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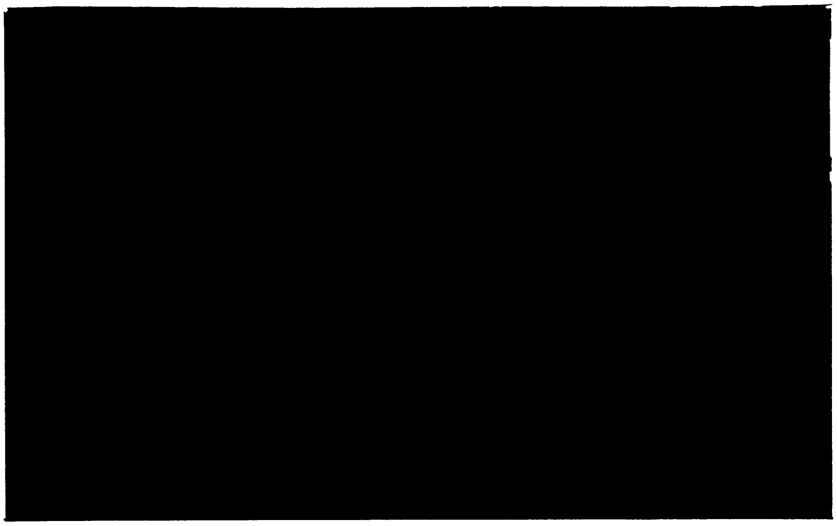

On Separation Results in Forward and Futures Markets

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A central result for the competitive, risk-averse firm with no production risk but facing price risk is that access to a futures market yields a separation of production and hedging decisions (Jean-Pierre Danthine, 1978; Duncan Holthausen, 1979; Gershon Feder et al., 1980; Ronald Anderson and Jean-Pierre Danthine, 1983): Production decisions are independent of the producer's risk preferences and depend only on the futures price. If the futures market is biased, hedging decisions depend upon the producer's risk preferences. But if the futures market is unbiased, the producer hedges completely regardless of the degree of risk aversion.

The key assumptions behind the Danthine-Holthausen-Feder et al.-Anderson and Danthine (DHFAD) "separation" result are the absence of production uncertainty and the identification of a "futures" market with a complete forward market for the commodity that the firm produces (Ronald Anderson and Jean-Pierre Danthine, 1981; Anderson and Danthine, 1983). That is, the firm can always execute a "futures contract" that permits it to buy or sell as much of the commodity in question (with zero transactions costs) for a delivery date that coincides exactly with the resolution of the price uncertainty. In the language of futures markets, there is no "basis risk", or put another way it implies that a free price-insurance contract is available to the producer. In the latter context, the DHFAD separation result manifests Karl Borch's (1989) rule for optimal risk sharing: the risk-neutral insurer (the "futures market") absorbs all of the risk implying that the producer's production decisions do not depend upon their risk preferences. If either assumption is relaxed, zero basis risk or certain production, the separation result disappears. Unfortunately, in the real world neither assumption is very plausible.

This paper revisits the separation question using a reformulation of the traditional model of production under uncertainty (Robert Chambers and John Quiggin, 1992). This reformulation allows us to relax simultaneously both of the key assumptions required for separation of production and hedging decisions-- zero basis risk and certain production. An important advantage of the Chambers-Quiggin (CQ) reformulation is that the producer facing price risk determines endogenously whether to use a technology with no production risk. This advantage is created by specifying a state-contingent production technology which always has zero production risk as a special case.

In what follows, we first introduce our model and briefly discuss the CQ specification of production uncertainty by comparing it to more traditional models of production uncertainty. Producer behaviour with both production and price risk in the absence of contingent markets is examined first to provide a backdrop to our latter results on hedging and production decisions. We then turn to the analysis of producer behavior with both price and production risk in the presence of a complete forward market for the commodity (i.e., zero basis risk) the producer produces. Our first major result in this section establishes that under plausible conditions with price uncertainty and the ability to hedge price risk in an unbiased forward market, *a risk-averse producer will never willingly adopt a certain technology* if given the alternative of adopting an uncertain production technology. Thus, even given the assumption of zero basis risk the other assumption underlying the DHFAD separation result is shown to be both unrealistic and overly restrictive. Thus, in a world where producers have some flexibility to choose the degree of production risk to which they expose themselves, the DHFAD separation will not apply. Our results on separation, however, are not all negative for we then show that the CQ production model affords a different and apparently unrecognized separation result. This separation result, unlike the DHFAD result, holds in the presence of both production and price risk, depends on the market's information structure, and closely parallels spanning results in finance theory.

The paper then analyzes a firm with an uncertain technology facing both price risk and basis risk. By recognizing that producers can operate in more than one contingent market to cross hedge we establish a separation result that applies even in the presence of basis risk. Like the result for a complete forward market, this separation result depends upon the informational structure of the market. The last section concludes.

The Model

Uncertainty is modelled by assuming that "Nature" makes a choice from among a finite set of alternatives. Each alternative is called a state and is indexed by a finite set of the form $\mathbf{W} = \{1,2,\ldots,S\}$. Production relations are governed by a technology set $\mathbf{T} \subseteq \mathbb{R}^n_+ \times \mathbb{R}^S_+$ defined by

$$\mathbf{T} = \{ (\mathbf{x},\mathbf{y}): \mathbf{x} \text{ can produce } \mathbf{y},\ \mathbf{x} \in \mathbb{R}^n_+ ,\ \mathbf{y} \in \mathbb{R}^S_+\}.$$

Here $\mathbf{x}$ is an input vector committed before Nature chooses the *ex post* state from $\mathbf{W}$ and $\mathbf{y}$ is a vector of state-contingent outputs with y_i corresponding to the amount of output that would occur if state i occurs. The most appropriate interpretation of **T** is as an *ex ante* technology: $(\mathbf{x},\mathbf{y}) \in \mathbf{T}$ implies that if input vector **x** is committed and nature chooses the j state from **W** then y_j occurs. Output price uncertainty is modelled similarly. The producer either knows or has subjective beliefs about the price distribution that are summarized by a state-contingent price vector $\mathbf{p} \in \mathbf{R}^S_{++}$, where $p_i > 0$ is the output price that occurs if state i is chosen by Nature from **W**. The firm is competitive in the sense that it treats **p** as independent of its actions.

