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Production Under Uncertainty

by

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Models of production under uncertainty are central to modern economic theory. Besides its obvious relevance to the theory of the firm facing uncertainty (Agnar Sandmo) and the literature on price stabilization (David M.G. Newbery and Joseph Stiglitz), production under uncertainty is also central to the incentives, contracting, and principal-agent literatures . Principal-agent models, in particular, are increasingly being applied to problems in all areas of economics. An incomplete sampling would include problems as diverse as insurance (Arthur Raviv), auction theory (John Riley and William Samuelson), the failure of labor markets to clear (the efficiency-wage literature), market regulation (Tracy Lewis and David Sappington), and optimal risk sharing in the face of moral hazard (Mark Pauly; Bengt Holmstrom).

While state-contingent commodities, production, and markets play a central role in general-equilibrium uncertainty models (Kenneth Arrow; Gerard Debreu; Roy Radner), the theory of production under uncertainty (Sandmo; Hayne Leland; Richard Hartman; Yasunori Ishii; Gershon Feder; and Newbery and Stiglitz) makes little use of these concepts. Broadly speaking, producers are there viewed as choosing an input vector or scalar output prior to the realization of a continuous random variable (either a random price, a random demand, or a random production input) to maximize expected utility. The combination of the input vector and the realization of the random variable uniquely determines the producer's *ex post* return. Suppose, for example, the random variable represents rainfall. Then for a given input vector, the output for each rainfall level is uniquely determined. Once producers select their input bundle, they have no control over the output they receive. If the random variable only assumes two values ("no rain" and "rain"), the transformation function between "no-rain" output and "rain" output is necessarily (explained below) of the fixed coefficient form illustrated in Figure 1.

The problem of production under uncertainty is, thereby, trivialized because producers are assumed to be incapable of arraying their available resources to prepare for different contingencies such as flood or drought. The traditional model arbitrarily and unnecessarily relegates producer decisionmaking to selecting an input bundle with desireable risk properties.[1]

This paper suggests an alternative approach to producer decisionmaking under uncertainty that is simultaneously more realistic, more general, and (perhaps most importantly for economists) more analytically tractable than the traditional approach. Moreover, the approach is congruent with the modern axiomatic approach to nonstochastic production analysis (Rolf Fare; Ronald Shephard), the state-contingent approach of Arrow and Debreu, and the modern approach to decisions under uncertainty, in which actions are represented as mappings from a state space to a space of outcomes. Thus, our approach offers a natural bridge between these apparently related (but previously disparate) literatures.

The central idea is that producers choose not only an input vector but a state-contingent output vector as well. For any input bundle, a large set of state-contingent output vectors may be feasible. In the rainfall example, producers can allocate capital and labor in a way which protects them against low rainfall. Alternatively, they may allocate the same capital and labor endowment in a way which yields high returns when rainfall is high, and low or negative returns when rainfall is low.

In what follows, we first develop a notion of a state-contingent production technology. The state-contingent technology is shown to generalize all existing models of production under both price and production uncertainty. The technology is then used to analyze the production decisions of a risk-averse producer under a very general (i.e. more general than is usually considered) version of the expected utility model. The first step is the development and characterization of an effort-cost function having properties in state-contingent revenues that are entirely analogous to properties usually possessed by multiple-output cost functions (e.g. nonnegative and increasing marginal cost). The effort-cost function is then used to characterize the production decisions of a risk-averse, expected-utility maximizing producer in terms of two expected-utility functions: One (the π-expected utility function) maps probability space into *ex post* returns, the other (the p-expected utility function) maps state-contingent price schedules into *ex post returns*.

The effort-cost function is observationally equivalent to a partial ordering that evaluates uncertain revenue alternatives exactly as a risk averter would. In the equal probability case, this latter result implies that the effort-cost function is always observationally equivalent to an S-concave function.

After characterizing the behavior of risk-averse producers, the power of our approach is illustrated by applying it to the special case of the additively separable utility structure studied extensively by David Newbery and Joseph Stiglitz. Our first result there establishes a duality between the effort-cost function and the π-indirect expected utility function. The remainder of the illustration develops comparative static results for the additively separable case that both generalize and extend existing results for this model. The final section concludes by summarizing and discussing some of the many possible future applications of our model to problems involving decisionmaking under uncertainty: moral hazard, adverse selection, insurance, futures market analyses are but a few examples.

