INEQUALITY MEASURES AND INTEGRABLE DEMAND SYSTEMS

by

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Abstract

Exact aggregation of demand systems using inequality indexes satisfying Kolm's central scaling and translation invariance axioms is considered. Only a single micro demand system, a form of Lewbel's LINLOG, is both integrable and aggregable using an inequality index satisfying scale invariance, and the only admissible class of inequality indexes is the class of transformations of the simple geometric mean of income shares. When the domain of the inequality index includes zero incomes and scaling invariance is imposed, only a trivial inequality index, with micro demands the Gorman Polar Form, aggregates. When the translation invariancaxiomom is imposed, integrable and aggregable demands must be a form of the QES developed by Howe, Pollak, and Wales, and the inequality index must be a transformation of the sample variance. When Kolm's compromise axiom is imposed, the inequality index is restricted to those proportional to the standard deviation of income. Welfare and representative consumer implications follow.

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This paper considers aggregation of demand systems using aggregate income and indices of income inequality. Letting $\mathbf{q}_i = \mathbf{d}^i(\mathbf{p}, y_i, \mathbf{z}_i)$ be the vector of individual i's demands where $\mathbf{p}$ is a vector of commodity prices, y_i is individual i's expenditure, and $\mathbf{z}_i$ is a vector of individual attributes, then aggregate demand can be represented $\mathbf{Q} = \Sigma_i \mathbf{q}_i$. The specific question that this paper asks is : 'When can one also write $\mathbf{Q} = g(\mathbf{p}, \mu, I)$?', where μ is aggregate income and $I = I(y_1, \ldots, y_n)$ is an inequality measure satisfying some of the standard axioms imposed upon inequality measures by Kolm (1976a, 1976b) and others. Demand forms permitting this operation are said to satisfy *inequality exact aggregation*.

One might reasonably ask why inequality exact aggregable demands are of interest. One answer is offered by recognizing that ever since the pathbreaking work of Atkinson (1970) on inequality measurement and social welfare, the routine approach to generating and calculating inequality measures is to induce them from explicit social-welfare functions. This approach is unabashedly normative. However, income inequality also has positive implications for consumer behavior as Engel's Law readily attests. And yet, to our knowledge, there does not exist a firm theoretical link between these social-welfare functions and observable consumer behavior[1]. This paper completes that link in the following fashion: First we posit basic properties that almost all inequality measures possess. To these basic properties we then append several additional criteria based upon the seminal work of Kolm (1976a, 1976b) on 'relative' and 'absolute' measures of inequality. We then deduce which inequality measures are consistent with these criteria and inequality exact aggregation. After that, it is a simple procedure to deduce social-welfare functions which are consistent with inequality exact aggregation.

Another reason to be interested in inequality exact aggregation is more pragmatic: The work of a long line of aggregation theorists (Gorman, 1953; Muellbauer 1975, 1976; Lau, 1982; Jorgenson, Lau, and Stoker, 1982; Stoker, 1984; and Lewbel, 1988, 1989) has taught us that nonlinearities in Engel curves imply that 'distribution matters'. So characterizing aggregate demand properly requires the use of some aggregate income statistics beyond aggregate income. However, the dispersion statistics that are most appropriate for, say, the aggregate AIDS system and its various permutations, are not routinely computed and reported by government agencies. Therefore, the theoretical results of previous aggregation studies have proven difficult to implement empirically. And because aggregate income is routinely computed and reported by government agencies, for practical reasons a fair amount of effort has been devoted to identifying restrictions on the income distribution (e.g., mean scaling) which can permit aggregate demand to be expressed solely in terms of aggregate income (Lewbel, 1988).

Another approach to this problem, which has been apparently neglected, is to determine which, if any, existing measures of income dispersion would be useful in characterizing aggregate demand. Beyond a doubt, the broadest existing class of income dispersion measures is the class of inequality indices which have been calculated at least since the time of Lorenz (1905). This paper tackles this latter problem of utilizing existing income dispersion measures in representing aggregate demand by focusing on the scaling and translation invariance properties suggested by Kolm (1976a). Alternatively, we ask: 'Which if any of the so-called 'relative' ('absolute') inequality measures can be used to exactly aggregate demand?'

We find that only a single class of integrable micro demand functions is consistent with exact aggregation using a relative inequality index--the LINLOG demand system: The only inequality index that aggregates for the relative case is some transformation of the geometric mean of income shares. When the domain of the inequality index is extended from R^n_{++} to R^n_{+}, the class of demand forms aggregable using a relative inequality measure reduces to the Gorman Polar Form implying that only trivial inequality indexes are consistent with exact aggregation. When Kolm's absolute (translation invariance) axiom is imposed, the only integrable class of demand functions consistent with exact aggregation is the quadratic expenditure system (Howe, Pollak, and Wales, 1979) suggesting the sample variance as an appropriate inequality index to use in aggregation. When Kolm's compromise axiom is imposed, the quadratic expenditure system remains the only integrable class of demands, but the inequality index is now identified up to a scale parameter as the sample standard deviation of income.

In each case, we also show that the intersection of *inequality exact aggregation* and integrability gives necessary and sufficient conditions for the inclusion of demographic effects into the general family of demand systems. Such conditions are essential for the specification and estimation of either micro demand systems or aggregate demand systems.

In what follows, we first specify our notation and our assumptions. Then we develop the class of demand forms and inequality indices which are consistent with inequality-exact aggregation using, respectively, a relative, absolute, and compromise index. Social welfare functions consistent with each respective class of demand forms and inequality index are then deduced, and in our closing section we point out how our results can be used to link Lewbel's (1989) representative-consumer results with a specific social-welfare function.