Production uncertainty is absent when $\mathbf{y} \in \mathbf{I^*}$ where $\mathbf{I^*} \subseteq \mathbf{R}^S_+$ is the set of output vectors with each element identical. That is **y** assumes the form

$$\mathbf{y^*} = (y,y,\ldots,y),$$

and price uncertainty is removed when $\mathbf{p} \in \mathbf{I^*}$, i.e., the price vector assumes the form:

$$\mathbf{p^*} = (p,p,\ldots,p).$$

The producer's beliefs about the relative likelihood of Nature picking a particular state are summarized by $\pi \in \Pi \subseteq \mathbf{R}^S_{++}$ where Π is the simplex

$$\Pi = \{\, \pi: \pi \in \mathbf{R}^S_{++} \text{ and } \Sigma_i\pi_i = 1\} .$$

No state occurs with zero probability.

We examine two alternative contingent market structures: a forward market and K futures markets. The forward market operates in the following fashion: at the time input decisions are made, the producer can take either a long or short position denoted by $h \in \mathbf{R}$ entitling him or her to receive or pay $q > 0$ for each unit of h. Only the commodity that the producer makes can be delivered on this contract. The futures markets operate as follows: there are K futures markets where the producer can take either a long or a short position. In each market at the current price futures price q_k, the producer can execute either a long or a short contract denoted by $h_k \in \mathbf{R}$ giving him or her the ability to sell or take delivery upon the commodity in question at some later date. These commodities need not be the commodity the producer produces. To allow specifically for basis risk, also presume that for each futures contract there exists a state-contingent price vector $\mathbf{f^k} \in \mathbf{R}^S_+$ of the same basic structure as **p**, i.e., if Nature chooses i from **W** then the *ex post* price of the kth commodity (or the futures contract) is f^k_i. (Special cases are where one of the futures commodities is the same commodity as the individual produces or where the futures contract is actually a forward contract for the commodity produced by the individual.) The basis vector, using a slight abuse of terminology, for the kth futures contract is denoted by $\mathbf{b}^k \in \mathbf{R}^S$ and has typical element $b^k_i = f^k_i - q_k$.

The producer's objective is,

$$\text{Max } \mathbf{U} = \Sigma_i \pi_i u(r_i),$$

where r_i denotes the producer's return in state i. Here $u: \mathbf{R} \rightarrow \mathbf{R}$ is a strictly concave and strictly increasing function satisfying the Von Neuman-Morgernstern postulates. u is differentiable. In the case of the forward contract

$$r_i = p_i y_i + h (q - p_i) - \mathbf{wx}$$

where $\mathbf{w} \in \mathbf{R}^n_{++}$ is a vector of input prices which are presumed known at the time of purchase. In the futures-markets case,

$$r_i = p_i y_i - \mathbf{wx} + \Sigma_k h_k b^k_i.$$

Chambers and Quiggin (1992) show that the above can reformulated in the forward market case as

$$\text{Max}_{y,h} \mathbf{U} = \Sigma_i \pi_i u(p_i y_i - c(\mathbf{w}, \mathbf{y}) + h (q - p_i)),$$

and in the futures market case as

$$\text{Max}_{\,y,h} \mathbf{U} = \Sigma_i \pi_i u(p_i y_i - c(\mathbf{w}, \mathbf{y}) + \Sigma_k h_k b^k_i),$$

where

$$c(\mathbf{w}, \mathbf{y}) = \text{Min } \{\mathbf{wx}: (\mathbf{x},\mathbf{y}) \in \mathbf{T}\}.$$

Under weak regularity conditions[1], c(**w**, **y**) will be positively linearly homogeneous in **w**, nondecreasing in **w**, concave in **w**, nondecreasing in **y**, convex in **y**, and continuous (Chambers and Quiggin, 1992). Moreover, c(**w**, **y**) can be used to recapture the input requirement sets associated with **T**. c(**w**, **y**) also satisfies Shephard's lemma. In short, c(**w**,**y**) has the same properties as a cost function for a multioutput technology with no uncertainty. For expositional convenience, presume that c(**w**,**y**) is differentiable.

[1]These assumptions are **T** exhibits free disposability of output, free disposability of input, **T** is a convex set, and for given **y** the set defined by the correspondence
V(**y**) = {**x**: (**x**,**y**) belongs to **T**} is nonempty.

c(**w**, **y**) offers a convenient method for pictorially isolating the differences between our model of production uncertainty and the one more usually used in the literature on uncertain production. Figure 1 portrays an isocost contour for c(**w**, **y**) in **y** space. Because c(**w**, **y**) is nondecreasing and convex in **y**, this isocost contour is negatively sloped and concave to the origin. The fact that the isocost contour is negatively sloped implies that increasing one state's output while maintaining constant costs requires lowering another state's output. Its concavity to the origin reflects the presumption that this substitution of one state-contingent output for another occurs at increasing marginal cost.

The traditional approach to production uncertainty, which specifies a production function depending upon **x** and the state of nature i (y(**x**,i)), does not allow for the possibility of substituting one state-dependent output for another. Rather, it assumes that once the input vector is chosen (costs are fixed), only one pair of state-contingent outputs in Figure 1 is possible (y(**x**,1), y(**x**,2)). That is producers have absolutely no ability to arrange their input utilization in different manners to prepare differentially for different states. This is unrealistic as can be illustrated by a simple example. Suppose there are only two states of nature, say rain and no-rain, as illustrated in Figure 1. The traditional model would then only permit the single rain-no-rain output pair (y(**x**,1), y(**x**,2)) in Figure 1 where more output is produced in the rain state than in the no rain state. The unrealistic implication is that when the rain state occurs the producer always gets the same output regardless of whether he devoted all of his inputs to digging irrigation ditches or to building irrigation dams. How the producer allocates inputs has no effect on the outputs that emerge for given **x**.

