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R.G. Chambers, Y. Chung, and R. Färe

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by

R.G. Chambers, Y. Chung, and R. Färe*

1. Introduction

Quite some time ago, Nerlove (1965) suggested a relative efficiency measure based on profit. The essential idea behind Nerlove's (1965) efficiency measure is to decompose profit maximization into two stages: In the first, profit is maximized for a given production function, while in the second stage, the *maximum maximorum* of profit is found by maximizing over all possible production functions. Overall efficiency is then judged by comparing observed profit for a decisionmaking entity to the *maximum maximorum* profit. And, following Farrell (1957), this overall efficiency measure is decomposed into two subsidiary measures: a measure of price or allocative efficiency which consists of comparing observed profit to the profit function for the given production function and technical efficiency which measures the difference between the profit function for the given production function and *maximum maximorum* profit.

A particularly striking aspect of Nerlove's (1965) efficiency measure is that, unlike virtually all other existing efficiency measures, it is expressed in difference (as opposed to ratio) form. Perhaps this explains why Nerlove's (1965) contribution, despite its obvious intuitive appeal, has remained dormant for so long. Apart from a passing reference by Lau and Yotopoulous (1971), the profession has apparently ignored this contribution. The purpose of this paper is to revisit and revitalize Nerlove's efficiency measure in a more modern framework that allows for multiple inputs and multiple outputs in a natural manner while using a representation of

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the technology for which difference (as opposed to ratio) measures are the most natural measures of relative efficiency. That representation is *the directional technology distance function* which generalizes Luenberger's (1992b, 1995) shortage function and Blackorby and Donaldson's (1980) translation function. The directional technology distance function is so general that it can be shown to encompass all known distance function representations of the technology as special cases.

The paper proceeds as follows: In the next section, we introduce the directional technology distance function and discuss its relationship to other functional representations of the technology (input and output distance functions, McFadden's (1978) gauge function, and the directional input distance function (Chambers, Chung, and Färe (1995)). Among other things, we show that the directional technology distance function is a complete function representation of a technology exhibiting free disposal of inputs and outputs. After that, we discuss the dual relationship between the directional technology, distance function and the profit function while providing a streamlined proof of the dual correspondence between the two. We also show that this dual correspondence has all previous dual correspondences as special cases. Then we take up efficiency measures and show how the directional technology distance function can be used to represent the Nerlovian efficiency measure. The final section concludes.

2. Directional and Radial Distance Functions

In this section we define and contrast Shephard's radial distance functions and McFadden's gauge function to three directional distance functions. Shephard's input and output distance functions[1] respectively, measure the largest radial contraction of an input vector and the largest radial

[1] See Shephard (1953, 1970).

expansion of an output vector consistent with each remaining technically feasible. McFadden's gauge function measure the largest radial expansion of a netput vector consistent with feasibility. The directional distance function measures the size of an input and or output vector radially from itself to the technology frontier in a preassigned direction. This direction can differ from the radial direction out of the origin, thus making the directional distance function, more general than Shephards' distance functions or McFadden's gauge function. The directional distance functions that we analyze are related to or derived from the shortage and the benefit functions introduced by Luenberger[2] and Blackorby and Donaldson's (1980) translation function. All functions are here defined in terms of a production technology.

Let $x \in \Re_{+}^{N}$ denote a vector of inputs and $y \in \Re_{+}^{M}$ a vector of outputs, the technology T is given by

(2.1) $T = \{(x, y) \text{ such that } x \text{ can produce } y\}$.

In this paper we make the following assumptions on T.

T.1 T is closed.

T.2 Input and outputs are freely disposable, i.e., if $(x, y) \in T$ and $(x', -y') \geq (x, -y)$ then $(x', y') \in T$.

T.3 There is no free lunch, i.e., if $(x, y) \in T$ and $x = 0$ then $y = 0$.

T.4 T is convex.

