Consumers' Surplus as an Exact and Superlative Cardinal Welfare Indicator

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1. Introduction

Consider the standard welfare problem: Determine whether a consumer is better off consuming the commodity bundle $\mathbf{x}^1$ at prices $\mathbf{p}^1$ than consuming the bundle $\mathbf{x}^0$ at prices $\mathbf{p}^0$. Nearly seven decades ago, Bowley (1928) showed that

$$\frac{1}{2}\left(m^1\mathbf{p}^1 + m^0\mathbf{p}^0\right)\left(\mathbf{x}^1 - \mathbf{x}^0\right) = \frac{1}{2}\sum_{i=1}^{n}\left(m^1 p_i^1 + m^0 p_i^0\right)\left(x_i^1 - x_i^0\right),$$

where m^i is the consumer's marginal utility in market situation i, exactly measures welfare change if the consumer's utility function is quadratic. Earlier Bennet (1920) had suggested a similar formula (with both marginal utilities of income set to 1) for use in cost-of-living measurement. The Bennet (1920) version of this formula corresponds to Hicks' (1945-46) many-market, consumer-surplus measure and to Harberger's (1971) measure of welfare change. While both Hicks' measure and Harberger's measure were based on approximation arguments, Diewert (1976b) has shown that a normalized version of the Bennet-Bowley measure provides the same ordinal welfare rankings as a homogeneous quadratic utility structure. Because of their simplicity and relatively sparse data requirements, versions of the Bennet-Bowley measure are commonplace in applied welfare computations.

Driven by its practical importance, several papers have investigated the ability of different versions of the Bennet-Bowley measure to approximate two standard cardinal welfare measures–the compensating and equivalent variations (Weitzman, 1989; Diewert, 1992). The compensating and equivalent variations, being cardinal, can provide aggregate welfare measures. Hence, if the Bennet-Bowley measure can approximate one or both of these measures closely, its approximation property will serve to rationalize a wide variety of practical welfare computations. This paper considers a related question: Can the Bennet-Bowley consumer surplus measure capture exactly the cardinal welfare measure that Allais (1943) has coined 'disposable surplus'? Below, it is shown that an appropriately normalized version of this consumer-surplus measure is an exact measure of disposable surplus if the consumer's utility function is of the translation–homothetic, generalized quadratic form. Hence, this normalized Bennet-Bowley measure, which can be computed using only price and quantity data, is a superlative cardinal welfare measure in Diewert's (1976) sense for an entire class of utility structures. An immediate corollary is that the Bennet-Bowley measure provides a good second-order approximation of Allais' disposable surplus if preferences are translation homothetic in the sense of Chambers and Färe (1997).

Perhaps one way to assess the importance of these results is to compare them to the welfare exactness result to which they are most closely related. Diewert's (1976b) demonstration that the Bennet-Bowley measure provides the same ordinal rankings as a homogeneous quadratic utility function is very helpful in making individual welfare comparisons. However, in most practical situations welfare comparisons need to made at an aggregate and not individual level. Because they rely upon ordinality, Diewert's (1976b) exactness results will, thus, be of limited value in making aggregate comparisons. Allais' (1943) disposable surplus, on the other hand, is a cardinal welfare measure upon which meaningful aggregate measures can be based. Hence, the exactness results presented in this paper are for a cardinal welfare measure, and under appropriate assumptions can be used as the basis for making aggregate welfare comparisons.

To prove these results, a few basic concepts are needed: Most important are Allais' (1943) notion of disposable surplus, Luenberger's (1992) benefit function, and Diewert's quadratic approximation lemma. In what follows, I first introduce my notation and assumptions, briefly discuss Luenberger's benefit function and its properties. After that I present my main results: First I consider the case for a single consumer, and then I show how the results extend to the important practical problem of making welfare comparisons for many consumers. After that I relate these results to closely related results on commodity aggregation and measurement. The final section concludes.