In Figure 1, the isocost contour intersects the bisector. This point of intersection represents the certainty outcome (no production uncertainty) for that level of cost. Perhaps the most important departure of our model from the traditional model of production under uncertainty is the presumption, technically free disposability of state-contingent outputs, that the producer can at appropriate cost choose to produce on the bisector. That is, faced with price uncertainty, the producer can always choose to remove production uncertainty by committing appropriate effort. Thus, although the presence of price uncertainty is exogenous in our model, the presence of production uncertainty is not. The producer is always free to produce a state-contingent output vector **y** $\in$ **I*** although this choice may be costly. For example, if rainfall is the sole source of the production uncertainty, a producer may always adopt irrigation as a means of eliminating this uncertainty. Production certainty, if it occurs at all, is endogenous.

Our analogue to the traditional assumption of no production uncertainty is a weaker restriction on c(**w**,**y**) which, in essence, guarantees distinct cost advantages to the adoption of a **y** ∈ **I***. Even so, this condition which we refer to as **Z** stops short of presuming that technology is nonstochastic. Formally,

(Z) $$c(\mathbf{w},\mathbf{Py}) \leq c(\mathbf{w},\mathbf{y}) \qquad \text{for all } \mathbf{P} \in \boldsymbol{\wp}$$

where $\boldsymbol{\wp} \subseteq \mathbf{R}^{SxS}_{+}$ is the set of row stochastic matrices, i.e. matrices whose row sums always equal one, for which the kth column sum equals $S\pi_k$. **Z** is a generalization of Schur convexity. A Schur-convex, h(x), has the property that h(Bx) ≤ h(x) for all B bistochastic (Albert Marshall and Ingram Olkin, 1979). In the equal probability case, **Z** and Schur convexity are equivalent. Cost structures satisfying **Z** have the unique property that absent price uncertainty, a risk-neutral individual will always find it advantageous to choose an output vector **y** ∈ **I*** . We summarize this property in the following lemma for later use (All proofs are in the Appendix):

Lemma 1: If condition **Z** is satisfied, then the solution, **y**(**p**), to the problem,

$$\mathrm{Max}_y\{\Sigma_i \pi_i p_i y_i - c(\mathbf{w},\mathbf{y}) : \mathbf{p} \in \mathbf{I}^*\}$$

satisfies **y**(**p**) ∈ **I***, or there exists an equivalent solution which satisfies this latter property. If condition **Z** is satisfied, then the solution, **y**(**p**,r), to the problem

$$\mathrm{Min}_y\{c(\mathbf{w},\mathbf{y}) : \Sigma_i \pi_i p_i y_i \geq r; \mathbf{p} \in \mathbf{I}^*\}$$

satisfies **y**(**p**,r) ∈ **I*** or there exists an equivalent solution which satisfies this latter property.

In the case of price uncertainty, cost functions satisfying **Z** always yield a state-contingent output vector for a risk-neutral producer that is positively correlated with the state-contingent price vector.

Lemma 2: If condition **Z** is satisfied, then the solution, **y**(**p**), to the problem

$$\mathrm{Max}_y\{\Sigma_i \pi_i p_i y_i - c(\mathbf{w},\mathbf{y})\}$$

satisfies

$$\Sigma_i \pi_i p_i [y_i(\mathbf{p}) - \mu(\mathbf{y}(\mathbf{p}))] \geq 0,$$

where $\mu(\mathbf{y}(\mathbf{p}))$ is the mean of $\mathbf{y}(\mathbf{p})$.

That the supply correspondence be positively sloped is a well-known, and understood, property of multioutput supply correspondences under conditions of perfect certainty. Typically, it is referred to as the law of supply. Generally speaking the law of supply requires each output to be nondecreasing in its own price. Lemma 2 establishes a subtly different result: under **Z,** the covariance between **p** and **y**(**p**) is nonnegative. That is, each state-contingent supply is not shown to be increasing in its own price (this, however, is easy to show), rather it is shown that state-contingent outputs tend to be higher for states that have higher state-contingent prices. Thus, Lemma 2 might be interpreted as a *probabilistic law of supply.*

Optimal Behavior in the Absence of Contingent Markets

We first consider the case where the producer does not have access to any contingent markets. The properties of u and c(**w,y**) guarantee that the Kuhn-Tucker conditions are necessary and sufficient. Hence, necessary and sufficient conditions for an optimum are given by

(1) $$\pi_i u'(r_i) p_i - c_i(\mathbf{w},\mathbf{y}) \Sigma_k \pi_k u'(r_k) \leq 0 \qquad y_i \geq 0,$$

(i =1,2,...,S). The notation in (1) denotes complementary slackness and subscript i's on c(**w,y**) denote the partial derivatives of the cost function with respect to state-contingent outputs. Expression (1) has the familiar interpretation that the marginal cost of producing the ith state contingent output is always greater than or equal to the marginal utility of increasing the ith state-contingent output divided by the expected marginal utility of income. A detailed analysis of the solution to (1) is provided in Chambers and Quiggin (1992) to which we refer the interested reader for details. However, one characteristic of producer equilbrium will prove especially useful in what follows. Thus,

Lemma 3: Producer equilibrium must satisfy:

$$\Sigma_k (c_k(\mathbf{w},\mathbf{y})/p_k) \geq 1,$$

with the inequality replaced by an equality in the case of an interior equilibrium.

For an interior solution, Lemma 3 has particularly interesting implications. Because both marginal costs and prices are nonnegative, the Lemma then implies that $c_k(\mathbf{w},\mathbf{y})/p_k \in (0,1)$ (k = 1, 2,

...,S). Hence, marginal cost for each state is always less than or equal to the corresponding state-contingent price. But this also implies that the ratios of marginal cost to state-contingent price all lie in Π, so that in fact, these ratios can be interpreted as probabilities, or perhaps more accurately as *shadow probabilities.* In fact, these shadow probabilities are the probabilities that would convince a risk-neutral individual (facing the same **p** and the same technology) to produce the same state-contingent output vector as the risk-averse individual chooses. Lemma 3, thus, has a natural interpretation as an arbitrage condition between the various states of nature because in terms of these shadow probabilities it implies that no way exists at the margin to raise expected profit systematically without increasing risk. To foreshadow our main results, it should also be noted that the condition in Lemma 3 is completely independent of the producer's risk preferences.