II. Assumptions and Notation

Let the domain of the income vector, $\mathbf{y}$, be Ω. Two different domain assumptions will be considered for Ω: $\mathbf{y} \in R^n_{++}$ and the less restricted domain $\mathbf{y} \in R^n_{+}$, where R_{++} is the set of positive reals and R_+ is the set of nonnegative reals. In both cases, the number of consumers, n, remains fixed.

Micro demands for commodity j by individual i are given by smoothly continuous functions $d^i_j(\mathbf{p}, y_i, \mathbf{z}_i)$ where $\mathbf{z}_i$ is a vector of individual-specific demographic variables, and $\mathbf{p} \in R^m_+$ are homogeneous prices. Thus, our method of incorporating individual specific effects in demands slightly generalizes that suggested by Lewbel (1989). However, it is often notationally cumbersome to carry all these arguments in the manipulations that follow. Therefore, for convenience sake and where there can be no confusion, the image of individual demands will be written as $d_j^i(\mathbf{p}, y_i, \mathbf{z}_i) = q_{ij}(y_i)$ with $\mathbf{q}_i(y_i) \in R_+^n$ denoting the vector of individual i's demands. $I(\mathbf{y})$ is a continuous inequality measure where $I: \Omega \to R$. Aggregate income is denoted by $\mu = \sum_i y_i$. The quantity-aggregation rule is:

$$Q_j(\mu, I(\mathbf{y})) = \sum_i^n q_{ij}(y_i). \tag{1}$$

where $Q_j: R_+ \times R \to R_+$ is the aggregate demand for commodity j. I does not have a j subscript: The same inequality index aggregates all j = 1, ..., m demands.

In determining which inequality indexes to use in aggregation, our focus is on Kolm's (1976a) scaling and translation invariance properties. A subset of Kolm's axioms includes:

(A1) Intensive inequality or positive homogeneity of degree zero (Scale Invariance):

$$I(\lambda \boldsymbol{y}) = I(\boldsymbol{y}) \quad \forall\ \lambda > 0.$$

(A2) Translation invariance:

$$I(\boldsymbol{y} + \psi \boldsymbol{e}) = I(\boldsymbol{y}) \quad \forall\ \psi \in R\ ,\ I(\boldsymbol{y} + \psi\ \boldsymbol{e}) > 0.$$

where $\mathbf{e}$ is an n-vector of ones.

Axioms (A1) and (A2) correspond, respectively, to the so-called 'relative' and 'absolute' inequality axioms. Many common inequality measures, such as the Gini and Theil's income entropy, satisfy (A1). However, as Kolm (1976a) strongly argues, many would view a doubling of incomes as increasing inequality, and this observation led him to consider indexes satisfying (A2).

Kolm (1976a,b), Blackorby and Donaldson (1980), and Bossert and Pfingsten (1990) among others have also investigated compromise inequality indexes. Here $I(\mathbf{y})$ is defined as a compromise absolute inequality index if:

(A3) $I(\lambda \mathbf{y} + \psi \mathbf{e}) = \lambda I(\mathbf{y}), \quad \lambda > 0, \quad \psi \geq 0.$

In addition to (A1)-(A3), Kolm[2] imposed two forms of additivity on $I(\mathbf{y})$ (one each for relative and absolute indexes) that enabledled him to deduce specific functional structures. Here, we impose no additivity condition directly on $I(\mathbf{y})$, but we do impose the form of additivity on aggregate demands that is most natural for empirical measurement in a regression context (Blinder,1975; Simmons, 1980; Lewbel,1989) and which implies structures closely paralleling Kolm's additivity assumptions[3]:

(A4) $Q_j(\mu, I(\mathbf{y})) = Q_{oj}(\mu) + Q_{1j}(I(\mathbf{y})), \ j = 1, \ldots, m,$

where Q_{oj} and Q_{1j} are continuous.

The only other condition that we impose on the inequality index is the condition of anonymity or symmetry:

(A5) I is left unchanged by a permutation of the y_i's, i.e., $I(\mathbf{Py}) = I(\mathbf{y})$ where $\mathbf{P}$ is any permutation matrix.

No axiomatization of an inequality index with which we are familiar fails to impose (A5). Notice, however, that (A5) is traditionally strengthened by the addition of several other axioms including Dalton's principle of transfers and a normalization property (e.g., all individuals having the same income implies $I(\mathbf{y})=0$). Anonymity and Dalton's principle of transfers together imply Schur-convexity (Marshall and Olkin, 1979; Dasgupta et al.,1973). Schur-convexity imposes an aversion to income inequality upon the index, namely, $I(\mathbf{By})\leq I(\mathbf{y})$ where $\mathbf{B}$ is any bistochastic matrix. However, in all instances our structural results only require anonymity of the index while still implying an inequality index that is a continuous transformation of a normalized, Schur-convex function, i.e., *normalized Schur-convexity is an implication of inequality-exact aggregation* with a symmetric index and need not be assumed. Thus, to preserve generality we only impose anonymity. However, as the paper progresses we shall remark upon how imposing something more than anonymity will sharpen results.

Summarizing, our criterion for inequality-exact aggregation is that the sum of individual demands be exactly aggregable using two indexes: aggregate income and an inequality measure satisfying either (A1), (A4) and (A5), (A2), (A4), and (A5), or (A3), (A4) and (A5).

In most instances, the demands that are consistent with inequality-exact aggregation are not generally integrable, and the conditions required to make them integrable are not transparent. Thus, integrable systems are stated as separate results and require the following additional assumption:

(A6) The once differentiable demand vector, $\mathbf{q}_i$ is obtained as

$$\mathbf{q}_i = \operatorname{argmax} \{U_i(\mathbf{q}_i) | \mathbf{p}\mathbf{q}_i \le y_i\} \quad (i = 1, \ldots, n),$$

where U_i is the i^{th} consumer's utility.

