Homotheticity as a Translation Property

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Abstract

Färe and Mitchell (1992) have shown that cost functions for a multi-output firm obey a particular output-scaling law if and only if the underlying production technology is ray-homothetic. Multi-output firms, however, frequently change their output mix in addition to their scale. Therefore, it is important to identify technologies which possess relatively tractable analytical characteristics when subjected to nonradial changes in the output vector. This paper considers additive changes in the output vector and shows that the cost function obeys an 'output-translation law' if and only if the input correspondence is input homothetic. Input homotheticity is thus shown to be more than just a scaling property.

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Perhaps the most common functional restriction employed in economics is homotheticity. Apparently introduced into economics by Shephard (1953), homotheticity is routinely used in virtually every field or subfield of economics. Because its most recognized characteristic is that the isoquants for homothetic production functions are radial blow ups of a reference isoquant, homotheticity is often portrayed as a scaling property. One important economic consequence is a linear expansion path through the origin. Another is that cost functions dual to homothetic technologies have output separable from input prices. Consequently, homothetic production and utility functions are very useful in the construction of input-price and cost-of-living indexes; this separable structure implies independence of the resulting indexes from the reference output or utility.

Färe and Mitchell (1993) have shown, however, that linear expansion paths do not imply cost function separability: As an example, multi-output technologies with input correspondences homogeneous in the output vector possess linear expansion paths but not separable cost structures. Properly defined "input-homothetic" input correspondences, however, preserve both linear expansion paths and separable cost structures. So, it turns out in a multi-output setting that input homotheticity is not a natural generalization of homogeneity: Instead, input homotheticity and multi-output homogeneous technologies are both special cases of ray-homothetic technologies whose defining characteristic is linear expansion paths.

The foregoing suggests that input homotheticity is more than just a scaling property: That is, it does more than just preserve marginal rates of substitution as one proceeds out along rays from the origin in input space. The purpose of this paper is to show that input homotheticity can emerge from considerations that do not relate directly to the scaling and rescaling of inputs or outputs. This is done by considering when one can define a meaningful functional restriction on the way in which cost is affected by a translation of the output vector, i.e., by adding a new vector of outputs to an existing vector of outputs. Take, for example, a multi-output technology whose input correspondence is homogeneous in the output vector. Its dual cost function is homogeneous in outputs, and so radial expansions of the output vector have no impact on marginal-cost ratios. But more than radial output movements are of interest to the firm. Firms routinely consider changes in output that are nonradial in character. From both an analytic and empirical perspective, therefore, it seems interesting to identify technologies which possess relatively tractable analytic characteristics when subjected to nonradial changes in the output vector. The ones that I consider are additive changes: It turns out that input homotheticity is the technical reflection of the ability to specify a meaningful functional restriction on how cost is affected by a translation of the output vector.

In what follows, basic notation and assumptions concerning input correspondences, input-distance functions, and cost functions are first introduced. Then the main result of the paper is demonstrated. An obvious reformulation of the present treatment in terms of output correspondences, output-distance functions, and revenue functions will demonstrate that output homotheticity can be viewed as a consequence of input-translation characteristics.

1 Notation and Assumptions

Technical possibilities are summarized by an input correspondence V: $\Re^m_+ \rightarrow \Re^n_+$:

$$V(\mathbf{y}) = \{\mathbf{x} \in \Re^n_+ : \mathbf{x} \text{ can produce } \mathbf{y} \in \Re^m_+\}.$$

Here $\mathbf{x} \in \Re^n_+$ denotes a vector of inputs and $\mathbf{y} \in \Re^m_+$ denotes a vector of outputs. $V(\mathbf{y})$ satisfies properties that guarantee the existence of a duality between cost and production:

V.1 $\mathbf{0}^n \notin V(\mathbf{y}), \mathbf{y} \geq \mathbf{0}^m, \mathbf{y} \neq \mathbf{0}^m$.

V.2 $\mathbf{z} \geq \mathbf{x} \in V(\mathbf{y}) \Rightarrow \mathbf{z} \in V(\mathbf{y})$ for all $\mathbf{y} \in \Re^m_+$.

V.3 V: $\Re^m_+ \rightarrow \Re^n_+$ is a closed correspondence.

V.4 For all $\mathbf{y} \in \Re^m_+$, $V(\mathbf{y})$ is convex.

Property V.1 says that there is no free lunch for any positive output. Property V.2 imposes strong disposability of inputs upon the technology, while property V.3 requires that input sets be closed. V.4 is self explanatory.