The fact that these shadow probabilities lead a risk-neutral individual to pick the same state-contingent output vector as a risk-averse individual has some interesting implications, one of which is summarized by our first Result:

Result 1: If condition **Z** is satisfied, then any interior solution, **y,** to (1) must satisfy:

$$\sum_i c_i(\mathbf{w},\mathbf{y})[y_i - \mu(\mathbf{y})] \geq 0.$$

Result 1 implies, loosely speaking, that marginal costs are positively correlated with divergences from expected output for a risk-averse individual when the technology satisfies **Z.** Also notice that the $c_i(\mathbf{w},\mathbf{y})$ are Peleg-Yaari (PY) efficiency prices. So Result 1 may be interpreted as requiring a positive correlation between outputs and PY efficiency prices. And if Result 1 is rewritten in terms of the shadow probabilities derived in Lemma 3, it implies that using the shadow probabilities there exists a positive correlation between the state-contingent prices and the state-contingent outputs. So under **Z,** one expects to observe higher outputs emerging in high price states than in low price states. Thus, what we referred to as the probabilistic law of supply applies to risk averters as well as risk-neutral individuals.

Result 1 also suggests that when price uncertainty is present, producers will not generally choose **y** such that $\mathbf{y} \in \mathbf{I}^*$ even if **Z** is satisfied. That is given price uncertainty, even strictly risk-averse producers will prefer an uncertain technology to a certain technology. The intuition is easy. Costwise **Z,** by Lemma 1, provides certainty as a lower bound to any risky production choice. However, given the presence of price uncertainty, choosing a certain technology, even though it is cheap, exposes the producer to the whole range of price risk, and gives him or her no chance to self insure. Choosing uncertain production, on the other hand, allows the producer to "self-insure" in the absence of a viable price insurance alternative. In fact, it is easy to see that the producer can always choose an output vector that completely removes all risk that he or she faces. If the producer chooses $y_i = kp_i^{-1}$, returns are stabilized at k - c(**w**,**y**) and the producer is fully insured. Whether the producer chooses to self insure fully in this fashion depends upon the relative costliness of providing self insurance, i.e, in picking this particular state-contingent output vector. But generally a producer will find it optimal to expose herself to some production risk in order to balance the price risk. This finding that a risk-averse producer does not prefer production certainty in the presence of price uncertainty is extended in the next section to encompass the possibility of hedging through unbiased forward markets.

Optimal Producer Behaviour in the Presence of A Forward Market

We now consider how the ability to hedge in a forward market changes producer behaviour. The Kuhn-Tucker conditions are again necessary and sufficient for producer equilibrium. Therefore, optimal behaviour is characterized by:

(2) $\pi_i u'(r_i)p_i - c_i(\mathbf{w},\mathbf{y})\Sigma_k\pi_k u'(r_k) \leq 0 \qquad y_i \geq 0, \qquad (i = 1,2,...,S)$, and

$$\Sigma_k\pi_k u'(r_k)(q - p_k) = 0.$$

Combining these expressions yields:

Lemma 4: For an interior equilibrium to (2), the producer's equilibrium must satisfy:

$$\Sigma_k c_k(\mathbf{w},\mathbf{y}) = q, \qquad \text{and}$$

$$\Sigma_k(c_k(\mathbf{w},\mathbf{y})/p_k) = 1.$$

The first condition in Lemma 4 has the straightforward interpretation of an arbitrage condition between production and hedging behavior. The left-side is the cost of increasing output in every state of nature by one unit. Because this additional output could always be sold on the forward market, interior equilibrium requires that the marginal cost should always equal the forward price, otherwise there would exist unexploited opportunities for raising expected profit while holding risk constant. The second condition is simply a repetition of Lemma 3 and has similar implications. Notice, however, that making use of the second condition permits rewriting the first condition (multiply each $c_k(\mathbf{w},\mathbf{y})$ by p_k/ p_k) to imply that the forward market price should equal the expected value of the state-contingent output prices as evaluated in terms of the shadow probabilities if an interior equilibrium is to exist. Because these shadow probabilities again have the interpretation of being the probabilities that lead a risk-neutral individual to choose the same output vector as a risk-averse individual, this last condition then implies that a risk-neutral individual facing these shadow probabilities should have no incentive to sell (or buy) any amount in the forward market. This is as it should be. But also notice that the Lemma places an important restriction on the domain from which q can be chosen:

Corollary 1: An interior equilibrium to (2) requires that q be a convex combination of the state-contingent prices.

Corollary 1 establishes that an interior equilibrium is not consistent with the forward price either: exceeding the largest state-contingent price, or being smaller than the smallest state-contingent price. Of course, if these conditions were violated the producer could make infinitely large expected profit by setting the hedge at plus infinity or minus infinity respectively.

The DHFAD separation result shows that production is independent of producer risk attitudes for an interior solution. Moreover, if the forward market is unbiased, i.e., $q = \sum_i \pi_i p_i$, forward sales just equal total output and the producer's forward sales are also independent of producer degree of risk aversion. This renders the risk-averse producer's production decisions trivial unless production uncertainty is also present. The producer simply produces at the point where marginal cost is equal to the forward price and sells the entire output forward.

Our more general model illustrates the origin of the DHFAD separation property. The certain production model used in the more traditional approach imposes that there is only one state contingent output and, hence, marginal cost is independent of the states. The first condition in Lemma 4 then shows that this marginal cost, in equilbrium, only depends upon q.

As is well-known the DHFAD separation result depends critically upon the assumption that producers can only choose one output. If that assumption is relaxed to **Z,** the following natural analogue of Result 1 shows that producers will not generally choose a $\mathbf{y} \in \mathbf{I^*}$ and that PY efficiency prices will be positively correlated with state-contingent outputs.