The first three properties are imposed throughout the paper, while convexity is only assumed when we discuss dualities. All assumptions are standard and need no further comments.[3]

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[2] See Luenberger (1992a, 1992b, 1994a, 1994b, 1995).

[3] For these and other axioms on the technology. See Färe (1988).

Shephard's input and output distance functions are defined in terms of T as

(2.2) $$D_i(y,x) = \sup_{\lambda}\{\lambda > 0 : (\frac{x}{\lambda}, y) \in T\}, \quad x \in \Re_+^N, y \in \Re_+^M.$$

and

(2.3) $$D_0(x,y) = \inf_{\theta}\{\theta > 0 : (x, y/\theta) \in T\}, \quad x \in \Re_+^N, y \in \Re_+^M$$

respectively. Each of the distance function is a complete characterization of the technology T,[4] i.e.,

(2.4) $$D_i(y, x) \geq 1 \Leftrightarrow (x, y) \in T,$$

(2.5) $$D_0(x, y) \leq 1 \Leftrightarrow (x, y) \in T.$$

Under constant returns to scale the following simple relation holds for the distance functions:

(2.6) $$D_i(y, x) = 1/D_0(x, y).$$

McFadden's gauge function is defined in terms of outputs with inputs entering with a negative sign and outputs with a positive sign. Define the 'mirror' technology

$$T^- = \{(-x, y) : (x, y) \in T\},$$

then McFadden's gauge function can be written as

(2.7) $$H(-x, y) = \inf\{\theta > 0 : \left(-\frac{x}{\theta}, \frac{y}{\theta}\right) \in T^-\} \quad x \in \Re_+^N, y \in \Re_+^M.$$

The most general directional distance function scales inputs and outputs simultaneously. This differs from the definitions of the above distance function. The input distance function is defined by scaling inputs and the output distance function is defined by scaling outputs.

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[4] Färe and Primont (1995, pp. 15, 22) show that weak disposability of inputs and outputs is necessary and sufficient for the input and output distance functions to completely characterize technology.

We define the directional technology distance function as

(2.8) $$\vec{D}_T(x, y; g_x, g_y) = \sup\{\beta: (x - \beta g_x, y + \beta g_y) \in T\},$$

where (g_x, g_y) is a nonzero vector in $\Re_+^N \times \Re_+^M$. This vector determines the *direction* in which $\vec{D}_T(\cdot)$ is defined. Actually because βg_x is subtracted from x, the direction is $(-g_x, g_y)$. Thus this function is defined by simultaneously contracting inputs and expanding outputs. Hence, the directional distance function is a variation on what Luenberger[5] calls the shortage function. Both functions measure the distance in a preassigned direction to the boundary of T, but Luenberger, (1992b, p. 242) sees the distance as a shortage of (x, y) to reach T, while we interpret the distance as an "efficiency measure", i.e., by how much output can be expanded and input contracted and still be feasible.

It is also possible to define a directional technology distance function in terms of the mirror technology T^-

(2.8a) $$\vec{D}_{T^-}(-x, y; g_x, g_y) = \sup\ \{\beta: (-x + \beta g_x, y + \beta g_y) \in T^-\}.$$

We now show that $\vec{D}_T(x, y; g_x, g_y)$ characterizes T.

(2.9) <u>Lemma</u>: Let $(g_x, g_y) \in \Re_+^N \times \Re_+^M$ with $(g_x, g_y) \neq 0$, then

$$\vec{D}_T(x, y; g_x, g_y) \geq 0 \text{ if and only if } (x, y) \in T.$$

Proof: Clearly if $(x, y) \in T$ then $\vec{D}_T(x, y; g_x, g_y) \geq 0$. Thus assume that $\vec{D}_T(x, y; g_x, g_y) \geq 0$. In this case, by definition

$$(x - \vec{D}_T(x, y, g_x, g_y)g_x, y + \vec{D}_T(x, y; g_x, g_y)g_y) \in T, \text{ and any}$$

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[5] Luenberger (1992b, 1995).

$$(x,-y) \geq (x - \vec{D}_T(x,y;g_x,g_y)g_x, -y - \vec{D}_T(x,y;g_x,g_y)g_y)$$

must also belong to T by T.2.