1. A State-Contingent Technology

The standard approach to production uncertainty specifies a production function that depends upon physical inputs committed prior to the resolution of uncertainty and a random input that indexes the state of nature. In that model, output price uncertainty is equivalent to having the random input shift the production function multiplicatively. Letting $x \in \mathbb{R}^n_+$ denote the physical inputs, $\theta \in \mathbb{R}_+$ denote the random input, and $f : \mathbb{R}^{n+1}_+ \rightarrow \mathbb{R}$ denote the production function, random output $z(\theta)$ is then defined by

$$z(\theta) = f(x,\theta).$$

Our approach follows Arrow and Debreu by dealing in state-contingent commodities. (There is no requirement, however, that a complete set of contingent markets exist.)

Following Debreu, assume that "Nature" makes a choice from among a finite set of alternatives. Each of these alternatives is called a "state" and is indexed by a finite set of the form $\Omega = \{1,2,3,\ldots,S\}$. S, thus, denotes the number of different states of nature. Once the index is given, all possible factors determining production conditions (weather, etc.) are known.

Production relations are governed by a technology set $T \subseteq \mathbb{R}^n_+ \times \mathbb{R}^{(mxS)}_+$ defined by

$$T = \{ (x,z) : x \text{ can produce } z,\ x \in \mathbb{R}^n_+,\ z \in \mathbb{R}^{mxS}_+\}.$$

Here x is an input vector committed prior to the realization of the index of the state of nature and z is a matrix of state-contingent outputs with typical element z_{ij} (i=1,...,m) (j=1,...,S) corresponding, *ex ante,* to the amount of the ith output that would be produced if the jth state of nature occurs. Multiple outputs are explicitly allowed. Notice, however, that if m = 1 (a single output) the technology, *ex ante,* is formally identical to the standard case of multiple-output production under certainty where n inputs are used to produce S outputs. The only difference is that the outputs are now state-contingent. Thus, only one output level actually occurs *ex post.* In the more general case, for each state there is a distinct output vector with m separate entries each corresponding to a distinct output.

In line with the traditional approach, all inputs are assumed to be chosen prior to the resolution of uncertainty. However, it is easy to generalize T to cover the more realistic case of sequential resolution of uncertainty by redefining z to include negative entries corresponding to inputs committed after the resolution of uncertainty. Similarly, it is also possible to extend T to cover the case where some outputs are produced under conditions of certainty by redefining x to include negative entries corresponding to outputs produced under certainty.

To relate T to the more traditional approach, consider the output correspondence $Z: \mathbb{R}^n_+ \rightarrow \mathbb{R}^{mxS}_+$ that maps an input bundle into subsets $Z(x) \subseteq \mathbb{R}^{mxS}_+$ of state-contingent outputs:

$$Z(x) = \{ z : (x,z) \in T \}.$$

In words, Z(x), the *state-contingent output set,* represents all feasible combinations of state-contingent outputs for input committal, x. Z(x) is easiest visualized by considering the traditional case of a scalar output generated by the production function f(x, θ) when θ can only assume two discrete values (1 and 2). Then,

$$Z(x) = \{ z \in \mathbb{R}^2_+ : f(x,1) \geq z_1 \text{ and } f(x,2) \geq z_2 \}.$$

Z(x), in this case, is depicted graphically by the shaded area in Figure 1. The outer boundary of Z(x), formally the efficient subset of Z(x), might be heuristically thought of as the transformation function between state-1 and state-2 contingent outputs. The state-contingent output set in Figure 1 corresponds to what would be derived from a fixed-coefficient transformation function (e.g. Robert Chambers, p. 266).

In the traditional model, only the vertex of Z(x) in Figure 1 can ever be observed. This happens because the standard model explicitly forces the inequalities in the definition of Z(x) to be equalities. Once inputs are chosen the range of outputs available effectively degenerates to a single point in $\mathbb{R}^{mxS}_+$. Even if the producer wished to operate at a point like A in Figure 1, it is precluded by assumption. This restriction, which departs markedly from most modern representations of technology, necessarily circumscribes the analytical results that emerge by imposing an overly narrow notion of technical efficiency.