2. Notation and Assumptions

Let $\mathbf{x} \in \Re^n_+$ denote a vector of commodities. Consumer preferences are represented by a twice continuously differentiable, quasi-concave function $u : \Re^n_+ \to \Re$ that is nondecreasing in each of its arguments. Prices of commodities are given by $\mathbf{p} \in \Re^n_{++}$. Consumers are presumed to be rational and to maximize utility subject to a fixed budget constraint, or equivalently to minimize expenditure required to achieve a given level of utility. The consumer's benefit function (Luenberger, 1992) $B : \Re^n_+ \times \Re \times \Re^n_+ \to \Re$ is defined by:

$$B(\mathbf{x}, u; \mathbf{g}) = \max\left\{\beta \in \Re : u\left(\mathbf{x} - \beta\mathbf{g}\right) \geq u\right\}, \mathbf{g} \in \Re^n_+, \mathbf{g} \neq \mathbf{0},$$

if $u\left(\mathbf{x} - \beta\mathbf{g}\right) \geq u$ for some β and $B(\mathbf{x}, u; \mathbf{g}) = -\infty$ otherwise. In words, the benefit function represents the largest amount of a reference commodity bundle, $\mathbf{g}$, that can be subtracted from another commodity vector and still achieve a reference utility, u. Geometrically, it represents the farthest one can move from the commodity vector $\mathbf{x}$ in the direction of $\mathbf{g}$ and still achieve the reference utility,

u. The benefit function is depicted in Figure 1 for $\mathbf{g} = (1, 1)$ by the ratio OA/OB. The benefit function is nonincreasing in u, nondecreasing and concave in $\mathbf{x}$, and satisfies the translation property $B(\mathbf{x}+\alpha\mathbf{g}, u; \mathbf{g}) = B(\mathbf{x}, u; \mathbf{g}) + \alpha$ (Luenberger, 1992). For our purposes, however, its most important property is that it provides a complete function representation of consumer preferences in that (Luenberger, 1992; Chambers, Chung, and Färe, 1996):

$$u(\mathbf{x}) \geq u \Leftrightarrow B(\mathbf{x}, u; \mathbf{g}) \geq 0.$$

That $B(\mathbf{x}, u; \mathbf{g}) \geq 0$ if $u(\mathbf{x}) \geq u$ is obvious from its definition. To go the other way, suppose that $B(\mathbf{x}, u; \mathbf{g}) \geq 0$. Using the definition of the benefit function and the presumption that utility is nondecreasing in commodities implies:

$$u(\mathbf{x}) \geq u(\mathbf{x} - B(\mathbf{x}, u; \mathbf{g})\mathbf{g}) \geq u.$$

In particular, it is important to recognize that $u(\mathbf{x}) = u \Leftrightarrow B(\mathbf{x}, u; \mathbf{g}) = 0$ (Luenberger, 1992).

The fact that the benefit function offers a complete function representation of preferences implies that the consumer's expenditure minimization problem, yielding the expenditure function $E(\mathbf{p}, u)$, can be reformulated as the unconstrained optimization problem:

$$E(\mathbf{p}, u) = \min_{x}\{\mathbf{p}\cdot(\mathbf{x} - B(\mathbf{x}, u; \mathbf{g})\mathbf{g})\}$$

so long as the set $\{\mathbf{x} : B(\mathbf{x}, u; \mathbf{g}) \geq 0\}$ is nonempty (Chambers, 1996). Because the solution to the expenditure minimization problem is generally independent of the choice of the reference vector $\mathbf{g}$, which is arbitrary,[1] the indirect objective function, $E(\mathbf{p}, u)$, does not depend upon $\mathbf{g}$.

Assuming differentiability any strictly interior solution to the expenditure minimization problem satisfies:

$$\frac{\mathbf{p}}{\mathbf{p}\cdot\mathbf{g}} = \nabla_x B(\mathbf{x}, u; \mathbf{g}) \tag{2.1}$$

[1] However, as will be apparent, under translation homotheticity, an obvious choice for the reference vector $\mathbf{g}$ is given by the vector of slopes of the compensated demands in utility.

where the notation $\nabla_x B(\mathbf{x}, u; \mathbf{g})$ denotes the gradient of the benefit function in the commodities.

Allais (1943) defined the disposable surplus associated with an allocation $\mathbf{x}$ relative to utility level u as the amount of the first commodity that could be subtracted from the allocation $\mathbf{x}$ and still achieve utility level u. Clearly, Allais' disposable surplus equals $B(\mathbf{x}, u; \mathbf{g})$ at $\mathbf{g} = (1, 0, ..., 0)$. Notice, following Groves (1979) and Luenberger (1995), that by appropriately reinterpreting the utility function to cover the case where $\mathbf{g} = (1, 0, ..., 0)$ represents a unit of money, then Allais's disposable surplus can be interpreted as a willingness to pay measure. I slightly generalize Allais' disposable surplus and define the disposable surplus of a subvector of the commodities that may or may not include the first commodity. To that end, I introduce the following notation: Let $I = \{1, 2, ..., n\}$ denote the set of commodity subscripts. Define the partition of this set by $\hat{I} = \{I^a, I^b\}$ where $I^a \cap I^b = \emptyset$. The *disposable surplus of commodity subvector* $\mathbf{x}^a$ *relative to utility level* u[2] is given by

$$B(\mathbf{x}, u; \mathbf{g}^a)$$

[2]This slight renormalization of Allais' disposable surplus is sufficient to ensure that the measure, by appropriate choice of a reference vector, can be made insensitive to changes in the units of measurement. Luenberger (1996) has recently provided a generalization of this disposable surplus measure which he terms the equivalent benefit.