Q.E.D.

There are a few special cases of $\vec{D}_T(x,y;g_x,g_y)$ that are of interest. First, if we take $g_y = 0$, then we get

(2.10) $$\vec{D}_T(x,y;g_x,0) = \vec{D}_i(y,x;g_x),$$

where $\vec{D}_i(y,x;g_x)$ is the directional input distance function as defined by Chambers, Chung and Färe (1995). This distance function is the multioutput version of the benefit function, introduced by Luenberger.[6] Furthermore, if $g_x = x$, then

(2.11) $$\vec{D}_T(x,y;x,0) = 1 - 1/D_i(y,x),$$

thus the directional technology distance function collapses to the input distance function.

By choosing $g_x = 0$, an output oriented directional distance function is obtained,[7] if further $g_y = y$ thus we have a relation between $\vec{D}_T(\cdot)$ and Shephard's output distance function, namely

(2.12) $$\vec{D}_T(x,y;0,y) = (1/D_0(x,y)) - 1.$$

It also follows immediately that

(2.13) $$\vec{D}_{T^-}(-x,y;-x,y) = \sup\{(\beta+1):(\beta+1)(-x,y) \in T^{-1}\} - 1$$
$$= \frac{1}{H(-x,y)} - 1.$$

[6] Luenberger (1992a, 1995).

[7] See Chung (1996) for directional output distance functions.

Expressions (2.9) - (2.13) show the directional technology distance function is a complete generalization of Shephard's distance functions, McFadden's gauge function, and the directional input and output distance functions.

Before we give $\vec{D}_T(\cdot)$'s basic properties, we illustrate pictorially how it is defined. Assume that one input x is used to produce output y, thus the technology T can be illustrated as in Figure 1. For simplicity, we choose the "direction" as (1, 1) so that $\vec{D}_T(x, y; 1, 1)$ is given by the ratio OA/OB.

We now turn to the most important properties that the directional distance function has. These are summarized by the following lemma.

(2.12) Lemma: $\vec{D}_T : \Re_+^N \times \Re_+^M \times \Re_+^N \times \Re_+^M \rightarrow \Re$

a) $\vec{D}_T(x - \alpha g_x, y + \alpha g_y; g_x, g_y) = \vec{D}_T(x, y; g_x, g_y) - \alpha$,

b) $\vec{D}_T(x, y; \lambda g_x, \lambda g_y) = \frac{1}{\lambda} \vec{D}_T(x, y; g_x, g_y), \ \lambda > 0$

c) $y' \geq y \Rightarrow \vec{D}_T(x, y'; g_x, g_y) \leq \vec{D}_T(x, y; g_x, g_y)$

d) $x' \geq x \Rightarrow \vec{D}_T(x', y; g_x, g_y) \geq \vec{D}_T(x, y; g_x, g_y)$

e) if T is convex, $\vec{D}_T(x, y; g_x, g_y)$ is concave in (x, y).

The proofs are similar to those in Chambers, Chung and Färe (1995) and thus omitted here. Also note a similar lemma could be proved easily for $\vec{D}_{T^-}(-x, y; g_x, g_y)$.

3. Dualities

Shephard (1953, 1970) proved that the input distance function is dual to the cost function and that the output distance function and the revenue are duals. McFadden later demonstrated that the gauge function is dual to the restricted profit function. In this section we define the profit function and show that it is dual to the directional technology distance function.