As is common (e.g., Lewbel, 1987), the integrability results focus on homogeneity and Slutsky-symmetry rather than the negative semidefiniteness condition. Notice, in particular, that the only important restriction, beyond integrability of course, imposed by (A6) is that demands be differentiable in $\mathbf{p}$. As will be apparent from what follows, all demands that are inequality-exact aggregable are differentiable in y.

III. Results

Previous work on aggregation indicates that only one micro demand form will generally be consistent with (1)--some form of Muellbauer's PIGL system (Lau, 1982; Gorman, 1990; Chambers and Pope, 1992). This section studies how each of the three invariance properties (A1 - A3) under inequality exact aggregation imposes more specific structure upon micro demands, macro demands, and the inequality index.[4]

Relative Indexes and Aggregation

Even though there is disagreement about which of (A1)-(A3) should characterize an inequality index (or poverty index), the relative axiom (A1) seems to have been of the most historical interest. Indexes like the Gini, and its pictorial analogue--the Lorenz curve, which are positively homogeneous of degree zero, long formed the core of empirical inequality measurement. Economists also have a natural affinity for measures which are unit-free making relative indexes both very intuitively appealing and very convenient for measuring inequality across various currencies and income levels. This predilection is even shared by noneconomists who have considered inequality measures: For example, Marshall and Olkin (1979) uncritically accept (A1) as a property that an inequality index should possess

Under relative aggregation, only one form of micro demand structure will aggregate, and the inequality index must be a monotonic transformation of the simple geometric mean of income shares: (All proofs are in the Appendix)

Theorem 1: Under (A1), (A4), and (A5), there exists a $Q_{oj}: R^n \to R$, $Q_{1j}: R \to R$, $I : R^n_{++} \to R$ such that (1) holds iff:

$$q_{ij}(y_i) = \alpha_{ij} + B_j\, y_i + \frac{c_j}{n}\log(y_i), \quad i=1,\ldots,n;\ j=1,\ldots,m,$$

$$Q_j = \alpha_j + \beta_j\, \mu + \frac{c_j}{n}\left(\sum_{i=1}^{n} \log(y_i)\right) \quad j=1,\ldots,m$$

$$Q_{1j}(I) = (\alpha_j - D_j) + c_j\left[\frac{1}{n}\left(\sum_{i=1}^{n} \log(y_i)\right) - \log(\mu)\right]; \quad j=1,\ldots,m,$$

where $\alpha_j = \Sigma_i \alpha_{ij}$, β_j, D_j, and c_j ($i = 1, \ldots, n$; $j = 1, \ldots, m$) all belong to R.

Several comments are pertinent here: First, assuming that the difference between mean income and the product of mean income and I(**y**) is additively separable, Kolm (1976a) isolates two possible relative inequality indexes--one minus the geometric mean of relative incomes (defining i's relative income by ny_i/μ), and one minus the Hardy-Littlewood-Polya (1952) generalized mean of relative incomes. Theorem 1 essentially eliminates the latter of these two inequality measures from consideration because the square bracketed term in Q_{1j} is the logarithm of the geometric mean of relative incomes minus log(n). Because the extended PIGL demand system aggregates to the Hardy-Littlewood-Polya generalized mean, Theorem 1 effectively rules out the extended PIGL demand system as inequality-exact aggregable. Second, the only role that anonymity plays in Theorem 1 is to ensure that the slope coefficient for the log y_i expression is the same for all consumers. Absent anonymity, these slope coefficients could differ across individuals. Thus, there would be more room for individual specific effects to play an important role in the micro demands if anonymity were abandoned.

For positive incomes, each micro demand in Theorem 1 is a generalization of the Gorman Polar Form. Thus, marginal propensities to consume are not constant with respect to individual income. However, marginal propensities to consume are identical across individuals at identical income levels. And even though there is a unique form of micro and macro behavior, there is not a unique representation of I. However, $Q_{1j}(I)$ is unique. So with little true loss of generality, taking I to be the negative of the square bracketed term in Q_{1j}, for example, confirms that it is homogeneous of degree zero in income and that it satisfies Schur-convexity and normalization.

Removing some of the ambiguity about the form of $I(\mathbf{y})$ can always be achieved by imposing further structure upon the inequality-exact aggregation problem. Suppose, for example, that (A4) is strengthened from anonymity to Schur-convexity, and that we also require that each Q_{1j} be strictly monotonic in $I(\mathbf{y})$. Because the square bracketed term in Q_{1j} is strictly Schur-concave, it then follows immediately from Marshall and Olkin (p. 61, Table 1) that c_j must be negative if Q_{1j} is increasing and positive if Q_{1j} is decreasing. In any case, either $(\log(\mu) - (1/n)\sum_i \log(y_i))$ or minus the geometric mean of income shares can always be interpreted as a Schur-convex inequality index with almost no true loss of generality since $I(\mathbf{y})$ must always be a transformation of either one. Taking $I(\mathbf{y})$ to be $(\log(\mu) - (1/n)\sum_i \log(y_i))$ simply amounts to a slight renormalization of the problem.