where the notation $\mathbf{g}^a$ denotes the vector with a strictly positive entry corresponding to each element of I^a and zero elsewhere. Obvious special cases are Allais' disposable surplus defined by $I^a = \{1\}$, $\mathbf{g}^\alpha = (1, 0, ..., 0)$. and the case where all commodities are in the subvector, i.e., $I^b = \emptyset$. I denote the latter case by setting $\mathbf{g}^a$ to $\mathbf{g}$, which I then take to be a strictly positive n-dimensional vector. In what follows, $\mathbf{g}^\alpha$ will be referred to as the reference commodity bundle.

Our generalized disposable surplus measure offers a convenient means for comparing two consumer situations. Suppose we wish to evaluate the welfare consequences of having the consumer change his or her consumption pattern from $\mathbf{x}^0$ to $\mathbf{x}^1$. A cardinal measure of the associated change in welfare is the disposable surplus of $\mathbf{x}^{1a}$ relative to the original utility $u(\mathbf{x}^0)$:

$$B(\mathbf{x}^1, u(\mathbf{x}^0); \mathbf{g}^a) = B(\mathbf{x}^1, u(\mathbf{x}^0); \mathbf{g}^a) - B(\mathbf{x}^0, u(\mathbf{x}^0); \mathbf{g}^a),$$

where the equality follows from the fact that $B(\mathbf{x}^0, u(\mathbf{x}^0); \mathbf{g}^a) = 0$. In words, this measure gives the number of units of the reference commodity bundle $\mathbf{g}^a$ that the consumer could subtract from $\mathbf{x}^1$and still be no worse off than at $\mathbf{x}^0$. If this measure is positive, welfare has improved as a result of the change, if it is negative welfare has fallen.

The utility function is defined to be $\mathbf{g^a}$-*translation homothetic*[3] if it can be expressed as:

$$u(\mathbf{x}) = F(\hat{u}(\mathbf{x}, \mathbf{g^a})),$$

where F is a strictly increasing and continuous function, and $\hat{u}(\mathbf{x}, \mathbf{g^a})$is a utility function that satisfies the translation property in $\mathbf{g^a}$, $\hat{u}(\mathbf{x}+\alpha\mathbf{g}^a, \mathbf{g^a}) = \hat{u}(\mathbf{x}, \mathbf{g^a})+\alpha$. If the utility function is $\mathbf{g^a}$-translation homothetic, it follows that:

$$\begin{aligned} B(\mathbf{x}, u; \mathbf{g}^a) &= \max\left\{\beta \in \Re : u\left(\mathbf{x}-\beta\mathbf{g}^a\right) \geq u\right\} \\ &= \hat{B}(\mathbf{x}, \mathbf{g}^a) - F^{-1}(u), \end{aligned}$$

where[4] $\hat{B}(\mathbf{x}, \mathbf{g}^a) = \max\left\{\beta \in \Re : \hat{u}\left(\mathbf{x}-\beta\mathbf{g}^a, \mathbf{g^a}\right) \geq 0\right\}$. Translation homothetic preference structures are a special case of the quasi-homothetic family of preferences. A distinguishing characteristic of this class of preferences is that they are dual to

[3] Chambers and Färe (1997) define a function as translation homothetic if it can be written as a monotonic transformation of a function that satisfies the translation property for an arbitrary reference vector.

[4] Hence, so long as utility is ordinal, no generality is lost in taking $\hat{B}(\mathbf{x}, \mathbf{g}^a)$ as the consumer's utility function.

expenditure functions of the general form (Chambers and Färe, 1997):

$$E(\mathbf{p}, u) = u\left(\mathbf{p} \cdot \mathbf{g}^{a}\right) + e\left(\mathbf{p}\right).$$

Here $e(\mathbf{p})$ is a well-behaved expenditure function. Hence, translation homothetic preference structures offer a natural choice of the reference commodity bundle: the slopes of the compensated demands in utility. Notice, that like homothetic and quasi-homothetic preference structures, translation homotheticity requires all commodities to be normal in consumption. Blackorby, Boyce, and Russell (1978) refer to this preference structure as being homothetic to minus infinity, while Dickinson (1980) refers to it as linear parallel preferences and has shown how they can be paricularly useful in the modelling of labour supply. Economically, they can be interpreted in terms of compensated demands consisting of two components: The first, which can be thought of as subsistence compensated demand, is independent of the reference utility and is given by the gradient of the function $e(\mathbf{p})$. The second consisting of how compensated demands vary with real income, u, is independent of the price level. As with all quasi-homothetic structures, the associated Engel curves are linear in expenditures.