Let $p \in \Re^{M}_{++}$ denote a vector of output prices and $w \in \Re^{N}_{++}$ a vector of input prices. The profit function is defined for the technology T as

(3.1) $$\pi(p, w) = \sup_{(x,y)\geq 0} \{py - wx: (x, y) \in T\}.$$

This function models "maximal" feasible profit. By Lemma (2.8) we may also write this function in terms of the directional distance function, namely

(3.2) $$\pi(p, w) = \sup_{(x,y)\geq 0} \{py - wx: \vec{D}_T(x, y; g_x, g_y) \geq 0\}.$$

Constrained optimization problem (3.2) can be converted into the following unconstrained problem given that we impose differentiability of the solution:

(3.3) $$\pi(p, w) = \sup_{(x,y)\geq 0} \{py - wx + \vec{D}_T(x, y; g_x, g_y)(pg_y + wg_x)\}.$$

The proof is found in the appendix, but the intuition is straightforward: The input-output vector

$$(x - \vec{D}_T(x, y; g_x, g_y)g_x,\ \ y + \vec{D}_T(x, y; g_x, g_y)g_y)$$

belongs to the technology T, thus in general the following inequality holds,

(3.4) $$\pi(p, w) \geq py - wx + \vec{D}_T(x, y; g_x, g_y)(pg_y + wg_x).$$

From this inequality (3.3) can be obtained and if T is convex we also have

(3.5) $$\vec{D}_T(x, y; g_x, g_y) = \inf_{(p,w)\geq 0} \left\{ \frac{\pi(p, w) - (py - wx)}{pg_y + wg_x} \right\}.$$

Thus the following duality exists between the directional technology distance function and the profit function holds[8]

$$
(3.6)\qquad
\begin{aligned}
\pi(p,w) &= \sup_{(x,y)\geq 0}\{py - wx + \vec{D}_T(x,y;g_x,g_y)(pg_y + wg_x)\}\\
\vec{D}_T(x,y;g_x,g_y) &= \inf_{(p,w)\geq 0}\left\{\frac{\pi(p,w)-(py-wx)}{pg_y + wg_x}\right\}.
\end{aligned}
$$

Färe and Primont (1995) proved duality theorems between Shephard's distance functions and the profit function. These duality theorems are special cases of (3.6) as we show next. First take $g_x = 0$ and $g_y = y$, then it follows from (2.11) and (3.6) that

$$
(3.7)\qquad
\begin{aligned}
\pi(p,w) &= \sup_{(x,y)\geq 0}\{py - wx + \vec{D}_T(x,y;0,y)py\}\\
&= \sup_{(x,y)\geq 0}\left\{py - wx + \left(\frac{1}{D_0(x,y)} - 1\right)py\right\}\\
&= \sup_{(x,y)\geq 0}\left\{\frac{py}{D_0(x,y)} - wx\right\}
\end{aligned}
$$

and

$$
D_0(x,y) = \sup_{(p\cdot w)\geq 0}\left\{\frac{py}{\pi(p\cdot w)+wx}\right\}.
$$

Furthermore if we choose $g_x = x$ and $g_y = 0$ their second duality theorem between the input distance function and the profit function follows.

[8] The proof of this duality statement can be deduced from Luenberger (1992a).

$$\pi(p,w) = \sup_{(x,y)\geq 0}\left\{py - \frac{wx}{D_i(y,x)}\right\}$$

(3.8)

$$D_i(y,x) = \sup_{(p\cdot w)\geq 0}\left\{\frac{wx}{py - \pi(p,w)}\right\}.$$

4. Efficiency Measurement

Most of the efficiency-measurement literature measurement is based on the input and output distance function. This is in the tradition of Debreu (1951) and Farrell (1957) that used the inverse of these functions as measures of technical efficiency. Here we show how the directional distance function $\vec{D}_T(x, y; g_x, g_y)$ can be adopted as a measure of technical efficiency. The usual distance functions measure technical efficiency either in the input direction or in the output direction. The measure we adopt simultaneously contract inputs and expand outputs.[9]

In addition to technical efficiency we introduce new measures of profit and relate them to Nerlove's measures of relative efficiency. All measures are derived through the inequality (3.4).