An input correspondence is input-homothetic (Färe and Primont, 1995) if it satisfies:

$$V(\mathbf{y}) = h(\mathbf{y}) V(\mathbf{1}^m)$$

where $h : \Re^m_+ \rightarrow \Re_+$ is continuous and $V(\mathbf{1}^m)$ is a reference set satisfying V.1-V.4.

The input-distance function offers a function representation of the input correspondence. It is defined as

$$D_i(\mathbf{x}, \mathbf{y}) = \sup \left\{ \lambda > 0 : \frac{\mathbf{x}}{\lambda} \in V(\mathbf{y}) \right\}.$$

The input-distance function satisfies a number of properties (see, e.g., Färe, 1988), but most importantly it is positively linearly homogeneous in the inputs and a complete function representation of the technology in the sense that $D_i(\mathbf{x}, \mathbf{y}) \geq 1 \Leftrightarrow \mathbf{x} \in V(\mathbf{y})$. If the input correspondence is input homothetic, the input-distance function can be written

$$D_i(\mathbf{x}, \mathbf{y}) = \frac{\bar{D}_i(\mathbf{x})}{h(\mathbf{y})},$$

where $\bar{D}_i(\mathbf{x})$ is the input-distance function for $V(\mathbf{1}^m)$.

Under V.1 to V.4, for any nonempty $V(\mathbf{y})$, there exists a cost function for input prices $\mathbf{w} \in \Re^n_{++}$ defined by:

$$c(\mathbf{w}, \mathbf{y}) = \min_{\mathbf{x}} \{\mathbf{wx} : \mathbf{x} \in V(\mathbf{y})\}.$$

It is well-known that the input correspondence is input homothetic if and only if: $c(\mathbf{w}, \mathbf{y}) = h(\mathbf{y})\bar{c}(\mathbf{w})$ where $\bar{c}(\mathbf{w})$ is the cost function associated with $V(\mathbf{1}^m)$. The cost function satisfies a number of well-known properties which need not be repeated for our purposes. However, I will make particular use of the fact that the cost function is always positively linearly homogeneous in input prices, i.e., $c(\theta\mathbf{w}, \mathbf{y}) = \theta c(\mathbf{w}, \mathbf{y})$ for $\theta > 0$.

2 Homotheticity as a Translation Property

This section presents two new characterizations of input-homothetic input correspondences that emerge not from the scaling properties of outputs in the cost or distance function but from the translation of the output vector. The approach taken here has two main sources of inspiration: Färe (1975) shows that scalar-output production functions, $g(\mathbf{x})$, satisfying the functional equation:

$$g(\boldsymbol{\lambda} \bullet \mathbf{x}) = \Lambda(\boldsymbol{\lambda}, g(\mathbf{x}))$$

where $\boldsymbol{\lambda} \bullet \mathbf{x}$ denotes the n-dimensional vector obtained by multiplying the vector $\boldsymbol{\lambda}$ $(\boldsymbol{\lambda} \geq \mathbf{0}^n, \boldsymbol{\lambda} \neq \mathbf{0}^n)$ component-wise by $\mathbf{x}$, must be homothetic Cobb-Douglas production functions. Later Färe and Mitchell (1992) demonstrated that ray-homothetic input correspondences can be characterized by the class of cost functions satisfying the following functional equation for arbitrary Γ:

$$c(\mathbf{w}, \theta\mathbf{y}) = \Gamma(\theta, \mathbf{y}, c(\mathbf{w}, \mathbf{y})), \theta > 0$$

In words, ray homotheticity of the cost function is the technical requirement needed to be able to express the cost of producing a scaled output vector as a function of the cost of producing the unscaled output vector, the unscaled output vector, and the scaling factor. In what follows, I consider two extensions of the approach taken by Färe and Mitchell (1992). The first is

$$c(\mathbf{w}, \mathbf{y} + \mathbf{z}) = F(\mathbf{z}, \mathbf{y}, c(\mathbf{w}, \mathbf{y})) \tag{1}$$