The single-output, fixed-coefficient nature of this state-contingent output set illustrates the principal shortcomings of the production function approach: once producers have selected the input bundle they have no control over the single output they ultimately receive. After the input bundle is chosen there is no substitutability between state-contingent outputs. This is entirely unrealistic in most cases because it implies producers cannot organize their inputs in a manner that prepares differentially for different contingent outcomes. A more general and realistic approach allows producers this flexibility. Pictorially, this implies allowing the transformation function in Figure 1 to assume something other than a fixed-coefficient form. T affords this flexibility.

To develop analytical results, it is convenient to consider the natural inverse of Z(x) -- the input correspondence $V: \mathbb{R}_+^{mxS} \to \mathbb{R}_+^n$ that maps the state-contingent output array into subsets $V(z) \subseteq \mathbb{R}_+^n$ of inputs

$$V(z) = \{x: (x, z) \in T\}.$$

V(z), *the input set,* gives the input combinations that can produce the state-contingent output array z. Returning to the production-function representation where θ can only assume two values then

$$V(z) = \{ x : f(x,1) \geq z_1 \text{ and } f(x,2) \geq z_2\}.$$

Thus, V(z) is the intersection of the upper contour sets (in x) of the production function evaluated at θ = 1 and θ = 2.

Developing analytical results requires specifying properties of T (axioms). Our axioms are:

Properties of the Input Set (V):

1. V(z) is nonempty;
2. $\mu V(z') + (1-\mu) V(z^o) \subseteq V(\mu z' + (1-\mu)z^o)$; and
3. for $z' \geq z$, $V(z') \subseteq V(z)$.

Property V.1 requires that z be producible. Property V. 2, implies that T is a convex set (see Rolf Färe). In a static production model, convexity of T is equivalent to concavity of the scalar production function in inputs. Concavity in inputs is typically imposed in the standard model of production uncertainty. Property V.3 requires the input set to exhibit free disposability of output. In words, V.3 says that if an input bundle can be used to produce z that same input bundle is capable of producing any smaller output array, there is no congestion among outputs. Pictorially V.3 allows for points like A in Figure 1.

Before proceeding, several comments should be made about properties V. Perhaps most obviously, V contains no analogue of positive marginal productivities of inputs. Although intuitive and universally imposed, such an assumption is unnecessary to what follows. Second, all of the axioms are not necessary for all of the results that follow. For example, the main role of V.3 is to guarantee monotonicity (positive marginal costs) of the effort-cost function developed below. While intuitive, and graphically convenient, effort-cost monotonicity (and hence V.3) is required only to provide a lower bound for the indirect expected utility functions. V.2, on the other hand, is critical and represents a central assumption in what follows.

Often it is desireable to work in terms of monetary returns from the technology. All existing models of producer decisionmaking under uncertainty can be represented by a canonical version of the current model expressed in terms of state-contingent revenues. For the case of production uncertainty only, this requires introducing a vector $p \in \mathbb{R}^m_{++}$ of output prices and a fixed payment (cost, asset) $a \in \mathbb{R}$. Because there exists no price uncertainty, prices are not differentiated according to the state of nature that occurs. V(z) induces a representation of the technology in terms of state-contingent revenues. Formally,

$$V^{pa}(y) = \{ x : y_i = a + pz_{.i} \ (i = 1,\ldots,S) \text{ and } (x,z) \in T\}.$$

Here $z_{.i} \in \mathbb{R}^m_+$ is the i-state-contingent output vector. The notation $V^{pa}(y)$ reminds the reader that this representation of the technology is for fixed p and a.

In the case of pure price uncertainty (no production uncertainty), i.e., $z \in \mathbb{R}^m_+$ there exists a complete set of state-contingent prices $p \in \mathbb{R}^{mxS}_{++}$. A representation of the technology in terms of state-contingent returns is

$$V^{pa}(y) = \{ x : y_i = a + p_{.i}z \text{ and } (x,z) \in T,\ x \in \mathbb{R}^n_+,\ z \in \mathbb{R}^m_+\}.$$

Finally, in the case of joint production and contract uncertainty[2] there exists a set of state-contingent fixed payments (assets, costs) $a \in \mathbb{R}^S$ and a set of state-contingent output prices $p \in \mathbb{R}^{mxS}_{++}$. A representation of the technology in terms of state-contingent revenues is given by

$$V^{pa}(y) = \{ x : y_i = a_i + p_{.i}z_{.i} \text{ and } (x,z) \in T\}.$$

In an abuse of terminology and notation, the same notation is used for each of the three different types of uncertainty, and $V^{pa}(y)$ is referred to as the input set. This is done for two reasons: to reinforce the notion that $V^{pa}(y)$ is a canonical technology; and for given p and a, $V^{pa}(y)$ is easily shown to satisfy properties V when z's are replaced by y's. (A demonstration of this fact is left to the interested reader.)