Our first profit-based efficiency measure is:

$$NE = \frac{\pi(p,w) - (py - wx)}{pg_y + wg_x}. \tag{4.1}$$

This measure is the difference between the maximal potential profit $\pi(p, w)$ and realized profit (py - wx). The difference is normalized by the sum $(pg_y + wg_x)$. Hence, apart from the price normalization, this measure equals Nerlove's earlier notion of overall efficiency, and we shall refer to it as Nerlovian efficiency in what follows. The price normalization, which follows

[9] Independently of Luenberger, W. Briec (1995) has introduced a graph measure of technical efficiency that is a special case of the directional distance function.

naturally from the duality between $\pi(p, w)$ and $\vec{D}_T(x, y; g_x, g_y)$, conveniently solves the linear homogeneity problem that Nerlove recognized about his measure. The direction (g_x, g_y) can be chosen at the realized input output vector (x, y) in which case we need not preassign any direction.

Technical efficiency, as mentioned above, is measured by the directional distance function,

$$\text{(4.2)} \qquad TE = \vec{D}_T(x, y; g_x, g_y)$$

and again (g_x, g_y) can be chosen as the observed input output vector. In this case (4.2) coincide with Briec's approach. Recall, that $\vec{D}_T$ provides a direct measure of how far (x, y) must be projected along (g_x, g_y) to reach the frontier of T. Hence, it offers a natural measure of inefficiency. Also note that it does not correspond to Nerlove's measure of technical inefficiency.[10] Finally, allocative efficiency is defined as the gap in the inequality (3.4), namely

$$\text{(4.3)} \qquad AE = \frac{\pi(p, w) - (py - wx)}{pg_y + wg_x} - \vec{D}_T(x, y; g_x, g_y).$$

The three last expression yields the following decomposition of profit efficiency,

$$\text{(4.4)} \qquad NE = AE + TE.$$

Several comments about (4.2), (4.3) and (4.4) are appropriate: For any feasible (x, y) NE is always nonnegative. Moreover, any feasible (x, y) must also have both (4.2) and (4.3) nonnegative. Thus, if a feasible point is Nerlovian efficient, it must be both technically efficient and allocatively efficient.

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[10] Nerlove's measures of allocative and technical inefficiency are expressed in ratio and not difference form.

Appendix

Proof of (3.3)[11]

The Lagrangian problem associated with (3.2) is

$$\pi(p,w)=\pi(p,w;0)=\sup_{(x,y)\geq 0} py-wx+\lambda(0-\vec{D}_T(x,y;g_x,g_y)).$$

Now consider the problem,

$$\pi(p,w;\alpha)=\sup_{(x,y)\geq 0}\{py-wx: \vec{D}_T(x,y;g_x,g_y)\geq\alpha\}, \quad (*)$$

where α, g_x and g_y are chosen so that $(x-\alpha g_x, y+\alpha g_y)=(\hat{x},\hat{y})\geq 0$. Using part a) of Lemma (2.12) (*) can be written as

$$\pi(p,w;\alpha)=\sup_{(x,y)\geq 0}\{p(y+\alpha g_y)-w(x-\alpha g_x): \vec{D}_T(x-\alpha g_x, y+\alpha g_y; g_x,g_y)\geq 0\}-\alpha(pg_y+wg_x).$$

or

$$\pi(p,w;\alpha)=\pi(p,w;0)-\alpha(pg_y+wg_x).$$

Thus $$\frac{\partial\pi(p,w;\alpha)}{\partial\alpha}=-(pg_y+wg_x)$$

and from the Lagrangian expression associated with (*) we have

$$\frac{\partial\pi(p,w;\alpha)}{\partial\alpha}=\frac{\partial}{\partial\alpha}(py-wx+\lambda(\alpha-\vec{D}_T(x,y;g_x,g_y)))=\lambda$$

and since $\pi(p,w)=\pi(p,w;0)$ we have

$$\pi(p,w)=\sup_{(x,y)\geq 0}\{py-wx+\vec{D}_T(x,y;g_x,g_y)(pg_y+wg_x)\}.$$

Q.E.D.

[11] A more general proof can be provided along the lines of Luenberger (1992b, p. 243).

