_y, \mathbf{y}^1 + \alpha\mathbf{g}_y) = L(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1).$

In Figure 4, $L(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1)$measures the difference between the amount that $(\mathbf{x}^0, \mathbf{y}^0)$ can be projected in the direction of $(-1, 1)$ while keeping it in technology 1 and the amount that $(\mathbf{x}^1, \mathbf{y}^1)$ can be projected in the same direction and keep it in technology 0. The next result is now obvious from previous developments. All that is required is to follow the path established in the calculation of the input indicators and then in the calculation of the output indicators, but now under the presumption of profit maximization.

Theorem 3. *If firms maximize profit, the technology directional distance func-*

tion is quadratic with $\alpha_{ij}^0 = \alpha_{ij}^1$ *for all i and j*, $\beta_{ij}^0 = \beta_{ij}^1$ *for all i and j then*

$$L\left(\mathbf{x}^0, \mathbf{x}^1, \mathbf{y}^0, \mathbf{y}^1\right) = BP(\hat{\mathbf{p}}^1, \hat{\mathbf{w}}^1, \hat{\mathbf{p}}^0, \hat{\mathbf{w}}^0; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0),$$

where $\hat{\mathbf{p}}^k = \frac{\mathbf{p}^k}{\mathbf{p}^k\mathbf{g}_y + \mathbf{w}^k\mathbf{g}_x}$.

Exact and superlative productivity indicators can be computed as Bennet-Bowley measures of profit differences. And, as a moment's reflection will establish, these productivity indicators correspond to the indicators that would emerge if we first computed Luenberger input and output indicators using the technology directional distance function (instead of the input- and output-directional distance functions) while assuming profit maximization and then took the difference between the output indicator and the input indicator.

7. Comparing the Productivity Indicators

Computationally, it is obvious that the three Bennet-Bowley productivity measures differ by the way in which prices are normalized. Thus, generally they will be distinct, and, in fact, by inspection it follows that so long as all prices are strictly positive, the Bennet-Bowley productivity measure derived from the technology directional distance function, $BP(\hat{\mathbf{p}}^1, \hat{\mathbf{w}}^1, \hat{\mathbf{p}}^0, \hat{\mathbf{w}}^0; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0)$, will never equal either $BP(\bar{\mathbf{p}}^1, \bar{\mathbf{w}}^1, \bar{\mathbf{p}}^0, \bar{\mathbf{w}}^0; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0)$ or $BP(\vec{\mathbf{p}}^1, \vec{\mathbf{w}}^1, \vec{\mathbf{p}}^0, \vec{\mathbf{w}}^0; \mathbf{y}^1, \mathbf{y}^0, \mathbf{x}^1, \mathbf{x}^0)$. Before turning to a comparison of the Bennet-Bowley productivity measures, it is first instructive to compare the Luenberger output-based and input-based productivity indicators assuming that firms operate efficiently but without presuming profit maximization.

Theorem 1. *If firms operate efficiently, the input-based productivity indicator,*

$$\frac{1}{2}\left(\overrightarrow{D_T^1}\left(\mathbf{x}^0, \mathbf{y}^0; \mathbf{g}_x, \mathbf{0}^m\right) - \overrightarrow{D_T^0}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{g}_x, \mathbf{0}^m\right)\right)$$

and the output-based productivity indicator,

$$\frac{1}{2}\left(\overrightarrow{D_T^1}\left(\mathbf{x}^0, \mathbf{y}^0; \mathbf{0}^n, \mathbf{g}_y\right) - \overrightarrow{D_T^0}\left(\mathbf{x}^1, \mathbf{y}^1; \mathbf{0}^n, \mathbf{g}_y\right)\right)$$