3. Producer Preferences

The producer's information and or beliefs about the relative likelihood of Nature picking a particular state is summarized by $\pi \in \Pi \subseteq \mathbb{R}^{S}_{++}$ where

$$\Pi = \{ \pi : \pi \in \mathbb{R}^{S}_{++} \text{ and} \sum_{l=1}^{S} \pi_l = 1\}.$$

No state occurs with zero probability. The present paper only restricts itself to expected-utility maximization, although more general behavioral models can easily be accomodated. Producer preferences over state-contingent returns and inputs, therefore, are captured by $W: \mathbb{R}^{S}_{++} \times \mathbb{R}^{(m\times S)} \times \mathbb{R}^{n}_{+} \rightarrow \mathbb{R}$

$$W(y,x) = \sum_{l=1}^{S} \pi_l w(y_l, x)$$

where the elementary (*ex post*) utility function $w: \mathbb{R} \times \mathbb{R}^{n}_{+} \rightarrow \mathbb{R}$ satisfies

$$w(y_l,x) = F(y_l, g(x)).$$

Here $F:\mathbb{R}_{+} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, strictly increasing and concave in y_l, and nonincreasing and concave in g while $g: \mathbb{R}^{n}_{+} \rightarrow \mathbb{R}_{+}$ is nondecreasing, continuous, and convex. F satisfies the von-Neumann-Morgenstern postulates. Special cases of F include the expected utility of net return model

$$F(y_l, g(x)) = \bar{F}(y_l - g(x))$$

with $\bar{F}$ strictly increasing and strictly concave and the separable utility model

$$F(y_l, g(x)) = u(y_l) - g(x)$$

with u strictly increasing and concave.

4. The Effort-Cost Function

The function g measures the producer's disutility of committing the input bundle x to the uncertain production process. Special cases of g(x) include

$$g(x) = G(wx)$$

with $G: \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and strictly convex and $w \in \mathbb{R}^n_{++}$ a vector of input prices. The *effort-cost function*, $c: \mathbb{R}^S_+ \to \mathbb{R}$, is defined by

$$c(y) = \text{Min } \{g(x) : x \in V^{pa}(y)\}.$$

Result 1: The effort-cost function, c(y), satisfies:

1. $c(y') \geq c(y) \geq c(0_S)$ for $y' \geq y$;
2. $\mu c(y') + (1 - \mu)\, c(y^o) \geq c(\mu y' + (1 - \mu)\, y^o) \qquad 0 < \mu < 1$;
3. for y restricted to the domain $\mathbb{R}^S_{++}$, c(y) is continuous.

The effort-cost function measures in utility units the cost of producing a given state-contingent revenue vector. c(y) has essentially the same properties as are usually imposed on multiple-output cost functions: Marginal cost for each state-contingent revenue is nonnegative (property 1) and nondecreasing (property 2). Moreover, if g(x) = wx, the effort-cost function has all the properties traditionally associated with cost functions (homogeneity and concavity in w and Shephard's Lemma) (Shephard; Rolf Fare).

The effort-cost function derived here is based on the canonical representation of the technology, $V^{pa}(y)$, and thus holds for fixed p and a (suppressed notationally). One can also define an effort-cost function mapping the primitives, i.e., the state-contingent outputs, into effort units. The properties of such an effort-cost function, apart from its domain, are identical to those in Result 1 after replacing y with z. (The derivation of these properties is left to the reader but the method of proof is virtually identical to the proof of Result 1. An effort-cost function of this type is used in section 6.)

