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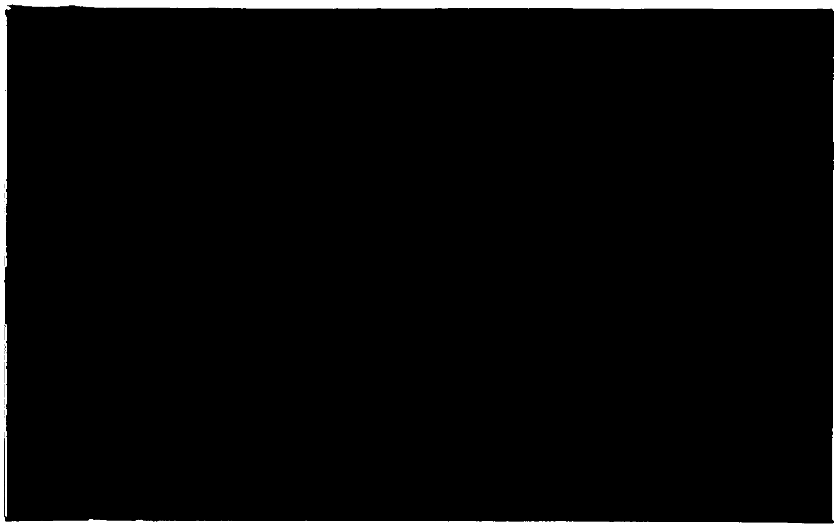

AGGREGATE HOMOTHETIC SEPARABILITY

by

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Aggregate Homothetic Separability

Magnus and Woodland have recently derived necessary and sufficient conditions for an aggregate cost function derived from industry cost functions exhibiting homothetic separability to inherit homothetic separability. This paper extends the Magnus and Woodland contribution, using weaker assumptions, and develops necessary and sufficient conditions for an aggregate sector cost function derived from general (i.e., ones that need not exhibit homothetic separability) industry cost functions to exhibit homothetic separability.

I. The Industry Cost Functions

Following the convention established in Magnus and Woodland, the analysis focuses on technologies with two types of net outputs: fuel inputs and nonfuel net outputs. It is assumed that there is a production sector with q industries, each industry possesses its own technology which is characterized by the production sets, $T^k \subset \Re^{n+m}$ $(k = 1, ..., q)$ where $\Re$ stands for the real numbers and n is the number of fuel inputs and m is the number of nonfuel net outputs. The fuel-input set is denoted:

$$X^k(y^k) = \{x : (-x, y^k) \in T^k\}.$$

Dual to each T^k is a sectoral cost function:

$$C^k(p, y^k) = \min_x\{ p'x : (-x, y^k) \in T^k\},$$

where p is the vector of strictly positive fuel-input prices. Following, Magnus and Woodland each industry cost function satisfies:

Assumption 1: The industry cost functions, $C^k(p, y^k)$ $(k = 1,...,q)$ satisfy:

a) $C^k(p, y^k)$ is defined for $p \in P$ and $y^k \in Y^k$ (the set of producible outputs for T^k).

b) Y^k is nonempty and convex.

c) $P = \{p: \ p > 0^n\}$.

d) For each $y^k \in Y^k$, $C^k(p, y^k)$ is a concave, positively linearly homogeneous, and closed function of $p \in P$.

e) $C^k(p, y^k) > 0$ for all $p \in P$, $y^k \in Y^k$ and $C^k(p, y^k) > 0$ if $y^k \neq 0^m$.

II. The Sectoral Cost Function

As Magnus and Woodland point out, each of the industry cost functions can, in principle, be estimated using data from the various industries. Unfortunately, as a general rule sufficient data do not often exist to estimate these industry-level cost functions, and lacking this data the more common empirical practice is for researchers to estimate sector-level cost functions using aggregate data. And when considering the demand for fuel inputs it is particularly common for researchers to assume that the technology underlying the aggregate sector-level cost function is consistent with homothetic separability of the fuel inputs (Fuss 1977; Griffin 1977; and Pindyck 1979). Magnus and Woodland (1990) examined the theoretical consistency of this practice under the presumption that each of the industry cost functions were also consistent with homothetic separability. They establish that this practice is consistent only if each industry fuel price index is proportional to the sectoral price index. In this section, I establish necessary and sufficient conditions for a sectoral cost function to exhibit homothetic separability when the only restrictions placed on the industry cost functions are those listed in Assumption 1. (In addition to Assumption 1 and homothetic separability in fuel inputs, Magnus and Woodland also impose a restriction on the variational properties of the net output component of the industry cost functions that is not needed in the current analysis.)

In what follows, denote

$$x = \Sigma_k x^k,$$

as the vector of fuel inputs applied across the q industries, and

$$X(y^1,...,y^q) = \{x : x \text{ can produce } (y^1,...,y^q)\}$$

as the aggregate input set . The sectoral cost function is then defined by:

$$C(p, y^1,...,y^q) = \text{Min }\{p'x : x \in X(y^1,...,y^q)\}.$$

Assumption 2: $C(p, y^1,...,y^q) = \Sigma_k C^k(p, y^k)$.

