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INPUT-OUTPUT SEPARABILITY IN PRODUCTION MODELS AND ITS STRUCTURAL CONSEQUENCES

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1. Introduction

The introduction of dual methods into empirical production analysis has brought an explosion of empirical models of supply-response and derived-demand systems. Even though dual studies of production systems have often proved more flexible than primal studies, econometric reality has often forced researchers to impose structural restrictions on the technology prior to empirical analysis. One of the most common is using a single aggregate output to represent a multioutput production technology. Formally, the existence of an aggregate output for a multioutput technology is justified by the presumption that the underlying production technology is *separable in outputs* (Chambers, 1988, p. 285). Chambers (1988, p. 286) shows that if the technology can be represented by a strictly monotonic product transformation function, output separability implies the existence of an aggregate input, i.e., output separability and input separability are equivalent. This paper extends that result by showing under two alternative assumptions that input separability and output separability are equivalent notions.

In what follows, we first introduce the technology, several scalar-valued representations of the technology, several basic duality results, and formal definitions of input and output separability. Section three shows that if either the input distance function or the output distance function satisfy a monotonicity condition, separability of outputs from inputs is equivalent to separability of inputs from outputs. Under the alternative assumption that if a profit function exists, output separability and input separability are also equivalent.

2. The Technology

The technology, i.e., the set of feasible input and output vectors, is denoted by T where

(2.1) $$T = \{(x, y) : \text{input } x \in \mathbb{R}_+^N \text{ can produce output } y \in \mathbb{R}_+^M\}.$$

The input distance function is defined on T as

(2.2) $$D_i\,(x, y) = \sup\,\{\lambda : (x/\lambda, y) \in T\}.$$

One can show (Färe, 1988) that this function completely characterizes T, i.e., T can be recovered from $D_i\,(x, y)$ by

(2.3) $$T = \{(x, y) : D_i\,(x, y) \geq 1\}.$$

Denote input prices by $w \in \mathbb{R}_+^N$. If the technology has convex, closed input requirement sets $L(y) = \{x : (x, y) \in T\}$ and its inputs are freely disposable, i.e., $x \in L(y), \hat{x} \geq x \rightarrow \hat{x} \in L(y)$, the following duality theorem holds (Shephard, 1953; 1970, p. 265).

(2.4) $$\begin{aligned} C(w, y) &= \min_x \{wx : D_i(x, y) \geq 1\} \\ D_i(x, y) &= \min_w \{wx : C(w, y) \geq 1\} \end{aligned}$$

(For some related technical conditions, see Färe (1988, pp. 83-87).) This theorem states that the cost and input distance functions are "dualistically determined from each other" (Shephard, 1970, p. 265). That is, given the distance function, the cost function is determined by finding the input vectors that minimize cost, and the distance function is obtained from the cost function by finding the (cost deflated) price vector that minimizes costs.

The second duality theorem that is used here is the duality between the revenue function and the output distance function. The latter function is defined by

(2.5) $$D_0\,(y, x) = \text{Inf}\,\{\theta : (x, y/\theta) \in T\},$$

and the technology can be recovered from $D_0\,(y, x)$ as

(2.6) $$T = \{(x, y) : D_0 (y, x) \leq 1\}.$$

Thus both the input and output distance functions are complete characterizations of the technology.

Denote output prices by $p \in \mathbb{R}^M_+$, and suppose that the output correspondence $P(x) = \{y : (x, y) \in T\}$, $x \in \mathbb{R}^N_+$, is compact, convex and that outputs are freely disposable, i.e., $\hat{y} \geq y \in P(x) \rightarrow \hat{y} \in P(x)$, then by Shephard (1970, p. 271) we have

(2.7)
$$R(p, x) = \max_y \{py : D_0(y, x) \leq 1\}$$
$$D_0(y, x) = \max_p \{py : R(p, x) \leq 1\}$$

The duality theorem (2.7) shows that the output distance function and the revenue function are dually determined from each other.