Members of the class of $\mathbf{g}^{a}$-translation homothetic utility structures include the translation homothetic quadratic:

$$\hat{B}(\mathbf{x},\mathbf{g}^a)=\sum_{i=1}^{n}a_ix_i+\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}b_{ik}x_ix_k, \tag{2.2}$$

where $b_{ik}=b_{ki}$, $\sum_{i\in I^a}a_ig_i^\alpha=1$, and $\sum_{i\in I^a}b_{ik}g_i^\alpha=0, k=1,..,n$, and the transcendental exponential:

$$\exp u(\mathbf{x})=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}\exp\left(\frac{x_i}{2}\right)\exp\left(\frac{x_k}{2}\right)$$

with $a_{ik}=a_{ki}$. The translation homothetic quadratic is $\mathbf{g}^a$-translation homothetic and provides a second-order approximation to any twice continuously differentiable $\mathbf{g}^a$-translation homothetic function (Chambers, 1996), while the transcendental exponential is translation homothetic in the special case $I^b=\emptyset$ and $\mathbf{g}=\mathbf{1}$.

3. Welfare measures for an individual consumer

This section demonstrates that the Bennet-Bowley consumer surplus measure, if appropriately rewritten, can be interpreted as an exact welfare measure for a $\mathbf{g^a}$-translation homothetic quadratic utility structure. I start by restating a result that can be found in Diewert's (1976) classic paper on index numbers:

Diewert's Lemma. $f(\mathbf{z})$ is expressible as

$$\sum_{i=1}^{n} a_i z_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik} z_i z_k$$

$b_{ik} = b_{ki,}$ if and only if:

$$f(\mathbf{z}^1) - f(\mathbf{z}^0) = \frac{1}{2}\left[\nabla_z f(\mathbf{z}^1) + \nabla_z f(\mathbf{z}^0)\right] \cdot \left(\mathbf{z}^1 - \mathbf{z}^0\right)$$

If the utility function is $\mathbf{g}^a$-translation homothetic and (2.2) holds:

$$\begin{aligned} B(\mathbf{x}^1, u(\mathbf{x}^0); \mathbf{g}^a) &= B(\mathbf{x}^1, u(\mathbf{x}^0); \mathbf{g}^a) - B(\mathbf{x}^0, u(\mathbf{x}^0); \mathbf{g}^a) = \hat{B}(\mathbf{x}^1, \mathbf{g}^a) - \hat{B}(\mathbf{x}^0, \mathbf{g}^a) \\ &= \frac{1}{2}\left[\nabla_x \hat{B}(\mathbf{x}^1, \mathbf{g}^a) + \nabla_x \hat{B}(\mathbf{x}^0, \mathbf{g}^a)\right] \cdot \left(\mathbf{x}^1 - \mathbf{x}^0\right). \end{aligned}$$

where [5]the second equality follows by applying Diewert's Lemma in terms of $\mathbf{x}$.

Consequently, for a price-taking consumer maximizing utility, it then follows from

[5]This manner of expressing the generalized disposable surplus measure also demonstrates that translation homotheticity guarantees its independence of the reference utility level. It is easily shown that this is a necessary and sufficient condition.

(2.1) that all strictly interior optima will be characterized by:

$$B(\mathbf{x}^1, u(\mathbf{x}^0); \mathbf{g}^a) = \frac{1}{2}\left(\bar{\mathbf{p}}^1 + \bar{\mathbf{p}}^0\right) \cdot \left(\mathbf{x}^1 - \mathbf{x}^0\right), \tag{3.1}$$

where $\bar{\mathbf{p}}^k = \frac{\mathbf{p}^k}{\sum_{i \in I^a} p_i^k g_i^a}$.