The last observation implies that the parameters c_j, which are estimable from micro data, play an important role in determining how aggregate demands respond to changes in inequality. To obtain more information on the parameters in

Theorem 1, note that the micro demand system in Theorem 1 is a special case of Lewbel's (1987) LINLOG form. Aggregability implies that only the "intercept" in the LINLOG form in Theorem 1 is individual specific. Thus:

Theorem 2: The micro demands in Theorem 1 satisfy (A6) iff

$$d_j^i(\mathbf{p}, y_i, \mathbf{z}_i) = (h(\rho(\mathbf{p})t(\mathbf{z}_i) + \Lambda(\mathbf{z}_i), \mathbf{z}_i) - \log(\Upsilon(\mathbf{p})/t(\mathbf{z}_i))$$

$\rho_j(\mathbf{p})\Upsilon(\mathbf{p}) + (\Upsilon_j(\mathbf{p})/\Upsilon(\mathbf{p}))y_i + \rho_j(\mathbf{p})\Upsilon(\mathbf{p}) \log(y_i), i=1, \ldots, n;$ *$j=1,\ldots, m$, and*

$\rho(\mu\mathbf{p}) = \rho(\mathbf{p})$, $\Upsilon(\mu\mathbf{p}) = \Upsilon(\mathbf{p})$ $\mu > 0$, *where $\rho(\mathbf{p})$ and $\Upsilon(\mathbf{p})$ are differentiable functions mapping $R_+^m \to R$, $(i = 1, \ldots, n; j = 1, \ldots, m)$, and j subscripts on ρ and Υ denote the derivative with respect to the j^{th} price.*

Theorem 2 shows that only two price functions are required to represent demands. That is, in Gorman's (1981) terminology, it is a rank-2 demand system. Demographic variables can enter the intercept, α_{ij} (see Theorem 1), in three ways: t, Λ, and h, and the impact of inequality upon the jth aggregate demand is effectively captured by the term $\rho_j(\mathbf{p})\ \Upsilon(\mathbf{p})$.

The domain of $I(\mathbf{y})$, however, is crucial to obtaining the generalization of the Gorman Polar Form in Theorems 1 and 2. In Theorem 1, no individual is allowed to have zero income. However, if we extend the domain of $I(\mathbf{y})$ to include the axes, we have the following corollary to Theorem 1.

Corollary: Under (A1), (A4), and (A5), there exists a $Q_{oj}: R^n \to R$, $Q_{1j}: R \to R$, $I: R^n_+ \to R$ *such that (1) holds iff:*

$$Q_j = \alpha_j + \beta_j \mu, \quad j=1,\ldots,m$$
$$q_{ij}(y_i) = \alpha_{ij} + \beta_j y_i, \quad i=1,\ldots,n; \; j=1,\ldots,m,$$

$$Q_{1j}(I) = (\alpha_j - D_j), \quad j=1,\ldots,m,$$

where $\alpha_j = \Sigma_i \alpha_{ij}$, D_j, *and* β_j $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ *are real constants with respect to* y_i.

Thus, if we permit zero incomes, only a noninformative, constant inequality index is consistent with rightist inequality-exact aggregation. Integrability under aggregability results for the demands in Corollary 1 are well-known and need not be repeated here.

Absolute Indexes and Aggregation

Kolm (1976a) has suggested that many might not accept axiom (A1). For example, if the mean and variance of income increase due to a proportional scaling, many might consider inequality to have risen. Kolm postulates that a reasonable definition of constant inequality is represented by the index being invariant to translations of the income distribution toward the equal-income ray. Put in more geometric terms, the income distribution represented by $\mathbf{y} + \psi\mathbf{e}$ is inequality-equivalent to its projection onto $\gamma\mathbf{y}$ ($\gamma > 0$) along e.

Interestingly, only a single family of individual demand functions is consistent with this axiom and inequality-exact aggregation:

Theorem 3: Under (A2), (A4), and (A5), there exists a $Q_{oj}: R^n \to R$, $Q_{1j}: R \to R$, $I: R^n_+ \to R$ such that (1) holds iff:

$$q_{ij}(y_i) = \alpha_{ij} + \beta_j y_i + \frac{c_j}{2} y_i^2, \quad i=1,\ldots,n;\ j=1,\ldots,m,$$

$$Q_j = \alpha_j + \beta_j \mu + \frac{c_j}{2}\sum_{i=1}^{n} y_i^2 \quad j=1,\ldots,m$$

$$Q_{1j}(I) = (\alpha_j - D_j) + \frac{c_j}{2}\left[\sum_{i=1}^{n} y_i^2 - \frac{\mu^2}{n}\right], \quad j=1,\ldots,m$$

where $\alpha_j = \Sigma_i\alpha_{ij}$, α_{ij}, β_j, c_j ($i = 1, \ldots, n$; $j = 1, \ldots, m$) all belong to R.

Inequality-exact aggregation with a absolute index requires that micro-demands be quadratic in income. Further, the bracketed term in Q_{1j} is n times the sample variance of income. Hence, aggregate demands depend on the mean income and the sample variance, or alternatively only on the first two moments about zero.

As with the results for relative indexes, the actual inequality index remains arbitrary to some extent. However, it is unique to the extent that it must be some transformation of the sample variance. This result is to be contrasted with Kolm's (1976a) finding that the only acceptable absolute inequality index is one minus the log of a sum of exponential functions of the difference between individual and mean income. The difference emerges directly from (A4). If (A4) were replaced with a requirement identical to Kolm's (1976a), i.e.,

$$\mu - I(\mathbf{y}) = F(\textstyle\sum_i q_{ij}(y_i)),$$

for all j, we would get the exact same form that Kolm (1976a) isolates. However, that would also require micro demands to be expressible as:

$$q_{ij}(y_i) = a_{ij} + b_j y_i + c_j \exp(g y_i).$$

Therefore, by the results reported in Lewbel (1987), the only integrable class of demands that could inequality-exact aggregate with an absolute inequality index would be the Gorman-Polar form and the inequality index would be the trivial index--a constant.

As with our discussion of the relative-inequality aggregation, because the sample variance is Schur-convex and satisfies normalization, little true generality is lost by simply taking $I(\mathbf{y})$ to be the sample variance. Moreover, the only role that anonymity plays in the derivation of $I(\mathbf{y})$ is to ensure that the slope coefficient for the quadratic term in individual demands is identical across individuals.