Assumption 3: $C(p, y^1,...,y^q) = c(p)h(y^1,...,y^q)$.

Assumption 2 corresponds to Assumption 3 in Magnus and Woodland, while Assumption 3, which is implicit in Magnus and Woodland, requires the aggregate cost function to be consistent with homothetic separability, i.e., one must be able to express $X(y^1,...,y^q)$ as $h(y^1,...,y^q)X(1)$ where $X(1)$ is a reference input set dual to $c(p)$.

My result is:

Theorem: If the industry cost functions satisfy Assumption 1 and the sector cost function satisfies Assumptions 2 and 3, each industry cost function must be expressible as:

$$C^k(p, y^k) = b^k(p) + c(p)h^k(y^k),$$

with

$$\Sigma_k b^k(p) = 0,$$

and

$$h(y^1,...,y^q) = \Sigma_k h^k(y^k).$$

Proof: That a sector cost function constructed from such industry cost functions satisfies Assumptions 2 and 3 is obvious. To go the other way, by Assumptions 2 and 3 it must be true that:

$$c(p)h(y^1,\ldots,y^q) = \Sigma_k\, C^k(p, y^k).$$

Pick a reference vector $p^* \in P$ and substitute it into the preceding equation to obtain using Assumption 1 (e) and Assumptions 2 and 3:

$$h(y^1,\ldots,y^q) = \Sigma_k\, C^k(p^*, y^k)/c(p^*) = \Sigma_k h^k(y^k)$$

after renormalization. Hence,

$$\text{(a)} \qquad c(p)\, \Sigma_k h^k(y^k) = \Sigma_k\, C^k(p, y^k).$$

Now set $y^k = 0$, a common reference vector, $k = 1,\ldots,q$. (Note, if the Y^k do not contain a common reference vector the proof remains unaffected if arbitrary reference vectors are chosen for each Y^k.) This yields:

$$\text{(b)} \qquad c(p)\, \Sigma_k h^k(0) = \Sigma_k\, C^k(p, 0).$$

By setting all $y^j = 0$ (their reference levels) for $j \neq k$ in (a), while using (b) it then follows that:

$$\begin{aligned} C^k(p, y^k) &= C^k(p, 0) + c(p)(h^k(y^k) - h^k(0)) \\ &= b^k(p) + c(p)h^k(y^k), \end{aligned}$$

using an obvious definition of $b^k(p)$. Summing $b^k(p)$ over k yields the final restriction in the Theorem. **QED**

The implication of the theorem is perhaps more intuitive if cast in terms of fuel input sets. Using McFadden's (1978) composition rules, the input sets associated with the industry cost functions must satisfy:

$$X^k(y^k) = B^k(1) + h^k(y^k)X(1).$$

The sector cost function exhibits homothetic separability in fuel inputs if and only each industry input set can be written as the sum of two input sets: One $B^k(1)$, dual to $b^k(p)$, is industry specific but independent of the level of the nonfuel net outputs produced in the industry. The other $h^k(y^k)X(1)$, dual to $c(p)h^k(y^k)$, is consistent with homothetic separability in fuel inputs and is proportional to the reference input set dual to $c(p)$, the sector fuel price index. One might, thus, refer to these input sets as *quasi-homothetically separable.* The adding-up restriction on the $b^k(p)$ functions implies that aggregation generally is only possible if the industry-cost functions exhibit externalities. (An exception occurs when the industry cost functions are homothetically separable in fuel inputs.) To see why this must be true, notice that the restriction that $\sum_k b^k(p) = 0$ implies that at least one $B^k(1)$ must be expressible as minus the sum of the remaining $B^j(1)$ $j \neq k$. This can be most easily seen by noting that by duality:

$$
\begin{aligned}
B^k(1) &= \{x: p'x \geq b^k(p) \text{ for all } p \in P\} \\
&= \{x: p'x \geq - \textstyle\sum_{j\neq k} b^j(p) \text{ for all } p \in P\} \\
&= \{x : p' \textstyle\sum_{j\neq k} x^j \geq - \sum_{j\neq k} b^j(p) \text{ for all } p \in P,\ x = \sum_{j\neq k} x^j \} \\
&= - \textstyle\sum_{j\neq k} B^j(1).
\end{aligned}
$$

Intuitively, the Theorem states that aggregate homothetic separability is only possible if each industry cost function consists of a price index, $b^k(p)$, that is specific to the industry, but which must obey the across industry constraint, and the aggregate price index, $c(p)$, multiplied by the industry's contribution to aggregate net output, $h^k(y^k)$.

Corollary: If the industry cost functions are homothetically separable in fuel inputs and satisfy Assumption 1, and the sector cost function satisfies Assumptions 2 and 3, each industry cost function must be expressible as:

$$C^k(p, y^k) = c(p)h^k(y^k).$$

The Corollary covers the special case considered by Magnus and Woodland under our weaker assumptions. The normalization, without loss of generality, is chosen to incorporate their factor of proportionality into $h^k(y^k)$.

III. Conclusion

This paper has deduced necessary and sufficient conditions for a sector cost function to be consistent with homothetic separability in fuel inputs.

References

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