Expression (3.1) establishes that a slight reworking of the Bennet-Bowley formula is, in fact, an exact measure of disposable surplus of the commodity subvector $\mathbf{x}^1$ relative to utility level $u(\mathbf{x}^0)$ when the benefit function assumes the form of (2.2). This reworking requires the normalization of the respective price vectors by the market value of the $\mathbf{g}^a$ bundle. So Allais' measure of disposable surplus is just our version of the Bennet-Bowley formula (3.1) with each price vector normalized by p_1. Several comments about (3.1) are in order: First, and most importantly, unlike Bowley's original formula, it can be computed only using observed price and quantity data. Second, it is an exact measure for a second-order flexible approximation to an arbitrary $\mathbf{g}^a$−translation homothetic utility structure. Therefore, it is superlative in Diewert's (1976) sense for this class of utility structures. Third, although it is linearly homogeneous in normalized prices, it is in fact homogeneous of degree zero in observed prices. Therefore, for example, it follows trivially that if $p^1 = \mu p^0, \mu > 0$ the measure degenerates to $\bar{p}^1 \cdot (x^1 - x^0)$, the difference in normalized expenditures. Next, the Bennet-Bowley consumer-surplus formula which now has an exact cardinal welfare interpretation, corresponds to the difference in expenditures on the two commodity bundles as evaluated at average normalized prices. So, rather intuitively, x^1 has a positive disposable surplus relative to x^0 if the former is the more valuable in terms of average normalized prices. And finally, it follows immediately that the Bennet-Bowley consumer surplus measure provides a good second-order approximation to disposable surplus for the entire class of g^a-translation homothetic functions. In the next section, we show that the Bennet-Bowley measure has the very desirable empirical property of providing exact cardinal welfare measures which can be aggregated over consumers.

4. Aggregate Welfare Measures

An especially attractive property of benefit functions is that they can be added across consumers to provide meaningful measures of aggregate benefits (Luenberger, 1992b). The importance of this property lies in the fact that in most practical situations involving welfare comparisons, one is not interested in assessing different market situations at the individual level. Rather, the more common case is the one where welfare comparisons need to be made at the aggregate level. This fact explains, for example, the tremendous amount of theoretical attention that has been devoted to understanding how closely various consumer surplus measures approximate the two most common cardinal welfare measures—the compensating and equivalent variations. Because, these are cardinal measures they can be meaningfully added across consumers. Here our concern is with the aggregation properties of the less common, but still meaningful, welfare measure associated with our generalization of Allais' disposable surplus.

Define$\mathbf{U} = (u_1, .., u_M)$ as the vector of utilities associated with M distinct individuals and $\mathbf{X} = (\mathbf{x}_1, ..., \mathbf{x}_M)$ as the associated commodity endowments of the M individuals.[6] Luenberger (1992) defines the total benefit function as the sum of the individual benefit functions, i.e.,

$$b(\mathbf{X}, \mathbf{U}, \mathbf{g}^a) = \sum_{k=1}^{M} B_k(\mathbf{x}_k, u_k; \mathbf{g}^a).$$

In words, $b(\mathbf{X}, \mathbf{U}, \mathbf{g}^a)$ is the number of units of $\mathbf{g}^a$ that the M consumers as a group would be willing to trade to move from the allocation $\mathbf{X}$ to $\mathbf{U}$. (Luenberger

[6]Notice that here we denote individual k's vector by the bold-faced $\mathbf{x}_k$ which is to be distinguished from the kth element of the vector x which we have denoted by the plain-faced x_k.

(1992b) shows that Pareto-efficient allocations correspond to a zero maximals for $b(\mathbf{X}, \mathbf{U}, \mathbf{g}^a)$.) As such, it offers a natural aggregate measure of consumer disposable surplus or willingness to pay as expressed in units of the reference commodity bundle. Specifically, define $\mathbf{X}^i = (\mathbf{x}_1^i, ..., \mathbf{x}_M^i)$ and $\mathbf{U}(\mathbf{X}^i) = (u_1(\mathbf{x}_1^i), .., u_M(\mathbf{x}_M^i))$, then the amount of the reference commodity bundle that the M consumers can dispose of and still achieve the utility levels associated with $\mathbf{X}^0$ from $\mathbf{X}^1$is given by $b(\mathbf{X}^1, \mathbf{U}(\mathbf{X}^0), \mathbf{g}^a)$. It is then a simple corollary of the preceding results that if all consumers face the same prices, maximize utility functions satisfying the translation property, and (2.2) applies for each consumer's benefit function that:

$$b(\mathbf{X}^1, \mathbf{U}(\mathbf{X}^0), \mathbf{g}^a) = \left(\frac{1}{2}\left(\bar{\mathbf{p}}^1 + \bar{\mathbf{p}}^0\right) \cdot \sum_{k=1}^{M} \left(\mathbf{x}_k^1 - \mathbf{x}_k^0\right)\right).$$

represents an exact measure of this aggregate disposable surplus that can be computed using only observed prices and quantities. Hence, this aggregate welfare measure can be used to make legitimate exact aggregate welfare comparisons. Moreover, it also follows immediately that this aggregate Bennet-Bowley measure provides a second-order approximation to the associated aggregate disposable surplus for the entire class of translation homothetic preferences.