Corollary 2: If condition **Z** is satisfied, then any interior solution, **y**, to (2) must satisfy:

$$\Sigma_i c_i(\mathbf{w},\mathbf{y})[y_i - \mu(\mathbf{y})] \geq 0.$$

Using Corollary 2 and Lemma 4

Corollary 3: If **Z** is satisfied, then for any interior solution to (2)

$$\Sigma_i c_i(\mathbf{w},\mathbf{y})y_i \geq q\mu(\mathbf{y}).$$

Hence,

Corollary 4: If **Z** is satisfied, $q \geq \Sigma_i \pi_i p_i$, then for any interior solution to (2),

$$\Sigma_i c_i(\mathbf{w},\mathbf{y})y_i \geq \mu(\mathbf{y})\Sigma_i \pi_i p_i.$$

Under **Z**, the value of the state-contingent output vector evaluated at PY efficiency prices must always be at least as large as the hedged value of expected output. And under the presumption that the forward price is at least as large as the expected price, the value of the state-contingent output vector evaluated at efficiency prices must always exceed expected value of the mean output.

The first equality in Lemma 4 has yet another interpretation. By that equality,

$$\Sigma_k \pi_k [c_k(\mathbf{w},\mathbf{y})/\pi_k] = q,$$

the expected value of marginal costs normalized by probabilities equals the forward price. This fact allows us to establish (a proof of this Corollary is in the Appendix):

Corollary 5: If the forward market is unbiased,

$$\Sigma_k \pi_k [c_k(\mathbf{w},\mathbf{y})/\pi_k - q](p_k - q)/p_k > 0.$$

Because q is the expected value of $c_k(\mathbf{w},\mathbf{y})/\pi_k$, Corollary 5 establishes that the covariance between $c_k(\mathbf{w},\mathbf{y})/\pi_k$ and the percentage divergence of the state contingent price from the forward price $(p_k - q)/p_k$ is positive. Hence, on average, one finds higher marginal costs associated with state-contingent prices higher than the forward price.

We now establish a key result, namely, that given an unbiased forward market and a reasonable restriction on the probabilities, a risk-averse producer *will never choose* ***y*** *belonging to* ***I****.

Result 2: If **Z** is satisfied, there exists an i and k such that $\pi_i p_i > \pi_k p_k$ and $\pi_k \geq \pi_i$, and the forward market is unbiased, then any interior solution to (2) will not satisfy **y**$\in$ **I***.

The intuition behind this result is clear. For a risk-neutral individual, $\pi_i p_i$ is (proportional to) the marginal return from raising y_i by one unit. Unless prices and probabilities are perfectly inversely correlated, the first condition on the probabilities is always satisfied, there will always exist at least one state having this marginal return higher than another states. The second condition on the probabilities just insures that this greater marginal return comes from having p_i higher than some other prices and not from one state always being more probable than other states. So, for example, if i and k are equally probable the assumption is always satisfied if i and k have two different prices associated with them. As such, it is a very plausible assumption to make on the state of the world. Now if the producer starts with a certain output vector, all of which is sold forward, the producer is effectively risk-neutral in relation to small risks. Hence, he or she is indifferent to small changes in the dispersion of returns. But it is also clear that an increase in expected profit can be obtained by reducing output in some relatively low return states and correspondingly increasing output in some high return states. Thus, he or she has to be better off. From Result 2, it follows immediately that

Corollary 6: If **Z** is satisfied, all states are equally probable, and the forward market is unbiased, then any interior solution to (2) will not satisfy **y**$\in$ **I*** unless **p**$\in$ **I***.

Thus, the cumulative effect of Corollary 2, Result 2, and Corollary 6 is to show that so long as true price uncertainty exists the DHFAD separation result of the existing literature on futures markets arises not because a forward market expands the opportunity set available to producers, but because the standard presumption of zero production risk is implausible. The DHFAD separation result is thus an artifact of the implausible assumption of production certainty.

In the standard formulation of production under uncertainty, no arbitrage conditions of the kind derived in Lemma 4 arise. This is because in the multiplicative uncertainty case (of which the usual formulation of price uncertainty is a special case) where **y** = zθ, the only option to the producer is to increase z, yielding an increase of θ_i in state i, and this cannot be offsett exactly by any change in the forward market position.

A New Separation Theorem

Our final result on forward contracts demonstrates, however, that even under production uncertainty, a separation result can hold given an appropriate information structure on the market.

Result 3: Suppose $S = 2$, and there is price uncertainty, then any interior production equilbrium for (2) will be independent of the producer's risk preferences.

This separation result is reminiscient of spanning arguments familiar from finance theory. To see the intuition, suppose that $S = 2$ but that there is no forward market. Then the best the producer can do is to make sure at the margin that there is no systematic loss in expected profitability, holding the riskiness of the production portfolio constant. This is the import of Lemma 3. Now if the producer is given access to a foward market satisfying the conditions of the Result, then the producer's production and hedging opportunities span the states of nature thus providing a complete set of contingent markets. Put another way, the spanning of the states of nature by the investment opportunities gives the producer a way of guaranteeing a certain return in each state of nature thus insuring local risk-neutrality.

By itself, Result 3 is of somewhat limited interest because it only applies to $S = 2$. However, it points the appropriate direction in which to search for more general separation results and that is in the direction of the information structure of the market. The next section pursues this point.