Examination of the consumer responses in Theorem 3 shows that demands are another special case of the Gorman-Engel forms--the Quadratic Expenditure System, or QES. Integrable versions of these demands have been developed by Howe, Pollak, and Wales (1979), and more recently by Van Daal and Merkies (1989). In Gorman (1981) and Lewbel's (1987) terminology, this demand system is full-rank because three independent functions of price must be used to determine the demands, expenditure, or indirect utility functions.

The counterpart to Theorem 2 yields the intersection of aggregability and integrability for the QES system.

Theorem 4. The micro demands in Theorem 3 satisfy (A6) iff

$$c_j /2 = h_j(\mathbf{p})/g(\mathbf{p})$$

$$\beta_j = g_j(\mathbf{p})/g(\mathbf{p}) - 2f(\mathbf{p})h_j(\mathbf{p})/g(\mathbf{p}), \text{ and}$$

$$\alpha_{ij} = F(\mathbf{p}, \mathbf{z}_i)^2 h_j(\mathbf{p})/g(\mathbf{p}) - F(\mathbf{p}, \mathbf{z}_i)g_j(\mathbf{p})/g(\mathbf{p}) + F_j(\mathbf{p}, \mathbf{z}_i) + \chi(H(\mathbf{p}, \mathbf{z}_i), \mathbf{z}_i)G(\mathbf{p}, \mathbf{z}_i)H_j(\mathbf{p}, \mathbf{z}_i),$$

$$h(\mu\mathbf{p}) = h(\mathbf{p}), \quad g(\mu\mathbf{p}) = \mu g(\mathbf{p}), \quad f(\mu\mathbf{p}) = f(\mathbf{p}) \quad \mu > 0,$$

where,

$$H(p, z_i) = \hat{H}(h(p), z_i) \ ,$$

$$G(p, z_i) = \hat{H}_h(h(p), z_i)\ g(p) \ ,$$

$$F(p, z_i) = f(p) + \frac{1}{2} g(p) \frac{\hat{H}_{hh}}{\hat{H}_h} \quad i=1, \ldots, n;\ j=1, \ldots, m,$$

and h and j subscripts denote the derivatives with respect to h and p_j.

Hence, when using an absolute inequality index to aggregate, micro demands must be quadratic in income and macro demands will depend upon mean income as well as the variance. The inequality index I is a transformation of the sample variance.

Compromise Absolute Indexes

(A3) offers a straightforward method of converting an absolute index into a relative index. Notice that when $\lambda = 1$, (A3) degenerates to (A2). When $\psi = 0$, a relative inequality index is obtained by dividing the $I(\mathbf{y})$ in (A3) by μ.

Imposing the compromise absolute index and strict monotonicity on Q_{1j} implies additional restrictions on the macro indexes obtained in Theorem 4:

Theorem 5: Under (A3)-(A5), strong monotonicity of Q_{1j}, there exists a Q_{oj}: $R^n \to R$, Q_{1j} : $R \to R$, I: $R^n_+ \to R$ such that (1) holds iff:

$$q_{ij}(y_i) = \alpha_{ij} + \beta_j y_i + \frac{c_j}{2} y_i^2, \quad i=1,\ldots,n;\ j=1,\ldots,m,$$

$$Q_j = \alpha_j + \beta_j \mu + \frac{c_j}{2}\sum_{i=1}^{n} y_i^2 \quad j=1,\ldots,m$$

$$I(y) = K\left(\sum_{i=1}^{n} y_i^2 - \frac{\mu^2}{n}\right)^{1/2}$$

where $K, \alpha_j = \Sigma_i \alpha_{ij}$, β_j, c_j, D_j $(i = 1, \ldots, N;\ j = 1, \ldots, m)$ *all*

$$Q_{1j}(I) = \alpha_j - D_j + \frac{c_j}{2K} I(y) \quad j=1,\ldots,m$$

belong to R.

As with absolute inequality aggregation, individual demand curves are quadratic in income. However, the range of transformations of the sample variance as acceptable inequality indexes is sharply reduced: the compromise axiom reduces the class of admissible inequality indexes to those proportional to the standard deviation of income, and so long as K > 0, the resulting inequality index is both Schur-convex and differentiable even though neither restriction was imposed in aggregation. Setting K = 1,

$$Q_{1j}(I(y)) = \alpha_j(p) - D_j(p) + \frac{c_j(p)}{2} n\, \sigma^2$$

where

$$\sigma = \frac{\left(\sum_{i=1}^{n} y_i^2 - \frac{\mu}{n}\right)^{1/2}}{\sqrt{n}},$$

j = 1, ..., m, would be a convenient specification.

Because the compromise absolute axiom only restricts further the form of I, the integrability conditions in Theorem 4 continue to apply here.

Social-Welfare Functions

Possessing the inequality measures which are consistent with inequality exact aggregation, it is now straightforward to obtain a social-welfare function that is consistent with the inequality index implied by inequality-exact aggregation. Both the geometric mean and the variance are 'reference-level-free' in the sense of Blackorby and Donaldson (1978, 1980). If the social-welfare function is presumed to be homothetic and consistent with (A1), it follows from our results and those of Blackorby and Donaldson (1978) that a social-welfare function consistent with inequality-exact aggregation can be represented as a transformation of $\mu - \mu\sum_i \log(ny_i/\mu)$.

Similarly, it follows immediately from our results and those of Blackorby and Donaldson (1980)that a translation-invariant, social-welfare function consistent with inequality-exact aggregation under (A2) can be derived as the implicit solution to

$$W(y) = Max\ w:\ \mu - (\sum_{i=1}^{n} y_i^2 - \frac{\mu^2}{n})^{1/2} \geq \phi(w)$$

for arbitrary continuous ϕ.