Although properties 1.1 - 1.3 are virtually identical to those of multiple-output cost functions under certainty, they now have a somewhat different economic meaning. Figure 2 depicts the isocost contour:

$$I(C) = \{ y \in \mathbb{R}^S_+ : c(y) = C \}$$

for S = 2. By Result 1.1, I(C) is negatively sloped. Result 1.2 implies that I(C) is concave to the origin as drawn. For concreteness sake, take the two states of nature to be "rain" (measured along the vertical axis) and "no rain" (measured along the horizontal axis). The point (A) where a bisector cuts $I(C_1)$ represents the certainty outcome (same revenue in both states) for that cost level. The slope of $I(C_1)$ at A measures the rate at which rain-state revenue must be sacrificed in order to compensate exactly (in effort-cost units) for increases in no-rain revenues. As such, it represents a local measure of technologically induced "risk" (or, alternatively, of the cost of self insuring). Suppose that both states are equally probable. In Figure 2, more than one unit of rain-state revenue must be sacrificed to increase no-rain revenue by one unit along $I(C_1)$. Thus, moving from the certainty outcome at A to, say, point B implies W(y,x) falls. At A the marginal utilities of both the "rain" and "no rain" revenues are equal but moving to B implies rain revenues fall more than no-rain revenues rise. Because cost is constant, moving from A to B always means a utility loss. Hence, no risk-averse individual would operate on $I(C_1)$ below the bisector.

By the same reasoning, moving from B to A always implies a utility gain. But A involves "complete self-insurance". Thus, the curvature of I(C) offers a natural measure of the insurance premium associated with points A and B.

Figure 2 also illustrates another important shortcoming of the standard model of production under uncertainty. That is, unlike the present model, it does not recognize that whether a particular state of nature would be classed as either "good" or "bad" in some generic context generally depends on the technology. Consider $I(C_2)$ in Figure 2. As drawn, the slope of $I(C_2)$ at the bisector just reverses the situation at A. Now, no risk-averse individual would operate on $I(C_2)$ above the bisector. Put another way, whereas on $I(C_1)$, the no-rain state is the one requiring insurance, now the rain state requires insurance. This might occur, for example, if the input bundle consistent with $I(C_2)$ was devoted mainly toward drought control. Only for very special functional structures, for example, c(y) homothetic, will it be true that the division between good and bad states is independent of the scale of operation.

5. The π-Indirect Expected Utility Function

The producer chooses a state-contingent revenue vector to solve

$$U(\pi) = \max_y \{\sum_{i=1}^{S} \pi_i F(y_i, c(y))\}.$$

$U: \Pi \to \mathbb{R}$ is the *π-indirect expected utility function.* The convexity of c(y) and the strict concavity of F guarantee an unique solution. Denote

$$y(\pi) = \operatorname{argmax}_y \{\sum_{i=1}^{S} \pi_i F(y_i, c(y))\}.$$

Our next result establishes the properties of U(π) and y(π).

Result 2: U(π) and y(π) satisfy:

1. $U(\pi) \geq F(0, c(0_S))$;
2. $\mu U(\pi') + (1 - \mu) U(\pi^o) \geq U(\mu\pi' + (1 - \mu)\pi^o) \quad 0 < \mu < 1$;
3. U(π) is continuous;
4. $\sum_{i=1}^{S} (\pi_i' - \pi_i^o)[F(y_i(\pi'), c(y(\pi'))) - F(y_i(\pi^o), c(y(\pi^o)))] \geq 0$; and
5. if $y \succ_S y(\pi)$ then $c(y) \geq c(y(\pi))$.

In the statement of the result $a \succ_S b$ is to be read "a second-order stochastically dominates b given π."

Result 2.1 establishes a lower bound for the π-indirect expected utility function. Property 2.2 is that the π-indirect expected utility function is convex in π. Convexity here is a well-known consequence of the producer's objective function being linear in the probabilities (i.e., the expected utility model). The economic implications of this result, however, are somewhat different than usually derived from the convexity properties of other indirect objective functions. Here convexity implies that the value of information is positive. Suppose the producer can observe a signal which takes the value 0 with probability μ and 1 with probability $(1-\mu)$. The producer's subjective probability distribution, given the observance of a signal of 0 (resp. 1) is given by π^o (resp. π'). Without a signal, the producer's subjective probability distribution is $\mu\pi^o + (1 - \mu)\pi'$. Result 2.2 implies that it is always beneficial to observe the signal.