Optimal Producer Behaviour in the Presence of Futures Markets

We now conclude our analysis by considering the possibility that producers do encounter basis risk. In doing so, it is important to recognize that just as producer's production alternatives under uncertainty are unduly limited by the traditional models, the producer's hedging alternatives have been unrealistically limited to contingent contracts for the commodity which he or she is produces. In reality, a much wider scope for the diversification of risk exists. There are futures markets for an increasing variety of commodities and financial instruments. Therefore, it is logical to examine the implications of this increased latitude for risk diversification on producer behavior when we consider the issue of basis risk. Hence, we now turn to an analysis of the producer's behavior when there exist an active set of K futures markets in which he or she can cross hedge. Again, the Kuhn-Tucker conditions are necessary and sufficient for producer equilibrium. Therefore, the producer's optimal behavior is characterized by:

(3) $\pi_i u'(r_i)p_i - c_i(\mathbf{w},\mathbf{y})\Sigma_k \pi_k u'(r_k) \leq 0 \qquad y_i \geq 0 \qquad (i = 1,2,...,S),$

$\Sigma_k \pi_k u'(r_k) b^j_k = 0, \qquad (j = 1,2,...,K).$

The first set of equations in (3) has exactly the same interpretation as the conditions given by (1) and (2). The second set of conditions in (3) represents a set of arbitrage conditions for the futures markets which requires the producer to make zero expected marginal utility in each of his futures operations. These arbitrage relations and the first set of equations, however, also yield an arbitrage relationship that connects the producer's production operation with his or her operations in the futures markets. We have:

Lemma 5: For an interior production equilibrium to (3), the producer's equilibrium must satisfy the arbitrage conditions:

$$\Sigma_k (c_k(\mathbf{w},\mathbf{y})/p_k) b^j_k = 0, \qquad (j = 1,2,...,K), \text{ and}$$

$$\Sigma_k (c_k(\mathbf{w},\mathbf{y})/p_k) = 1.$$

The second inequality is yet another manifestation of Lemma 3. To provide an alternative interpretation of the first condition to the requirement that the futures market be unbiased when evaluated in terms of the shadow probabilities, notice that $c_k(\mathbf{w},\mathbf{y})/p_k$ represents a risk-neutral individual's cost-benefit ratio from increasing the k-state output, b^j_k represents the k-state marginal profit on the jth futures contract. The result requires that state-contingent outputs be chosen so that these vectors be orthogonal to one another, or in other words uncorrelated. The intuition is simple: The existence of futures markets offers the producers K more markets in which to take a position. The producer being risk-averse would like to balance the risk that arises in the commodity market for which he produces (both price and production) against the risk present in these other markets. In doing so, however, he or she must insure that no systematic opportunites for increasing expected profit while holding its dispersion constant exist. The first K conditions in Lemma 5 insure this last result that all systematic opportunities for increasing expected profit have been exploited. To see this more clearly, suppose that instead of equalling zero, any one of the first K expressions in Lemma 5 were strictly positive. Let the contract for which this is true be the jth futures contract. Then the producer could increase his or her jth futures position by one unit. This generates a revenue change of b^j_k in each state k. The original state k revenue can now be restored by changing the k-state output by - b^j_k/p_k. Revenue would be unchanged but costs would now change by $-\Sigma_k(c_k(\mathbf{w},\mathbf{y})/p_k)b^j_k < 0$. Hence, the producer could achieve the exact same revenue vector but at reduced cost. Thus, the original allocation could not have been optimal.

Obvious analogues to Corollaries 1- 6 apply to Lemma 5. Therefore, we shall not repeat them but we shall leave their derivation and interpretation to the interested reader. Instead we focus on isolating sufficient conditions for a separation result. To that end we introduce the following spanning condition:

S: There exist S - 1 futures markets for which the matrix $\mathbf{B} \in \mathbf{R}^S \times \mathbf{R}^S$ is invertible:

$$\mathbf{B} = [\mathbf{e}, \mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^{S-1}]^T.$$

Here $\mathbf{e} \in \mathbf{R}^S$ is the unit vector. **S** requires **e** and the basis vectors, employing our original abuse of terminology, to span S space (quite literally the basis vectors provide a basis for S space). Notice in particular that the condition requires that no two basis vectors or any subset of the basis vectors be perfectly correlated with one another.

Result 4: Suppose **S** is satisfied then any interior solution to (3) must have the producer's production decisions independent of his or her risk preferences.

Here again the intuition is similar to that for Result 3. Therefore, it need not be repeated. However, we do note that Result 4 greatly strengthens Result 3 as it shows that as the number of independent futures markets proliferates, it becomes more likely that producers decisions will be independent of their risk preferences.

Concluding Remarks

This paper revisits the DHFAD separation results from the theory of the risk-averse, competitive firm facing price risk but not production risk using a more general formulation of production under uncertainty. Using this formulation, we show that that result hinges upon an overly restrictive representation of the technology. Hence, the DHFAD separation result will not generally apply. However, one can establish true separation results which, like the spanning results in the finance literature, depend upon the information structure of the market. In deriving these results, we also

(3) $\pi_i u'(r_i)p_i - c_i(\mathbf{w,y})\Sigma_k\pi_k u'(r_k) \leq 0 \quad y_i \geq 0 \quad (i = 1,2,...,S),$

$\Sigma_k\pi_k u'(r_k)b^j_k = 0, \quad (j = 1,2,...,K).$

The first set of equations in (3) has exactly the same interpretation as the conditions given by (1) and (2). The second set of conditions in (3) represents a set of arbitrage conditions for the futures markets which requires the producer to make zero expected marginal utility in each of his futures operations. These arbitrage relations and the first set of equations, however, also yield an arbitrage relationship that connects the producer's production operation with his or her operations in the futures markets. We have:

Lemma 5: For an interior production equilibrium to (3), the producer's equilibrium must satisfy the arbitrage conditions:

$$\Sigma_k(c_k(\mathbf{w,y})/p_k)b^j_k = 0, \quad (j = 1,2,...,K), \text{ and}$$

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The second inequality is yet another manifestation of Lemma 3. To provide an alternative interpretation of the first condition to the requirement that the futures market be unbiased when evaluated in terms of the shadow probabilities, notice that $c_k(\mathbf{w,y})/p_k$ represents a risk-neutral individual's cost-benefit ratio from increasing the k-state output, b^j_k represents the k-state marginal profit on the jth futures contract. The result requires that state-contingent outputs be chosen so that these vectors be orthogonal to one another, or in other words uncorrelated. The intuition is simple: The existence of futures markets offers the producers K more markets in which to take a position. The producer being risk-averse would like to balance the risk that arises in the commodity market for which he produces (both price and production) against the risk present in these other markets. In doing so, however, he or she must insure that no systematic opportunites for increasing expected profit while holding its dispersion constant exist. The first K conditions in Lemma 5 insure this last result that all systematic opportunities for increasing expected profit have been exploited. To see this more clearly, suppose that instead of equalling zero, any one of the first K expressions in Lemma 5 were strictly positive. Let the contract for which this is true be the jth futures contract. Then the producer could increase his or her jth futures position by one unit. This generates a revenue change of b^j_k in each state k. The original state k revenue can now be restored by changing the k-state output by $- b^j_k / p_k$. Revenue would be unchanged but costs would now change by $-\Sigma_k(c_k(\mathbf{w},\mathbf{y})/p_k)b^j_k < 0$. Hence, the producer could achieve the exact same revenue vector but at reduced cost. Thus, the original allocation could not have been optimal.