$\mathbf{z}, \mathbf{y} \neq \mathbf{0}^m$ where $c : \Re^n_{++} \times \Re^m_+ \rightarrow \Re_+$ is a well-behaved cost function, and $F : \Re^m_+ \times \Re^m_+ \times \Re_+ \rightarrow \Re_+$ is continuous in all arguments, but otherwise arbitrary. So my interest is in expressing the cost of producing a translated output vector as a function of the cost of producing the untranslated output vector, the untranslated output vector, and the vector in whose direction output is translated. I also consider the more general functional equation:

$$c(\mathbf{w}, \mathbf{y} + \mathbf{z}) = G(\mathbf{z}, \mathbf{y}, \mathbf{w}, c(\mathbf{w}, \mathbf{y})) \tag{2}$$

$\mathbf{z}, \mathbf{y} \neq \mathbf{0}^m$ where $G : \Re^m_+ \times \Re^m_+ \times \Re^n_{++} \times \Re_+ \rightarrow \Re_+$ is continuous in all arguments.

To better motivate our interest in expressions (1) and (2), suppose for the moment that output is a scalar. Because all scalar output changes can be represented as radial changes, a functional equation of the type considered by Färe and Mitchell (1992) can be derived directly from expression (1): For example, setting $\theta = \frac{y+z}{y}$, allows (1) to be rewritten as:

$$c(\mathbf{w}, \theta y) = F(\theta y - y, y, c(\mathbf{w}, y)).$$

From this rewriting of (1), it is apparent that expressions (1) and (2) place more stringent requirements on the technology than the functional restriction used by Färe and Mitchell (1992) suggesting that ray homotheticity may not be sufficient to guarantee a solution to either (1) or (2). This happens, of course, because in the multi-output case all changes in the output vector will not be expressible as radial changes. Instead all changes in the output vector can be depicted as consisting of two components, a rescaling of the output vector and a change in the output mix. The results of Färe and Mitchell (1992) only apply to the first component—the rescaling of the output vector.

As a practical matter, however, firms routinely contemplate and make changes in their production plans that cannot be depicted by rescaling of outputs. For example, many modern firms rather routinely change their output mix by moving in and out of different product lines in response to perceived market opportunities. And it is not unusual to encounter firms that were once highly specialized in a single product line and that have moved into entirely new product lines in an attempt to capture new markets or to prevent entry by potential competitors.

For example, Schmalensee (1978) documents that the six leading producers of breakfast cereals introduced roughly eighty new brands between 1950 and 1972. Output changes of this type, which do not preserve the output mix, are modeled by (1) and (2). Therefore, isolating families of functions which satisfy those functional equations could provide forms with convenient analytic and empirical properties for economic problems that involve analysis of a changing output mix.

My first result is:

Theorem 1 *The cost function satisfies (1) if and only if it is input homothetic.*

Proof *By the positive linear homogeneity of the cost function, (1) implies* $F(\mathbf{z},\mathbf{y},\theta c(\mathbf{w},\mathbf{y})) = \theta F(\mathbf{z},\mathbf{y},c(\mathbf{w},\mathbf{y}))$ *for* $\theta > 0$*, whence* $F(\mathbf{z},\mathbf{y},c(\mathbf{w},\mathbf{y})) = c(\mathbf{w},\mathbf{y})F(\mathbf{z},\mathbf{y},1) = c(\mathbf{w},\mathbf{y})f(\mathbf{z},\mathbf{y})$*. Using this result in (1) establishes that* $c(\mathbf{w},\mathbf{y})f(\mathbf{z},\mathbf{y}) = c(\mathbf{w},\mathbf{z})f(\mathbf{y},\mathbf{z})$*. Set* $\mathbf{z} = \bar{\mathbf{z}}$*, a reference value chosen so that* $\bar{\mathbf{z}} > 0$*. V.1 implies* $c(\mathbf{w},\mathbf{y}+\mathbf{z}) > 0$*, and it follows immediately that* $c(\mathbf{w},\mathbf{y}) = c(\mathbf{w},\bar{\mathbf{z}})\frac{f(\mathbf{y},\bar{\mathbf{z}})}{f(\bar{\mathbf{z}},\mathbf{y})} = h(\mathbf{y})\bar{c}(\mathbf{w})$*. This establishes necessity. Sufficiency follows by choosing* $f(\mathbf{z},\mathbf{y}) = \frac{h(\mathbf{y}+\mathbf{z})}{h(\mathbf{y})}$

Corollary 2 *The cost function satisfies (1) if and only if* $D_i(\mathbf{x},\mathbf{y}) = \frac{\bar{D}_i(\mathbf{x})}{h(\mathbf{y})}$.