Concluding Remarks

We have examined aggregation using the scaling or translation invariance criteria inherent in relative, absolute, and compromise indexes of inequality. In each case, only a single class of micro demand forms is consistent with inequality-exact aggregation: respectively, the LINLOG, the quadratic, and a more structured quadratic demand systems are obtained. When the domain of the relative inequality index is ext.nded to the whole nonnegative orthant, the LINLOG collapses to the Gorman Polar Form implying that only a trivial inequality index (a constant) is appropriate. In this case, relative indexes are largely inconsistent with aggregation and additivity. Using a function of the Gini coefficient or coefficient of variation plus a function of mean income to represent aggregate demand as is often attempted is problematic.

Our results have a number of other implications: We have used the inequality indices derived to deduce social-welfare functions consistent with inequality exact aggregation using both relative and absolute indices. The social-welfare function associated with inequality-exact aggregation using a relative index is some transformation of $\mu - \mu\sum_i \log(ny_i/\mu)$, and a social-welfare function consistent with inequality-exact aggregation using an absolute index can be written as some transformation of

$$\mu - \left(\sum_{i=1}^{n} y_i^2 - \frac{\mu^2}{n}\right)^{1/2}.$$

Our findings also usefully relate to Lewbel's (1989) notion of a representative consumer for aggregable demand systems. The inequality indexes derived here have the pleasantly intuitive property that they measure the difference between Lewbel's representative consumer and an 'average' or 'aggregate' consumer.

For example, consider the following special case of the Theorem 2 demand system:

$$d_j^i(p,\ y_i,\ z_i) = \rho_j(\mathbf{p})\Upsilon(\mathbf{p})[h + \log t(\mathbf{z}_i) + \log(y_i/\Upsilon(\mathbf{p}))] + (\Upsilon_j(\mathbf{p})/\Upsilon(\mathbf{p}))y_i$$

$$,\ i=1,\ldots,n;\ j=1,\ldots,m,$$

where h is now taken as a real constant. Then in Lewbel's sense, one can always find a representative consumer with taste parameter $\log T(z_1,\ldots,z_n;\mathbf{y}) = (1/n)\sum_i \log t(z_i) + (1/n)\sum_i \log(y_i) - \log\mu + \log n$, and whose demand equals average demand. This representative consumer's jth demand assumes the form:

$$\rho_j(\mathbf{p})\Upsilon(\mathbf{p})[h + \log T(z_1,\ldots,z_n;\mathbf{y}) + \log(\mu/n\Upsilon(\mathbf{p}))] + (\Upsilon_j(\mathbf{p})/\Upsilon(\mathbf{p}))\mu/n$$

$$j=1,\ldots,m,$$

These demands are integrable and have the empirically convenient property that they only depend upon average income and the taste parameter T. Taking $I(\mathbf{y}) = \log\mu - (1/n)\sum_i \log(y_i)$ from Theorem 1, then it follows immediately that $I(\mathbf{y}) = (1/n)\sum_i \log t(z_i) - \log T + \log n$. Hence, the inequality index is an exact measure of the difference between the average consumer and the integrable 'representative consumer'. And, as one would intuitively expect, the more unequal is the income distribution the more the representative consumer will depart from the average consumer. The major empirical implication, therefore, is that the less equal is the income distribution the less it makes sense to impose integrability upon average demand systems based only upon aggregate or average income. Framed in terms of the derived social-welfare function, we can also say that the more the representative consumer departs from the average consumer, the lower is social welfare. A similar interpretation applies in the QES case: Notice, however, that for the QES the representative consumer's taste parameter will generally depend upon both individual specific characteristics and the price vector.

APPENDIX: Proofs

Proof of Theorem 1: The proof supposes that $y_k > 0$ for all k. By (A1) and (A4), for v > 0, j = 1, ..., M,

$$Q_{oj}(v\mu)-Q_{oj}(\mu) = \sum_{i=1}^{n} q_{ij}(vy_i)-q_{ij}(y_i);$$

which is a Pexider equation with solution:

$$Q_{oj}(v\mu)-Q_{oj}(\mu)=a_j(v)+b_j(v)\mu$$

$$q_{ij}(vy_i) - q_{ij}(y_i) = a_{ij}(v) + b_j(v)y_i;$$

with

$$a_j(v) = \sum_{i=1}^{n} a_{ij}(v),$$

i = 1, ..., n; j = 1, ..., m.

Because

$$Q_{oj}(v\mu) - Q_{oj}(1) = Q_{oj}(v\mu)-Q_{oj}(\mu) + Q_{oj}(\mu) - Q_{oj}(1)$$

$$= Q_{oj}(v\mu) - Q_{oj}(v) + Q_{oj}(v) - Q_{oj}(1),$$

it follows that:

$$a_j(v) + b_j(v)\ \mu + a_j(\mu) + b_j(\mu) = a_j(\mu) + b_j(\mu)\, v + a_j(v) + b_j(v)$$

for v, $\mu > 0$, j = 1, ..., m. Whence,

$$b_j(v)\ (\mu - 1) = b_j(\mu)\ (v-1),$$

j = 1, ..., m. Now set $\mu \neq 1$, v = 1 to obtain $b_j(1) = 0$, j = 1, ..., m. In general,

$$b_j(v) = [b_j(\mu)/(\mu - 1)]\,(v - 1) \equiv k_j\ (v - 1),$$

where $k_j \in \mathbb{R}$, j = 1, ..., m. Now subtract

$$q_{ij}(vy_i) - q_{ij}(1) = a_{ij}(vy_i) + b_j(vy_i) = a_{ij}(vy_i) + k_j(vy_i - 1)$$