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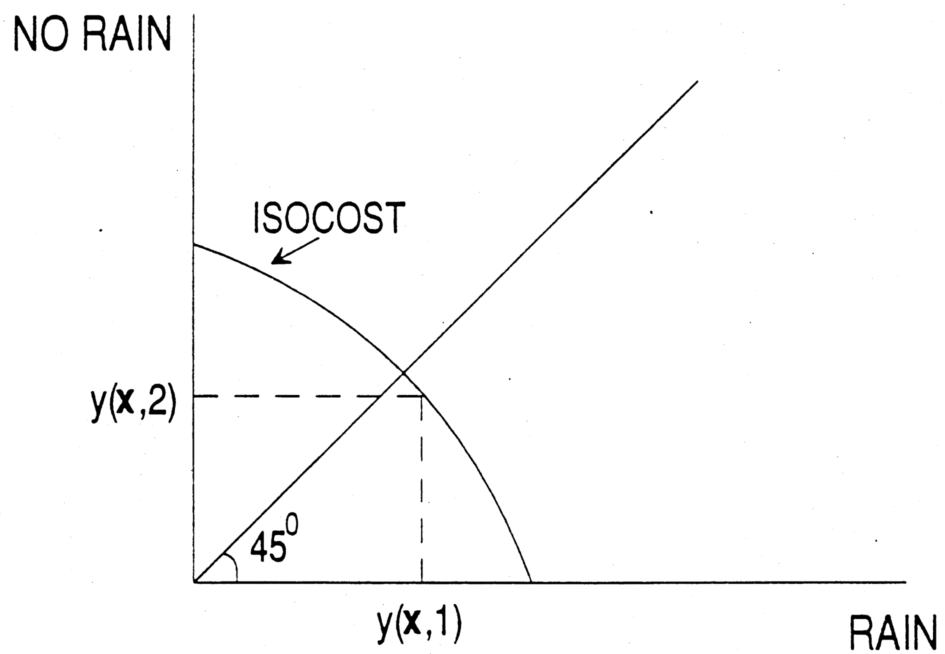

Figure 1

Proofs

Lemma 1: Both parts of the lemma are proved in an identical fashion, so the proof of the second part is left to the reader. To prove the first, suppose contrary to the lemma that $\mathbf{y(p)} \notin \mathbf{I^*}$. Then define, $\mathbf{y'} \in \mathbf{I^*}$ by $\mathbf{y'} = (y', ..., y')$ where $y' = \Sigma_i \pi_i y_i(\mathbf{p})$. Because $\mathbf{p} \in \mathbf{I^*}$, expected revenue for $\mathbf{y'}$ is the same as for $\mathbf{y(p)}$. Now notice that $\mathbf{y' = P'y(p)}$ where $\mathbf{P'}$ is the row stochastic matrix defined by having each row correspond to π . Hence, by $\mathbf{Z}$, $c(\mathbf{w,y'}) \leq c(\mathbf{w}, \mathbf{y(p)})$.

Lemma 2: $\mathbf{Z}$ implies $c(\mathbf{w}, \mathbf{P'y(p)}) \leq c(\mathbf{w}, \mathbf{y(p)})$, where $\mathbf{P'}$ is defined as in the proof of Lemma 1. Hence, for $\mathbf{y(p)}$ to be optimal it must satisfy

$$\Sigma_i \pi_i p_i y_i(\mathbf{p}) - c(\mathbf{w}, \mathbf{y(p)}) \geq \mu(\mathbf{y(p)}) \Sigma_i \pi_i p_i - c(\mathbf{w}, \mathbf{P'y(p)}).$$

Rearranging establishes

$$\Sigma_i \pi_i p_i [y_i(\mathbf{p}) - \mu(\mathbf{y(p)})] \geq c(\mathbf{w}, \mathbf{y(p)}) - c(\mathbf{w}, \mathbf{P'y(p)}) \geq 0.$$

Lemma 3: A necessary condition for a producer equilibrium is that

$$\pi_i u'(p_i y_i - c(\mathbf{w}, \mathbf{y}))p_i - c_i(\mathbf{w,y}) \Sigma_k \pi_k u'(p_k y_k - c(\mathbf{w}, \mathbf{y})) \leq 0$$

$(i = 1,2,...,S)$. Hence,

$$\pi_i u'(p_i y_i - c(\mathbf{w}, \mathbf{y})) \leq (c_i(\mathbf{w,y})/p_i) \Sigma_k \pi_k u'(p_k y_k - c(\mathbf{w}, \mathbf{y})).$$

Summing over all i gives

$$\Sigma_i \pi_i u'(p_i y_i - c(\mathbf{w}, \mathbf{y})) \leq \Sigma_i (c_i(\mathbf{w,y})/p_i) \Sigma_k \pi_k u'(p_k y_k - c(\mathbf{w}, \mathbf{y})),$$

and division using the fact that u is strictly increasing gives the result. The result for an interior solution follows immediately by using complementary slackness.

Result 1: Consider using the shadow probabilities, $c_i(\mathbf{w,y})/p_i$ $(i = 1, 2, ...,S)$, derived in Lemma 3 to maximize expected profit for the same $\mathbf{p}$ and cost structure faced by a risk-averse individual. Using these shadow probabilities leads to replacing $\pi_i p_i$ with $c_i(\mathbf{w,y})$ (the PY efficiency prices for $\mathbf{y}$) in the expected profit maximization problem. This leads a risk-neutral individual to pick the same $\mathbf{y}$ as the risk-averse individual chose. Now apply Lemma 2 using these shadow probabilities.