5. Commodity Aggregates And Disposable Surplus

Besides being an exact welfare measure, our reformulated version of the Bennet-Bowley formula also has a direct interpretation as an exact measure of a commodity aggregate for a second-order flexible commodity aggregator. We start our discussion of commodity aggregates by noting that in the one-commodity case there are at least two natural commodity indexes: The first, and by far more common, is the ratio of the commodity in, say, period 1 to the corresponding commodity in period 0. The second, is the difference between the commodity in period 1 and the corresponding commodity in period 0.

When more than one commodity exists, the natural generalization of the first approach is to specify a commodity index by taking ratios of aggregator functions in period 1 and period 0. In recent years, the quantity-aggregator function of choice has been the distance or deflation function, and most commodity aggregates have been expressed in the form of Malmquist (1953) quantity indexes. I wish to express my commodity index in a form that is natural for the benefit function. Because the benefit function is a directional (and not radial) characterization of preferences, the more natural definition of a commodity index is in terms of differences of the benefit function. Therefore, I define the *Luenberger commodity* *indicator:*

$$X(\mathbf{x}^1, \mathbf{x}^0, u) = B(\mathbf{x}^1, u; \mathbf{g}) - B(\mathbf{x}^0, u; \mathbf{g}).$$

In words, $X(\mathbf{x}^1, \mathbf{x}^0, u)$ represents the difference in the amounts that can be subtracted from each element of $\mathbf{x}^1$ and $\mathbf{x}^0$ and still leave each of them capable of achieving the reference utility level. If this measure is positive, then it makes sense to conclude that, at least in terms of being able to achieve u, $\mathbf{x}^1$ is bigger than $\mathbf{x}^0$. A few comments should be made about $X(\mathbf{x}^1, \mathbf{x}^0, u)$: First, $X(\mathbf{x}^1, \mathbf{x}^0, u)$ is obviously just the difference between two generalized disposable surplus measures. Hence, relative to u, it has welfare implications in its own right. If $X(\mathbf{x}^1, \mathbf{x}^0, u)$ is positive, one might judge the consumer as being better off with $\mathbf{x}^1$. But this judgment depends critically upon the choice of the reference utility level. Second, by the properties of the benefit function it is translation invariant, i.e.,

$$X(\mathbf{x}^1 + \alpha\mathbf{g}, \mathbf{x}^0 + \alpha\mathbf{g}, u) = X(\mathbf{x}^1, \mathbf{x}^0, u).$$

Translation invariance for indicators specified in difference form mirrors homogeneity of degree zero for indexes expressed in ratio form: It makes the choice of the normalized unit invariant to the choice of the origin. Third, $X(\mathbf{x}^1, \mathbf{x}^0, u)$ corresponds exactly to input indicators introduced in Chambers (1996) in the producer context. But most importantly, lacking some further assumptions on either the utility structure, $X(\mathbf{x}^1, \mathbf{x}^0, u)$ is not generally computable because it depends upon the unobservable u.

It turns out that the Luenberger commodity indicator is independent of the reference utility if and only if the utility structure underlying is *translation homothetic*, that is, satisfies the translation property in terms of the vector $\mathbf{g}$. Sufficiency is, of course, obvious. To see necessity, note that for the Luenberger commodity indicator to be independent of the reference utility, it has to be true that:

$$B(\mathbf{x}^1, u; \mathbf{g}) - B(\mathbf{x}^0, u; \mathbf{g}) = f\left(\mathbf{x}^1, \mathbf{x}^0, \mathbf{g}\right).$$

Set $\mathbf{x}^0$ to a reference vector $\bar{\mathbf{x}}^0$to obtain:

$$\begin{aligned} B\left(\mathbf{x}^1, u; \mathbf{g}\right) &= B\left(\bar{\mathbf{x}}^0, u; \mathbf{g}\right) + f\left(\mathbf{x}^1, \bar{\mathbf{x}}^0, \mathbf{g}\right) \\ &= \hat{B}(\mathbf{x}^1, \mathbf{g}) - F^{-1}(u), \end{aligned}$$

which establishes necessity (also see Chambers and Färe (1997)).