Provided the profit function exists, the following dualities are also well-known (Chambers, 1988):

(2.8)
$$\pi(p, w) = \text{Max}_y \{py - C(w, y)\}$$
$$C(w, y) = \text{Max}_p \{py - \pi(p, w)\}, \text{ and}$$

(2.9)
$$\pi(p, w) = \text{Max}_x \{R(p, x) - wx\}$$
$$R(p, x) = \text{Min}_w \{\pi(p, w) + wx\};$$

and we see clearly that

(2.10)
$$\pi(p, w) = \max_{x,y} \{py - wx : D_0(x, y) \leq 1\}$$
$$D_0(x, y) = \inf \{\theta : \pi(p, w) \geq p(y/\theta) - wx \text{ for all } p \text{ and } w\}, \text{ and}$$

$$
(2.11) \qquad \begin{aligned} \pi(p, w) &= \max_{x,y} \{py - wx : D_i(x, y) \geq 1\} \\ D_i(x, y) &= \sup \{\lambda : \pi(p, w) \geq py - w(x/\lambda) \text{ for all } p \text{ and } w\}. \end{aligned}
$$

Definition 1: A technology is separable in outputs if T is representable by

$$\hat{T} = \{(x, h) : x \text{ can produce } h\}$$

where $h : \mathbb{R}_+^M \to \mathbb{R}_+$ is an aggregate output $h(y)$.

Definition 2: A technology is separable in inputs if T is representable by

$$\hat{T} = \{(g, y) : g \text{ can produce } y\}$$

where $g : \mathbb{R}_+^N \to \mathbb{R}_+$ is an aggregate input $g(x)$.

The following is now obvious.

Lemma 1: The technology is separable in outputs if and only if the input distance function can be written:

$$D_i(x, y) = \Delta_i(x, h(y)),$$

where $\Delta_i(x, h(y))$ is an input distance function; and the technology is separable in inputs if and only if the output distance function can be written:

$$D_0(y, x) = \Delta_0(y, g(x))$$

where $\Delta_0(y, g(x))$ is an output distance function.

3. The Results

Our results are summarized by two theorems:

(3.1) **Theorem:** If the technology is separable in inputs and $\Delta_0(y, g(x))$ is strictly monotonic in g, the technology must be separable in outputs. If the technology is separable in outputs and $\Delta_i(x, h(y))$ is strictly monotonic in h, the technology must also be separable in inputs.

(3.2) **Proof:** We only show the first part of the theorem, the second is proved in a similar fashion. Suppose the technology is separable in inputs, then by the Lemma the output distance function must be expressible as $\Delta_0(y, g(x))$. From (2.8)

$$\begin{aligned} D_i(x, y) &= \sup \{\lambda : \Delta_0(y, g(x/\lambda)) \le 1\} \\ &= \sup \{\lambda : g(x/\lambda) \le \Delta_0^{-1}(y, 1)\} \\ &= \Delta_i(x, h(y)). \end{aligned}$$

(3.3) **Remark:** If the monotonicity condition is satisfied, then the cost function is of the form $C(w, h(y))$ if and only if the revenue function is of the form $R(p, g(x))$. Hence, the presumption of output separability now implies that the technology can be modelled as if only a single input existed to the production process. Just as the assumption of output separability heuristically implies that there exists an aggregate production function, $f(x)$, satisfying $h(y) = f(x)$, we see that it also implies that there exists an input requirement function, $m(y)$, satisfying $g(x) = m(y)$.

(3.4) **Theorem:** If the profit function exists, the technology is separable in inputs if and only if the technology is also separable in outputs.

(3.5) **Proof:** Suppose the profit function exists. We establish the result by showing that output separability and input separability yield the same profit function. Hence, they must be equivalent. Suppose the technology is output separable, then by (2.4) $C(w, y) = \hat{C}(w, h(y))$. The associated profit function is

$$\begin{aligned} \pi(p, w) &= \mathrm{Max}_y \{py - \hat{C}(w, h(y))\} \\ &= \mathrm{Max}_m \{\mathrm{Max}_y \{py : h(y) \le m\} - \hat{C}(w, m)\} \\ &= \mathrm{Max}_m \{\hat{R}(p, m) - \hat{C}(w, m)\}. \end{aligned}$$

Now suppose the technology is input separable, then by (2.4) $R(p, x) = \hat{R}(p, g(x))$. Hence,

$$
\begin{aligned}
\pi(p, w) &= \mathrm{Max}_x \, \{ \hat{R}(p, g(x)) - wx \} \\
&= \mathrm{Max}_g \, \{ \hat{R}(p, g) - \mathrm{Min}_x \, \{ wx : g(x) \geq m \} \} \\
&= \mathrm{Max}_g \, \{ \hat{R}(p, g) - \hat{C}(w, g) \}.
\end{aligned}
$$

(3.6) **Remark:** If the technology is compact, then by the Weierstrass theorem the profit function exists. Both input separability and output separability lead to conditionally additive profit functions of the same form. Intuitively, therefore, input and output price effects can always be decomposed in the following fashion: the effect of input price variation upon profit maximizing supplies is always manifested through its effect on the aggregate input g; the effect of output price variation on profit maximizing derived demands is always manifested through its effect on the aggregate output m. That both theorems hold is of course due to the duality theorems.

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