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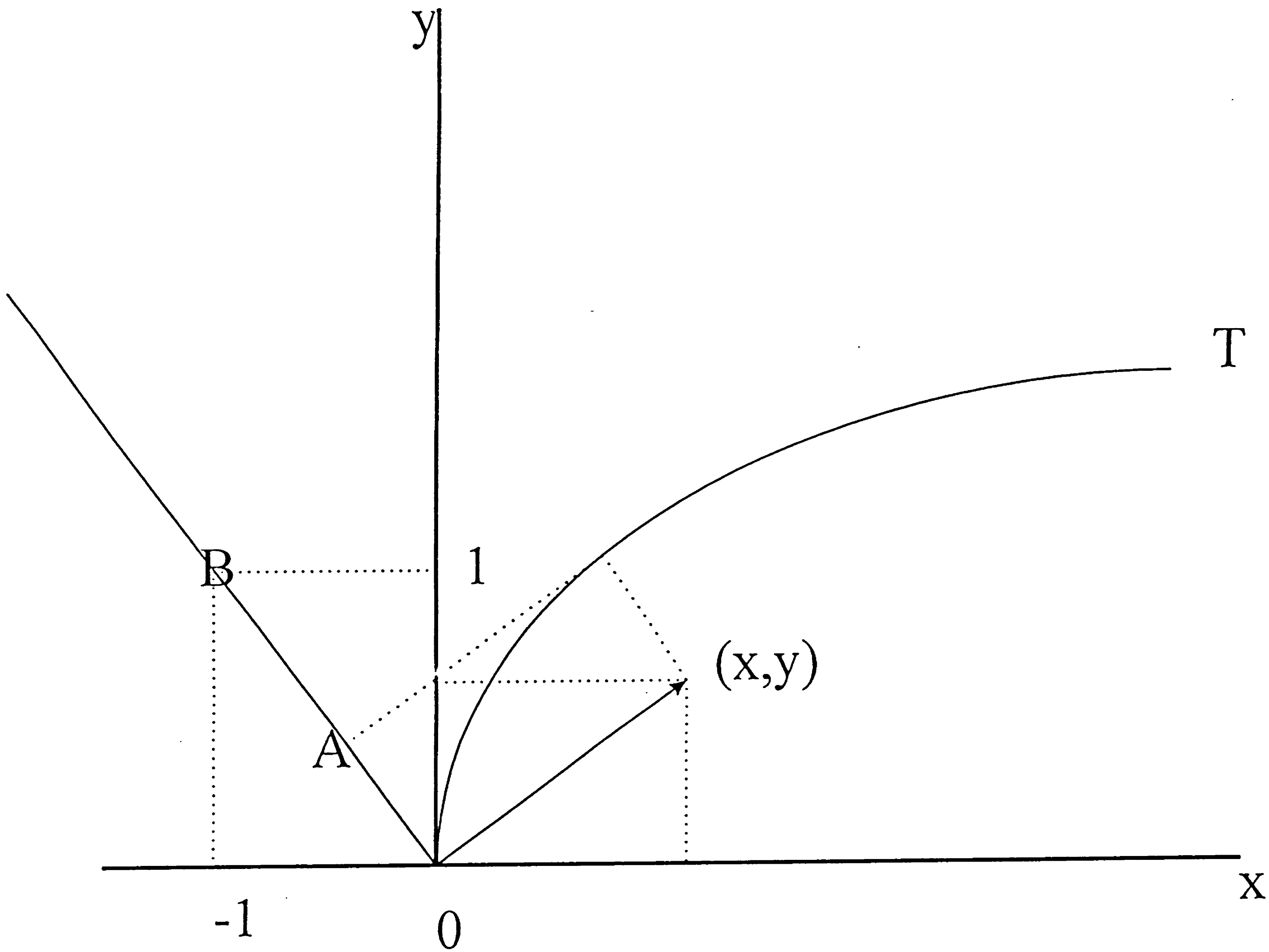

Figure 1: Directional Technology Distance Function