from

$$q_{ij}(v) - q_{ij}(1) = a_{ij}(v) + b_j(v) = a_{ij}(v) + k_j(v-1)$$

to establish that

$$a_{ij}(vy_i) = a_{ij}(v) + a_{ij}(y_i),$$

which is Cauchy's third equation with solutions

$$a_{ij}(v) = c_{ij} \log(v),$$

where $v \in \mathbb{R}_{++}$, $y_i \in \mathbb{R}_{++}$ i = 1, ..., n; j = 1, ..., m. Hence

$$Q_{oj}(v\mu) - Q_{oj}(\mu) = c_j \log v + k_j(v-1)\mu$$

$$q_{ij}(vy) - q_{ij}(y) = c_{ij} \log v + k_j(v-1)\,y$$

where,

$$c_j = \sum_{i=1}^{n} c_{ij} \ .$$

Now set $\mu = 1$, and $y = 1$ to obtain

$$Q_{oj}(v) = Q_{oj}(1) + c_j \log v + k_j(v-1)$$
$$= \hat{D}_j + c_j \log v + k_j(v-1)$$
$$q_{ij}(v) = q_{ij}(1) + c_{ij} \log v + k_j(v-1)$$
$$= a_{ij} + c_{ij} \log v + k_j(v-1)$$

$v \in \mathbb{R}_{++}$ yielding the LINLOG form interpreting α_{ij} in the theorem as $a_{ij} - k_j$ and $D_j = \hat{D}_j - k_j$.

By (1) and (A4):

$$Q_{1j}(I(y)) = (\sum_{i=1}^{n} a_{ij}) - D_j + \sum_{i=1}^{n} c_{ij} \log y_i + c_j \ln \mu,$$

$j = 1, \ldots, m$. However, anonymity requires that

$$Q_{1j}(I(y_1, \ldots, y_i, y_{i+1}, \ldots y_n)) = Q_{1j}(I(y_1, \ldots, y_{i+1}, y_i, \ldots, y_n)).$$

implying that

$$c_{ij} \log y_i + c_{i+1,j} \log y_{i+1} = c_{ij} \log y_{i+1} + c_{i+1,j} \log y_i, \quad i=1, \ldots, n-1.$$

Thus, of necessity $c_{ij} = c_j/n$ $(i=1,\ldots,n; j=1,\ldots,m)$. Sufficiency is clear. Q.E.D.

Proof of Theorem 2.

Lewbel (1987, 1989) derived the integrability conditions for the LINLOG form as:

$$q_{ij} = (h^i(C^i(p)) - \ln B^i(p))B^i(p)C_j^i(p)) + B_j^i(p)/B^i(p)\ y_i$$
$$+ B^i(p)C_j^i(p) \log y_i,\quad i=1,\ldots,n; j=1,\ldots,m,$$

where i superscripts denote individual specific functions and j subscripts on functions denote differentiation with respect to p_j. Thus, two functions $C^i(p)$ and $B^i(p)$ completely characterize the demands. We rule out the trivial or undefined cases where either $C^i{}_j(p)$ or $B^i(p)$ is zero. In our notation, $B^i(p) = B(p,z_i)$ and $C^i(p) = C^i(p,z_i)$. The demographic variables z_i can also enter h. In what follows, it will be notationally convenient to ignore the subscript upon z. From Theorem 1, $B_j(p,z)/B(p,z) = B_j(p,z^o)/B(p,z^o)$ for an arbitrary reference vector z^o. Hence, $B_j(p,z)/B(p,z) = B_j(p,z^o)/B(p,z^o) = Y_j(p)/Y(p)$ in an obvious notation. Integrating establishes $B(p,z) = Y(p)V(z)$. Theorem 1 also requires $C_j(p,z)Y(p)V(z) = C_j(p,z^o)Y(p)V(z^o)$ for the reference vector z^o. Hence $C_j(p,z) = \rho_j(p)/V(z)$ in an obvious notation. Whence $C(p,z) = \rho_j(p)/V(z) + \Lambda(z)$. Adding up requires:

$$\sum_{j=1}^{m} p_j\, Y_j(p) = 0\ ;\ \sum_{j=1}^{m} p_j\, \rho_j(p) = 0\ ,$$

from Theorem 1. Sufficiency is clear.

Q.E.D.

Proof of the Corollary: After recognizing from the proof of Theorem 1 that

$$q_{ij}(vy_i) - q_{ij}(y_i) = a_{ij}(v) + b_j(v)y_i$$

set $y_i = \underline{0}$ to obtain $Q_{ij}(v) = 0$.

Proof of Theorem 3: Without using differentiability of Q_{oj}, we could proceed along the lines of the proof of Theorem 1. However, assuming once differentiability of Q_{oj} and q_{ij} yields a simpler more direct proof. By (A2),

$$Q_{oj}(\mu+n\psi) - Q_{oj}(\mu) = \sum_{i=1}^{n} q_{ij}(y_i+\psi) - q_{ij}(y_i),\ j = 1, \ldots, m.$$

Differentiating with respect to Ψ yields

$$nQ_{oj}'(\mu+n\psi) = \sum_{i=1}^{n} q_{ij}'(y_i+\psi), j=1, \ldots, m,$$

which is a Pexider equation with respect to $w_i = y_i + \Psi$. Without loss of generality, set Ψ to zero to obtain the solution

$$nQ_{oj}'(\mu) = B_j + c_j\,\mu$$

$$q_{ij}'(y_i) = B_{ij} + c_j\,y_i.$$

with $B_j = \Sigma_i B_{ij}$, $j = 1, \ldots, m$, where primes now denote derivatives. Integrating:

$$Q_{oj}(\mu) = D_j + B_j(\frac{\mu}{n}) + (\frac{c_j}{2n})\mu^2$$
$$q_{ij}(y_i) = \alpha_{ij} + B_{ij}\, y_i + (\frac{c_j}{2})\, y_i^2,$$

$i = 1, \ldots, n$; $j = 1, \ldots, m$. Hence, from (A4),

$$Q_{1j}(I(y)) = \alpha_j - D_j + \sum_{i=1}^{n} B_{ij}\, y_i - \frac{B_j}{n}\mu + \frac{c_j}{2}(\sum_{i=1}^{n} y_i^2 - \frac{\mu^2}{n}),$$

$j = 1, \ldots, m$. Anonymity (A5) requires that $B_{ij} = \beta_j$ (independent of i) or $\Sigma_i B_{ij} = n\beta_j$ ($i = 1, \ldots, n$; $j = 1, \ldots, m$) yielding the form in the Theorem. Sufficiency is clear. Q.E.D.