My next result extends this theorem to the more general case covered by functional equation (2). As in Färe and Mitchell (1992), functional equation (2) has proven too general for me to solve[1]. However, following Färe and Mitchell (1992) one can solve (2) after placing some further structure on its behavior in input prices.

Theorem 3 *If* $G(\mathbf{z},\mathbf{y},\mathbf{w},c(\mathbf{w},\mathbf{y}))$ *is nondecreasing in* $\mathbf{w}$*, then a cost function satisfies functional equation (2) if and only if the technology is input homothetic*

Proof *I demonstrate that functional equation (2) is equivalent to functional equation (1) under this restriction. The positive linear homogeneity of the cost function and (2) establishes that*

$$G(\mathbf{z},\mathbf{y},\theta\mathbf{w},\theta c(\mathbf{w},\mathbf{y})) = \theta G(\mathbf{z},\mathbf{y},\mathbf{w},c(\mathbf{w},\mathbf{y}))$$

for $\theta > 0$*. Hence,* $G(\mathbf{z},\mathbf{y},\mathbf{w},c(\mathbf{w},\mathbf{y})) = c(\mathbf{w},\mathbf{y})G(\mathbf{z},\mathbf{y},\frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})},1) = c(\mathbf{w},\mathbf{y})g(\mathbf{z},\mathbf{y},\frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})})$*. Substitute this result in (2) to obtain*

$$c(\mathbf{w},\mathbf{y}+\mathbf{z}) = c(\mathbf{w},\mathbf{y})g(\mathbf{z},\mathbf{y},\frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})}),$$

[1] Färe and Mitchell (1992) (their footnote 11) point out that Aczél (1969) has described functional equations of this type as being "too general" to solve.

or

$$\frac{c(\mathbf{w},\mathbf{y}+\mathbf{z})}{c(\mathbf{w},\mathbf{y})} = g(\mathbf{z},\mathbf{y},\frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})}).$$

Now multiply the left-hand side of this last expression by one in the form of $\frac{c(\mathbf{w},\mathbf{y})}{c(\mathbf{w},\mathbf{y})}$ *and then use the positive linear homogeneity of the cost function in input prices to obtain*

$$\frac{c(\frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})},\mathbf{y}+\mathbf{z})}{c(\frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})},\mathbf{y})} = \frac{c(\bar{\mathbf{w}},\mathbf{y}+\mathbf{z})}{c(\bar{\mathbf{w}},\mathbf{y})} = g(\mathbf{z},\mathbf{y},\bar{\mathbf{w}}) = g(\mathbf{z},\mathbf{y},\frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})}),$$

where $\bar{\mathbf{w}} = \frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})}$. *Because the cost function is positively linearly homogeneous in prices, the left-most expression is homogeneous of degree zero in* $\frac{\mathbf{w}}{c(\mathbf{w},\mathbf{y})}$ *establishing that* $g(\mathbf{z},\mathbf{y},\bar{\mathbf{w}})$ *is homogeneous of degree zero in* $\bar{\mathbf{w}}$. *Now use the lemma reported in Färe and Mitchell (1992) to establish that* $g(\mathbf{z},\mathbf{y},\bar{\mathbf{w}})$ *must be independent of* $\bar{\mathbf{w}}$, *and hence functional equation (2) is equivalent to (1).*

Although no formal proof is presented, it is obvious from the preceding theorems and the corollary that input homotheticity can also be characterized by requiring the input distance function to satisfy equations analogous to (1) and (2) with inputs replacing input prices and the distance function replacing the cost function. Results completely analogous to Theorems 1 and 2 with obvious corollaries can be easily established. Moreover, a completely parallel argument demonstrates that an output correspondence is output-homothetic if and only if its dual revenue function satisfies functional equations in inputs, output prices, and revenue which are analogous to (1) and (2).

3 Conclusion

Färe and Mitchell (1992) showed that an input correspondence is ray homothetic if and only if its dual cost function satisfies the functional equation:

$$c(\mathbf{w},\theta\mathbf{y}) = \Gamma(\theta,\mathbf{y},c(\mathbf{w},\mathbf{y})), \theta > 0$$

regardless of whether the technology has a single or multiple outputs. Following in this line of research, this paper shows that an input correspondence must be input homothetic if its dual cost function is to satisfy the more restrictive functional equation:

$$c(\mathbf{w},\mathbf{y}+\mathbf{z}) = F(\mathbf{z},\mathbf{y},c(\mathbf{w},\mathbf{y})).$$

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