Then it follows immediately from what we have already done that if $\hat{B}(\mathbf{x}, \mathbf{g})$ satisfies (2.2), an exact measure of the Luenberger commodity indicator is given by:

$$X(\mathbf{x}^1, \mathbf{x}^0, u) = \frac{1}{2}\left(\hat{\mathbf{p}}^1 + \hat{\mathbf{p}}^0\right) \cdot \left(\mathbf{x}^1 - \mathbf{x}^0\right) \tag{5.1}$$

where $\hat{\mathbf{p}}^k = \frac{\mathbf{p}^k}{\mathbf{p}^k \mathbf{g}}$.

What I wish to demonstrate now, however, is that expression (5.1) applies in an even more general case. Suppose that the consumer's benefit function is given by:

$$B(\mathbf{x}, u; \mathbf{g}) = \sum_{i=1}^{n} a_i x_i + a_u u + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik} x_i x_k + b_{uu} u^2 + u \sum_{k=1}^{n} b_{uk} x_k \tag{5.2}$$

where $b_{ik} = b_{ki}, \sum_{i=1}^{n} a_i g_i = 1$, and $\sum_{i=1}^{n} b_{ik} g_i = 0, k = 1, .., n, and\ \sum_{k=1}^{n} b_{uk} g_k = 0$. Now following Diewert's (1976) approach in the translog price index case (especially result (2.16)), consider evaluating $X(\mathbf{x}^1, \mathbf{x}^0, u)$ at $\hat{u} = \frac{1}{2}\left(u(\mathbf{x}^1) + u(\mathbf{x}^0)\right)$, the average of the two utilities realized from $\mathbf{x}^1$ and $\mathbf{x}^0$, respectively. Applying

Diewert's lemma to (5.2) establishes:

$$X\left(\mathbf{x}^1, \mathbf{x}^0, \hat{u}\right) = \frac{1}{2}\left[\nabla_x B\left(\mathbf{x}^1, \hat{u}; \mathbf{g}\right) + \nabla_x B\left(\mathbf{x}^1, \hat{u}; \mathbf{g}\right)\right] \cdot \left(\mathbf{x}^1 - \mathbf{x}^0\right).$$

Evaluating this expression shows:

$$\begin{aligned} X\left(\mathbf{x}^1, \mathbf{x}^0, \hat{u}\right) &= \frac{1}{2}\left[\nabla_x B\left(\mathbf{x}^1, u(\mathbf{x}^1); \mathbf{g}\right) + \nabla_x B\left(\mathbf{x}^1, u(\mathbf{x}^0); \mathbf{g}\right)\right] \cdot \left(\mathbf{x}^1 - \mathbf{x}^0\right) \\ &= \frac{1}{2}\left(\hat{\mathbf{p}}^1 + \hat{\mathbf{p}}^0\right) \cdot \left(\mathbf{x}^1 - \mathbf{x}^0\right). \end{aligned}$$

Hence, as claimed (5.1) represents an exact commodity indicator for the class of benefit functions given by (5.2). Because this class of benefit functions is second-order flexible, this implies that (5.1) is, in fact, a superlative commodity indicator in addition to being a superlative welfare indicator.

6. Conclusion

Almost seventy years ago, Bowley (1928) derived an exact formula for computing welfare change when the consumer's utility function is quadratic. This formula, given the assumption that marginal utility of income is approximately constant, is easy to compute and can be used to assess the direction of welfare change using only observable data on prices and quantities. As a result, over the years many variations of it have been offered as measures of consumers'surplus or consumer welfare, and it has been routinely used in applied cost-benefit analysis and back-of-the-envelope welfare calculations for generations now.

This paper shows that a slightly rewritten version of the Bennet-Bowley formula, which can be expressed solely in terms of normalized observed prices, is an exact measure of a generalized version of Allais' notion of disposable surplus for an entire class of quadratic utility structures. Hence, the rewritten version of the Bennet-Bowley formula is a superlative cardinal welfare measure, for that general class of utility structures. As such, it can be legitimately used as the basis of aggregate welfare computations. Prior to this finding, it was only known that a suitably normalized version of the Bennet-Bowley measure was an exact ordinal welfare measure, and in some instances, a close second-order approximation to the equivalent or compensating variaton (Diewert, 1976b; Diewert, 1992). It also turns out that the Bennet-Bowley formula can be made an exact commodity indicator for the class of quadratic benefit functions, and thus the Bennet-Bowley formula is a superlative commodity indicator for the class of twice continuously differentiable benefit functions.