Proof of Theorem 4:

Van Daal and Merkies present necessary and sufficient conditions for integrability of the QES. Under (A6), in our notation the condition is that there exist functions H(p,z), G(p,z), and F(p,z) (again ignoring subscripts on z) such that:

$$q_{ij} = \frac{H_j(p,\ z)}{G(p,\ z)}\ y_i^2 + [\frac{G_j(p,\ z)}{G(p,\ z)} - 2F(p,\ z)\frac{H_j(p,\ z)}{G(p,\ z)}]\ y_i$$
$$+ \frac{F(p,\ z)^2 H_j(p,z)}{G(p,\ z)} - F(p,\ z)\frac{G_j(p,\ z)}{G(p,\ z)}$$
$$+ \chi(H(p,\ z))G(p,\ z)H_j(p,\ z) + F_j(p,\ z),$$

$i = 1, \ldots, n$; $j = 1, \ldots, m$. We presume that each of these functions are non-zero implying the system is rank-three. By Theorem 3, $H_j(p,z)/G(p,z) = H_j(p,z^o)/G(p,z^o)$ for an arbitrary reference vector z^o. Hence, in an obvious notation $H_j(p,z)/G(p,z)=h_j(p)/g(p)$ for all z and j. It follows immediately that $H_j(p,z)/H_k(p,z) = h_j(p)/h_k(p)$ for all j, k and z. Thus,

$$H(p,z) = \hat{H}(h(p),z).$$

Because $H_j(p,z)/G(p,z)=h_j(p)/g(p)$,

$$G(p,z) = \hat{H}_h(h(p),z)g(p).$$

It also follows from Theorem 3 and Van Daal and Merkies that:

$$G_j(p,z)/G(p,z)-2F(p,z)H_j(p,z)/G(p,z)=G_j(p,z^o)/G(p,z^o)-$$
$$2F(p,z^o)H_j(p,z^o)/G(p,z^o),$$

for any arbitrary reference vector z^o. Thus, in an obvious notation:

$G_j(p,z)/G(p,z)-2F(p,z)H_j(p,z)/G(p,z)=g_j(p)/g(p)-$

$2f(p)h_j(p)/g(p)$.

Using the preceding part of the proof establishes:

$$F(p,z) = f(p) + (1/2)(g(p)/h_j(p))[G_j(p,z)/G(p,z)-g_j(p)/g(p)].$$

Direct calculation using the form of $G(p,z)$ isolated above establishes:

$$F(p,\ z) = f(p) + \frac{1}{2}\frac{\hat{H}_{hh}}{\hat{H}_h}g(p).$$

Finally, adding up (see Proof of Theorem 2) implies the homogeneity results and the form in the Theorem necessarily follows. That the form is integrable is easily verified.

Proof of Theorem 5: Taking $\lambda = 1$ in (A3) implies that all of the conditions derived in Theorem 3 must continue to apply. However, setting $\psi = 0$ shows that $I(\lambda y) = \lambda I(y)$ for all $\lambda > 0$.

Now from Theorem 3

$$Q_{1j}(I(y)) = \alpha_j - \frac{D_j}{n} + \frac{c_j}{2}\left(\sum_{i=1}^{n} y_i^2 - \frac{\mu^2}{n}\right)$$

$$\equiv t + \frac{c_j}{2}T, \quad j = 1,\ \ldots,\ m.$$

Because Q_j is monotonic in $I(y)$ and $I(y)$ is anonymous it follows that

$$I(y) = f(T).$$

Use the fact that $I(\lambda y) = \lambda I(y)$ with $\lambda = \mu^{1/2}$ for $\mu > 0$ to establish,

$$f(\mu T) = \mu^{1/2} f(T)$$

for $\mu > 0$. Hence f is positively homogeneous of degree 1/2 in T establishing that

$$f(T) = KT^{1/2}$$

with $K \in \mathbb{R}$. Q.E.D.

ENDNOTES

1. Jorgenson and Slesnick (1984), however, use a demand-systems approach to estimate the parameters of consumer indirect utility functions, and then use the estimated indirect utility functions in the construction of a translog relative inequality measure of individual welfares.

2. For absolute indexes, Kolm (1976a) required that the mean of income minus the inequality index be strongly separable in the extended income partition. For relative indexes, Kolm (1976a) required that the mean of income minus the product of the mean and the inequality index be strongly separable in the extended income partition. In both cases, Kolm's assumptions imply that the social-welfare functions consistent with the respective inequality indexes, as determined using the methods of Blackorby and Donaldson (1978, 1980), are either additively separable or strongly separable in the extended partition.

3. As will become clear from what follows, little true generality is lost by taking $-Q_{1j}$ as $I(y)$. In this specification, (A4) then reduces to

$$Q_{oj}(\mu) - I(y) = \sum_{i=1}^{n} q_{ij}(y_i) .$$

which is very similar to Kolm's criterion.

4. Several readers have pointed out that one can always start, say, from the two-index results of Chambers and Pope (1992) and then impose (A4) along with the appropriate invariance property to derive our results. However, starting from (A4) actually makes the proofs simpler and more transparent.

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