coincide if and only if for all $(\mathbf{x}^1+\alpha\mathbf{g}_x, \mathbf{y}^1+\alpha\mathbf{g}_y) \geq (\mathbf{0}^n, \mathbf{0}^m)$, $\overrightarrow{D_T^0}(\mathbf{x}^1 + \alpha\mathbf{g}_x, \mathbf{y}^1 + \alpha\mathbf{g}_y; \mathbf{0}^n, \mathbf{g}_y) = \overrightarrow{D_T^0}(\mathbf{x}^1, \mathbf{y}^1; \mathbf{0}^n, \mathbf{g}_y)$, *and for all* $(\mathbf{x}^0+\alpha\mathbf{g}_x, \mathbf{y}^0+\alpha\mathbf{g}_y) \geq (\mathbf{0}^n, \mathbf{0}^m)$, $\overrightarrow{D_T^1}(\mathbf{x}^0 + \alpha\mathbf{g}_x, \mathbf{y}^0 + \alpha\mathbf{g}_y; \mathbf{0}^n, \mathbf{g}_y) = \overrightarrow{D_T^1}(\mathbf{x}^0, \mathbf{y}^0; \mathbf{0}^n, \mathbf{g}_y)$.

Proof *For the indicators to coincide, it must be true that:*

$$
\begin{aligned}
& \frac{1}{2}\left(\overrightarrow{D_T^1}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{0}^n,\mathbf{g}_y\right)-\overrightarrow{D_T^0}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{0}^n,\mathbf{g}_y\right)\right) \\
= \; & \frac{1}{2}\left(\overrightarrow{D_T^1}\left(\mathbf{x}^0,\mathbf{y}^0;\mathbf{g}_x,\mathbf{0}^m\right)-\overrightarrow{D_T^0}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{g}_x,\mathbf{0}^m\right)\right)
\end{aligned}
$$

Set $(\mathbf{x}^0,\mathbf{y}^0)$*equal to the reference vector* $(\bar{\mathbf{x}}^0,\bar{\mathbf{y}}^0)$*so that*

$$
\begin{aligned}
\overrightarrow{D_T^0}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{0}^n,\mathbf{g}_y\right) &= \overrightarrow{D_T^0}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{g}_x,\mathbf{0}^m\right)+\overrightarrow{D_T^1}\left(\bar{\mathbf{x}}^0,\bar{\mathbf{y}}^0;\mathbf{0}^n,\mathbf{g}_y\right)-\overrightarrow{D_T^1}\left(\bar{\mathbf{x}}^0,\bar{\mathbf{y}}^0;\mathbf{g}_x,\mathbf{0}^m\right) \\
&= \overrightarrow{D_T^0}\left(\mathbf{x}^1,\mathbf{y}^1;\mathbf{g}_x,\mathbf{0}^m\right)+z \qquad (7.1)
\end{aligned}
$$

Now consider the set $Z \subset \Re_+^n \times \Re_+^m$*of translations of* $(\mathbf{x}^1,\mathbf{y}^1)$,

$$
Z=\left\{(\hat{\mathbf{x}}\in\Re_+^n,\hat{\mathbf{y}}\in\Re_+^m):(\hat{\mathbf{x}},\hat{\mathbf{y}})=(\mathbf{x}^1+\alpha\mathbf{g}_x,\mathbf{y}^1+\alpha\mathbf{g}_y),\alpha\in\Re\right\}.
$$

Now pick $(\hat{\mathbf{x}}^1,\hat{\mathbf{y}}^1)\in Z$ *and note that* $(\mathbf{x}^1,\mathbf{y}^1)=(\hat{\mathbf{x}}^1-\alpha\mathbf{g}_x,\hat{\mathbf{y}}^1-\alpha\mathbf{g}_y)$ *for some* α*. Substituting into (7.1) gives:*

$$
\begin{aligned}
\overrightarrow{D_T^0}\left(\hat{\mathbf{x}}^1-\alpha\mathbf{g}_x,\hat{\mathbf{y}}^1-\alpha\mathbf{g}_y;\mathbf{0}^n,\mathbf{g}_y\right) &= \overrightarrow{D_T^0}\left(\hat{\mathbf{x}}^1-\alpha\mathbf{g}_x,\hat{\mathbf{y}}^1-\alpha\mathbf{g}_y;\mathbf{g}_x,\mathbf{0}^m\right)+z \\
&= \overrightarrow{D_T^0}\left(\hat{\mathbf{x}}^1,\hat{\mathbf{y}}^1-\alpha\mathbf{g}_y;\mathbf{g}_x,\mathbf{0}^m\right)+z-\alpha \\
&= \overrightarrow{D_T^0}\left(\hat{\mathbf{x}}^1,\hat{\mathbf{y}}^1-\alpha\mathbf{g}_y;\mathbf{0}^n,\mathbf{g}_y\right)+\alpha
\end{aligned}
$$

$$= \overrightarrow{D_T^0}\left(\hat{\mathbf{x}}^1, \hat{\mathbf{y}}^1; \mathbb{0}^n, \mathbf{g}_y\right)$$