The arguments in the current paper have all been developed in terms of $\mathbf{g}^a$ -translation homotheticity. However, it is a routine matter to show that these same exactness and approximation results, but with a slightly different normalization, apply for translatability and disposable surplus measures defined in terms of an arbitrary reference vector. Examples here would include Luenberger's (1996) compensating and equivalent benefits. Hence, appropriately normalized versions of the Bennet-Bowley consumer surplus measures are capable of providing good cardinal welfare measures for a wide variety of economic situations, and thus should be extremely useful in making practical welfare computations.

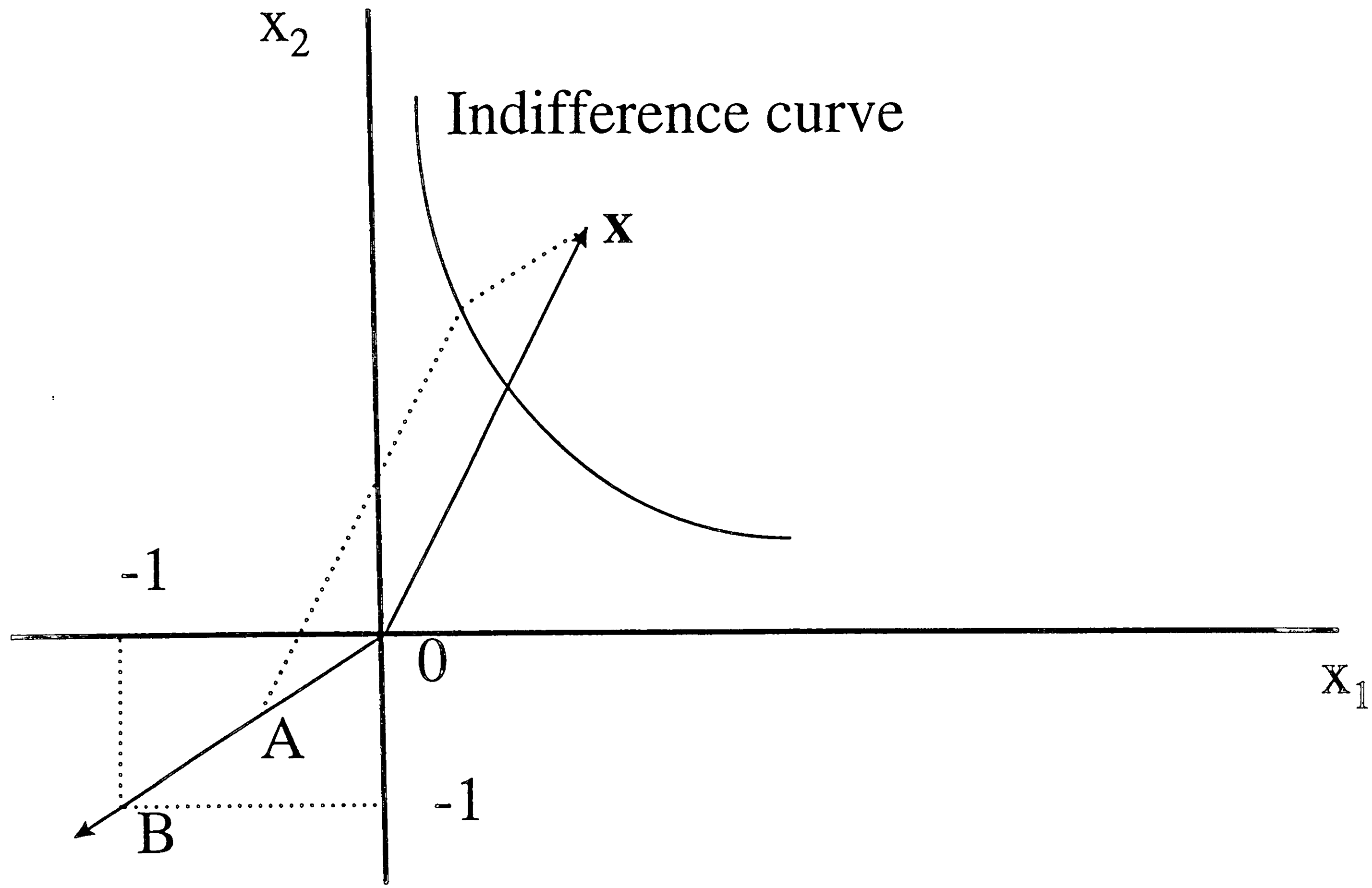

Figure 1: The benefit function

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