Lemma 4: The proof of the second equality is the same as the proof of the second equality in Lemma 3 and will not be repeated. To prove the first equality, notice that for an interior solution (2) requires that

$$\Sigma_i \pi_i u'(r_i) p_i = \Sigma_i c_i(\mathbf{w},\mathbf{y}) \Sigma_k \pi_k u'(r_k).$$

Substituting this result into the second expression in (2) yields the desired equation.

Corollary 5: If the forward market is unbiased then Theorem 2.16 of Hardy, Littlewood, and Polya (p.26) applied to the harmonic and ordinary means implies $q\Sigma_k\pi_k p_k^{-1} > 1$, whence, $\Sigma_k\pi_k(p_k - q)/p_k < 0$ and $q\Sigma_k\pi_k(p_k - q)/p_k < 0$. $\Sigma_k\pi_k(c_k(\mathbf{w},\mathbf{y})/\pi_k)(p_k - q)/p_k = \Sigma_k c_k(\mathbf{w},\mathbf{y}) - q\Sigma_k(c_k(\mathbf{w},\mathbf{y})/p_k) = 0$ using Lemma 4.

Result 2: The proof is by contradiction. Suppose contrary to the result that the interior optimizer $\mathbf{y}^* \in \mathbf{I}^*$, i.e., $y_i = \mu(\mathbf{y}^*)$ for all i. Under **Z**, the second part of Lemma 1 implies that $\mathbf{y}^*$ also solves

$$\text{Min}_{\,y}\{c(\mathbf{w},\mathbf{y}): \Sigma_i\pi_i y_i = \mu(\mathbf{y}^*)\}.$$

Hence from the first-order conditions, which are necessary and sufficient, for this problem it must be true that $c_i(\mathbf{w},\mathbf{y})/\pi_i = c_k(\mathbf{w},\mathbf{y})/\pi_k$ for all i and k. Also, because the forward market is unbiased it must be true that if this $\mathbf{y}^*$ is optimal, the optimal hedge must involve selling the entire amount forward. Thus the producer is risk-neutral in a neighborhood of the certainty outcome $\mathbf{y}^*$. Now suppose states i and k satisfy the condition stated in the Result. An increase in output by the small but positive amount δ in state i and a corresponding decrease in output by δ in state k leads to the following change in expected utility, $u'(r)\delta\,[\pi_i p_i - \pi_k p_k - c_i(\mathbf{w},\mathbf{y}^*)(1 - \pi_k/\pi_i)]$, where $r = q\,\mu(\mathbf{y}^*) - c(\mathbf{w},\mathbf{y}^*)$. Under the conditions of the Result this change in expected utility must be strictly positive contradicting the supposition that $\mathbf{y}^*$ was optimal.

Result 3: Suppose S = 2, then Lemma 4 implies

$$c_1(\mathbf{w},\mathbf{y}) + c_2(\mathbf{w},\mathbf{y}) = q, \text{ and}$$

$$c_1(\mathbf{w},\mathbf{y})/p_1 + c_2(\mathbf{w},\mathbf{y})/p_2 = 1.$$

Provided that there is price uncertainty ($p_1 \neq p_2$), this set of equations has a solution given by

$$c_1(\mathbf{w},\mathbf{y}) = p_1(q - p_2)/(p_1 - p_2), \text{ and}$$

$$c_2(\mathbf{w},\mathbf{y}) = p_2(p_1 - q)/(p_1 - p_2).$$

Corollary 1 now establishes that marginal costs are nonnegative as required and the above can be used to determine the production equilibrium independent of the producer's risk preferences.

Lemma 5: The second equation follows as in Lemma 3 and Lemma 4. For an interior solution, the first-order conditions require

$$\pi_i u'(r_i)p_i - c_i(\mathbf{w},\mathbf{y})\Sigma_k\pi_k u'(r_k) = 0 \qquad (i = 1,2,...,S), \text{ and}$$

$$\Sigma_k\pi_k u'(r_k)(f^j_k - q_j) = 0, \qquad (j = 1,2,...,K).$$

The first S conditions imply

$$\pi_i u'(r_i) = (c_i(\mathbf{w},\mathbf{y})/p_i)\Sigma_k\pi_k u'(r_k) \qquad (i = 1,2,...,S).$$

To get the first K equalities substitute this last expression into each of the K futures market conditions recognizing that $u'(r_k)$ is strictly positive.

Result 4: Under **S** the set of equalities in Lemma 5 are an invertible system of equations, independent of the producer's risk attitudes, which can be solved for the equilibrium $c_i(\mathbf{w},\mathbf{y})/ p_i$ $(i = 1,2,...,S)$.

References

Anderson, Ronald and Jean-Pierre Danthine. "Hedger Diversity in Futures Markets" **Economic Journal** 93 (1983), 370-389.

________. "Cross Hedging". **Journal of Political Economy** 89 (1981):1182-97.

Borch, Karl. **The Economics of Insurance**. North-Holland/Elseveir: New York, 1989.

Chambers, Robert G. and John Quiggin. "Production under Uncertainty". University of Maryland, May 1992.

Danthine, Jean-Pierre. "Information, Futures Prices, and Stabilizing Speculation." **Journal of Economic Theory** 17(1978), 79-98.

Feder, Gershon, Richard E. Just, and Andrew Schmitz. "Futures Markets and the Theory of the Firm under Price Uncertainty." **Quarterly Journal of Economics** (1980), 317-28.

Hardy, G., J.E. Littlewood, and G. Polya. **Inequalities** (Second Edition). Cambridge University Press: Cambridge, 1989.

Holthausen, Duncan M. "Hedging and the Competitive Firm under Price Uncertainty." **American Economic Review** 69 (1979): 989-995.

Marshall, Albert and Ingram Olkin. **Inequalities: Theory of Majorization and its Applications.** Academic Press: New York, 1979.

Peleg, B. and M.E. Yaari. "A Price Characterization of Efficient Random Variables." **Econometrica** 43(1975): 283-292.