The second equality follows by the translation property of input-directional distance functions while the fourth follows by the translation property of output-directional distance functions. This result and an exactly parallel argument for the 1-technology establishes necessity. Sufficiency follows from Lemma 2.

The technologies characterized in the theorem, in a sense, satisfy a translated version of constant returns in the direction of the reference vectors. So this result parallels Caves, Christensen, and Diewert's (1982a) finding that their input-based and output-based productivity indicators coincide in the presence of constant returns to scale. Under profit maximization, this basic result becomes particularly transparent for differentiable technologies. Consider the derivative of the technology directional distance function in the direction $(\mathbf{g}_x, \mathbf{g}_y)$:

$$\begin{aligned}
\eta(\mathbf{x}, \mathbf{y}) &= \frac{\partial}{\partial \alpha} \overrightarrow{D_T}\left(\mathbf{x}+\alpha \mathbf{g}_x, \mathbf{y}+\alpha \mathbf{g}_{\mathbf{y}}; \mathbf{g}_x, \mathbf{g}_y\right) |_{\alpha=0} \\
&= \nabla_x \overrightarrow{D_T} \cdot \mathbf{g}_x + \nabla_y \overrightarrow{D_T} \cdot \mathbf{g}_y.
\end{aligned}$$

If the directional derivative, $\eta(\mathbf{x}, \mathbf{y})$, is positive, we might say that the technology exhibits increasing returns to a reference-vector translation in the sense that outward movement in the direction $(\mathbf{g}_x, \mathbf{g}_y)$places a technically feasible input-output combination farther away from the boundary of T, while movement in the direction $-(\mathbf{g}_x, \mathbf{g}_y)$places a technically feasible input-output combination closer to the boundary of the technology. And if it is negative, we might say that the technology exhibits decreasing returns to a reference-vector translation. Finally, if $\eta(\mathbf{x}, \mathbf{y})$ is zero one might say that the technology exhibits constant returns to a reference-vector translation. Using (2.6) establishes for a profit maximizer that:

$$\eta(\mathbf{x}, \mathbf{y})(\mathbf{p}\cdot\mathbf{g_y} + \mathbf{w}\cdot\mathbf{g_x}) = \mathbf{w}\cdot\mathbf{g_x} - \mathbf{p}\cdot\mathbf{g_y},$$

from which it follows immediately that for a profit maximizer,

Theorem 2. *If firms maximize profits, the Bennet-Bowley input- and output-based productivity measures coincide if* $\eta(\mathbf{x}^1, \mathbf{y}^1) = \eta(\mathbf{x}^0, \mathbf{y}^0) = 0$.

8. Conclusion

I have shown that appropriate Bennet-Bowley measures offer exact indicators of technical change, output production, input usage, and comparative productivity for a quadratic representation of the technology. Because the quadratic is second-order flexible. these Bennet-Bowley indicators, thus, are all superlative. Perhaps the most attractive thing about these results is that they are simple and economically intuitive. Consider, for example, the productivity indicator: $(\mathbf{x}^1, \mathbf{y}^1)$ is judged to have higher productivity than $(\mathbf{x}^0, \mathbf{y}^0)$ if it is more profitable using an average of the normalized prices. So an input-output bundle is judged more productive if it profit dominates another. Similarly, an input bundle is judged to be larger than another if, on average, it is costlier, and an output bundle is larger if it yields, on average, higher revenue.

Up until the work Luenberger (1992b), almost all function valued representations of technology had been radial measures and not translation measures. Lacking the notion of a directional distance function, the type of indicators that I have derived are not obvious. However, having the notion of a directional technology distance function, these indicators become relatively transparent upon pursuing the path set by the earlier work of Diewert (1976) and Caves, Christensen, and

Diewert (1982a, 1982b). These indicators are not interpretable as indexes in their usual sense. For example, if one wanted to construct a time series of aggregate input use using traditional input indexes, one would start by picking a base period and developing later observations multiplicatively. Here, one would start by finding a base period and then constructing later time periods by addition or subtraction. This is a distinctly different way of looking at things. Perhaps the most obvious mathematical reflection of this fact is that our indicators are not homogeneous of degree zero in inputs and outputs as Malmquist and related indexes are. Instead they are translation invariant: a property which they inherit from the basic properties of directional technology distance functions. However different they may be, their very simplicity suggests they will prove extremely useful in practical productivity comparisons.

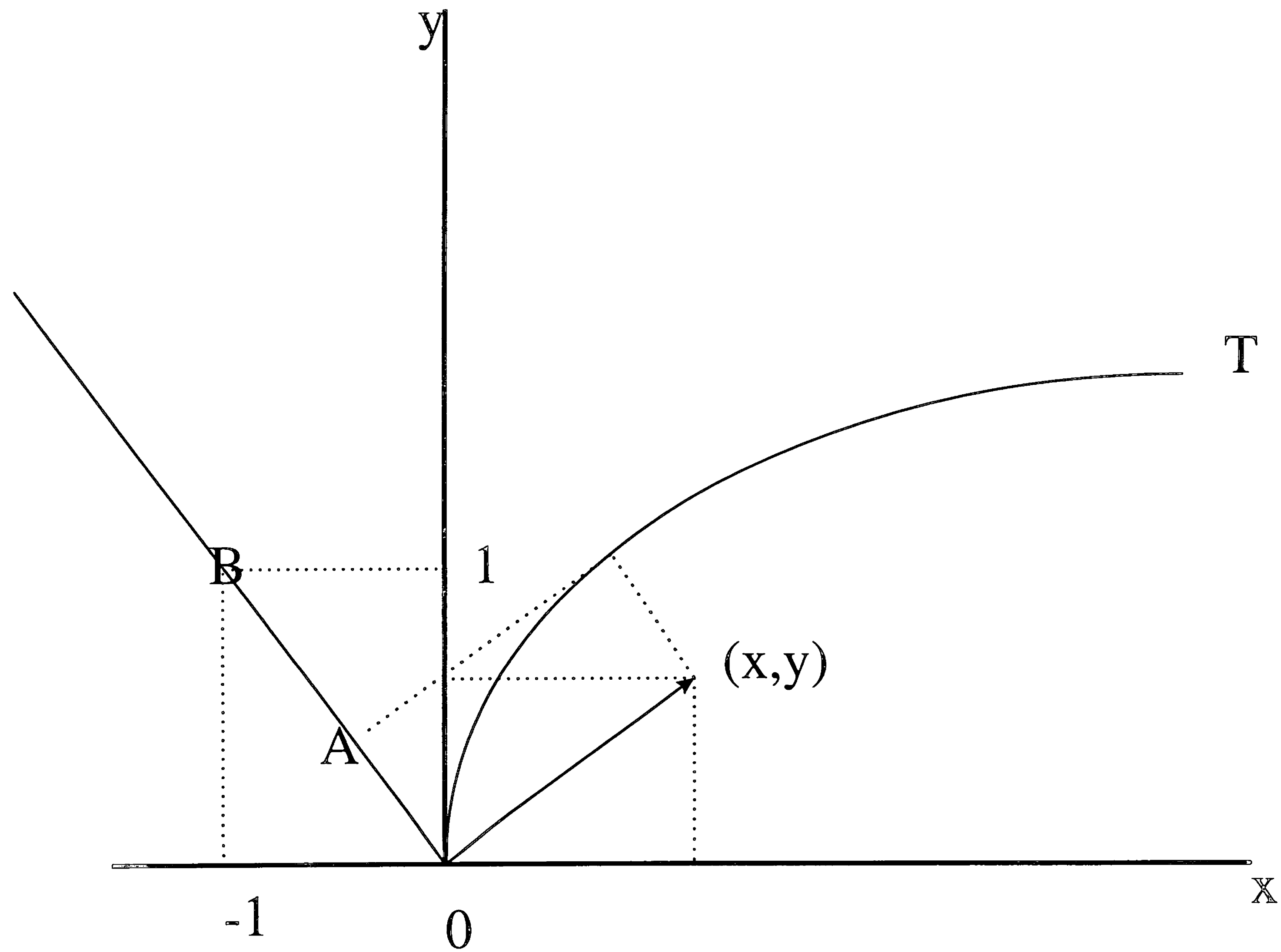

Figure 1: Directional Technology Distance Function

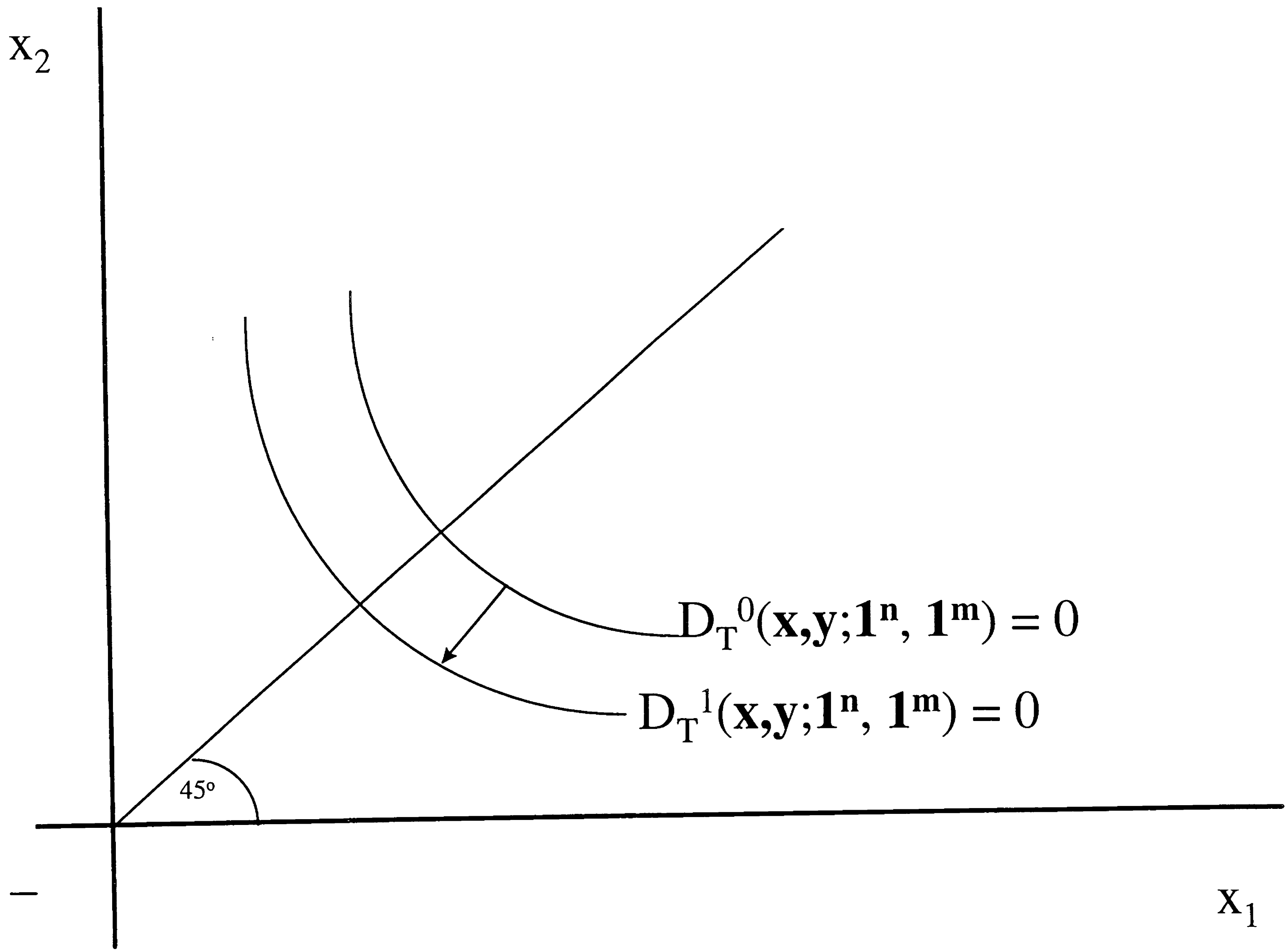

Figure 2: Input-translation neutral technical change

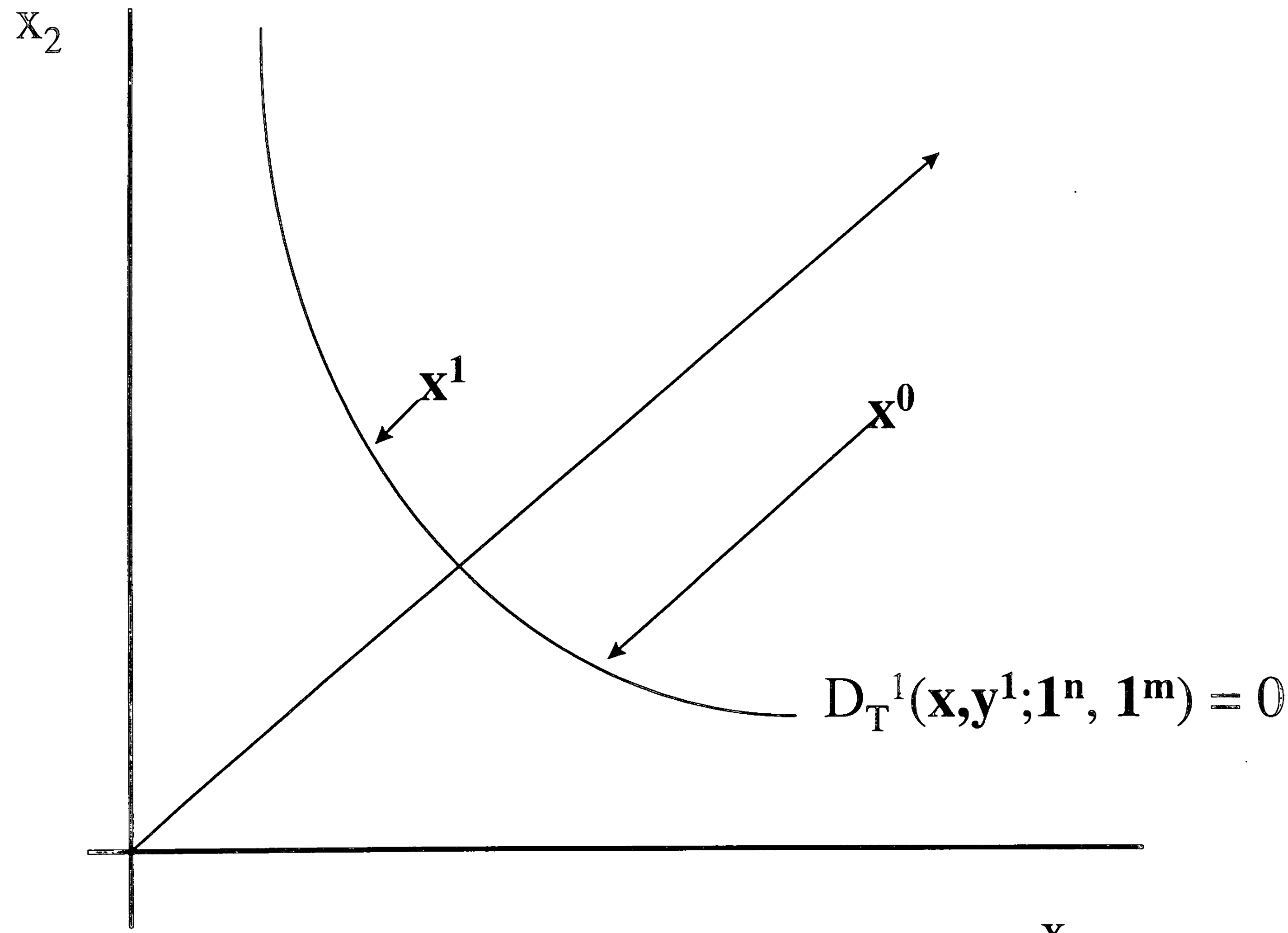

Figure 3: Luenberger 1-technology input indicator

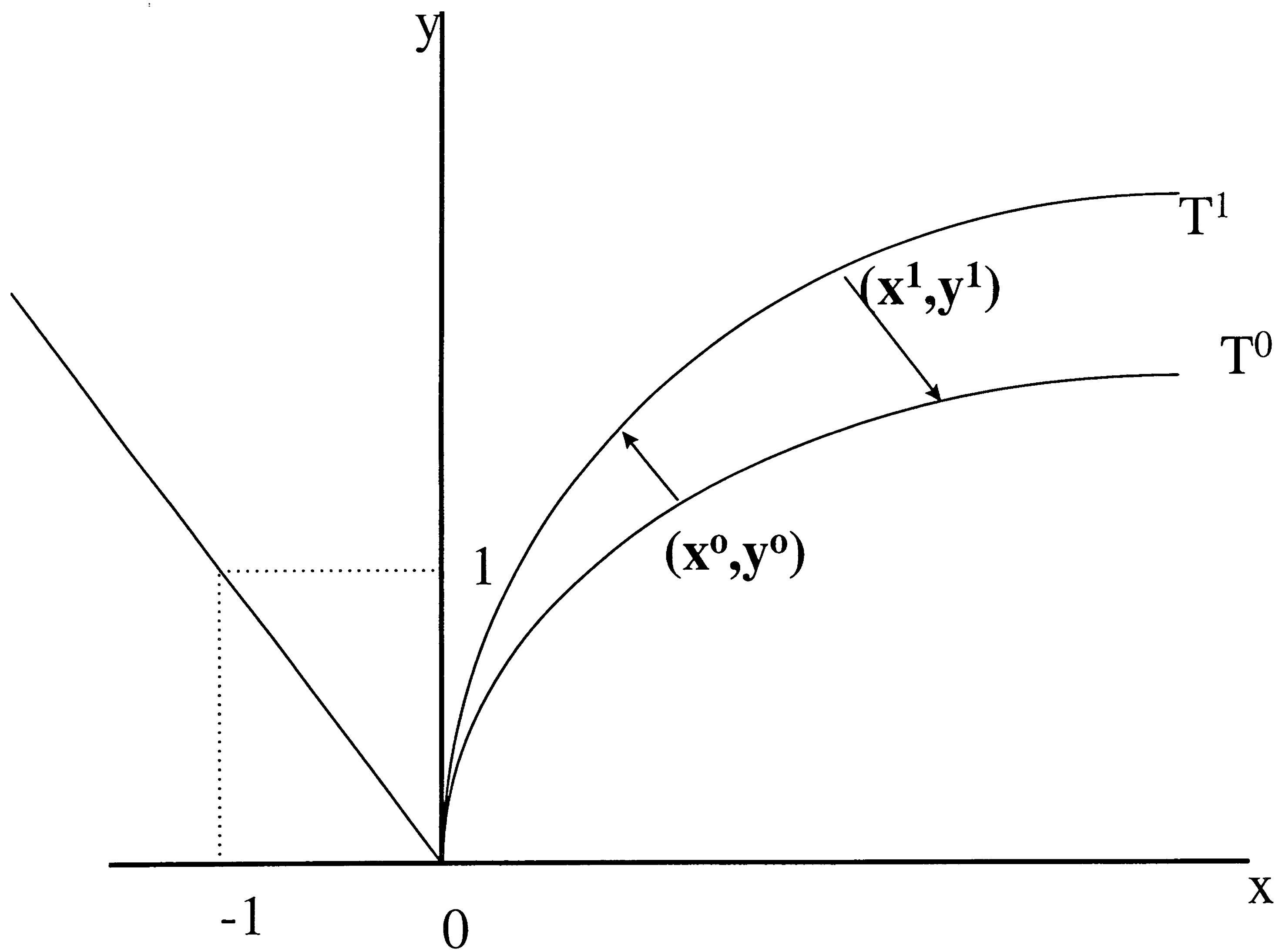

Figure 4: Luenberger Productivity Indicator

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