Result 2.3 says that $U(\pi)$ has no breaks. Result 2.4 implies that changes in the probability vector and changes in the elementary revenue utility function, at the optimum, are positively "correlated". Intuitively, therefore, one expects an increase in a particular state's probability of occurrence to be associated with an increase in the utility maximizing revenue for that state once cost levels are compensated. It is misleading, however, to infer from 2.4 that the π-indirect expected utility function is increasing or nondecreasing in any particular probability. The simplicial nature of Π precludes any single probability from changing in isolation.

Finally, 2.5 shows that in a neighborhood of the equilibrium, the effort-cost function defines a partial ordering over uncertain revenue alternatives that is equivalent to what a risk-averse individual would choose. To understand 2.5 note that if $y \succ_s y(\pi)$ but $c(y) < c(y(\pi))$, then a risk-averse producer should prefer y to $y(\pi)$ violating the definition of $y(\pi)$. Moreover, in the equal probability case, i.e., $\pi_i = 1/S$ $(i = 1,\ldots,S)$, the following corollary follows immediately from property 5 in Result 2:

Corollary 2.1: If $\pi_i = 1/S$ $(i = 1,\ldots,S)$ then $c(y) \geq c(y(\pi))$ if $y \succ_m y(\pi)$.

The notation $a \succ_m b$ is to be read "a is majorized by b" or more simply "b majorizes a". Therefore, the Corollary implies that in a neighborhood of the equilibrium, the effort-cost function is always consistent with Schur concavity (abbreviated as S-concavity) in the equal probability case. (For the definition and discussion of majorization, S-concavity, and related concepts, consult Albert Marshall and Ingram Olkin, notice, however that our notation differs slightly from their notation.) Intuitively, for y to majorize y′ means that both these state contingent revenue vectors have the same mean but that y′ is "more evenly" distributed than y. Or, more simply, y is riskier than y′. If regions of c(y)'s domain exist for which 2.5 is not satisfied, an expected-utility maximizer will never produce in those regions.

6. The p-Indirect Expected Utility Function

As noted earlier, just as one can define an effort-cost function in terms of revenue, one can develop an effort-cost function in terms of the primitives, i.e. the state-contingent outputs. Define this effort-cost function by

$$C(z) = \text{Min } \{ g(x) : x \in V(z)\}.$$

The reader can easily verify that C(z) satisfies properties 1.1 - 1.3 in Result 1 (apart from the obvious change in domain). Frequently, one is interested in determining how the state-contingent vectors a and p affect the allocation of state-contingent outputs by the producer. The π-indirect expected utility function, which suppresses these vectors, is inappropriate this case. This section develops a representation that can be used. To conserve on notation and to emphasize the role of p and a, we revert to the equal-probability case. And for simplicity we also concentrate on the case of a scalar output (the results easily extend to the case of vector outputs)

$$y_l = a_l + p_l z_l.$$

Define the *p-indirect expected utility function* $\mathcal{U}$: $\mathbb{R}_+^{S\times 2} \to \mathbb{R}$ by

$$\mathcal{U}(p,a) = \text{Max } S^{-1} \sum_{l=1}^{S} F(a_l + p_l z_l, C(z)).$$

The strict concavity and monotonicity of F() and the convexity of C(z) insure that a unique global solution exists to this problem. Denote the optimizer

$$z(p,a) = \operatorname{argmax} S^{-1}\sum_{i=1}^{S} F(a_i + p_i z_i, C(z)).$$

Our next result develops the properties of $\mathcal{U}(p,a)$ and $z(p,a)$:

Result 3: $\mathcal{U}(p,a)$ and $z(p,a)$ satisfy:

1. $\mathcal{U}(p,a) \geq S^{-1}\sum_{i=1}^{S} F(a_i, C(0_S))$;

2. $\mathcal{U}(p,a)$ is nondecreasing in a;

3. $\mathcal{U}(p,a)$ is nondecreasing in p;

4. (i) $\mathcal{U}(p,a)$ is concave in a;

   (ii) if F is jointly concave in p and z, $\mathcal{U}(p,a)$ is concave in p;

5. for a restricted to $\mathbb{R}^S_{++}$ $\mathcal{U}(p,a)$ is continuous in a;

6. $$\sum_{i=1}^{S} F[a_i' + p_i' z_i(p', a'), C(z(p',a'))] - \sum_{i=1}^{S} F[a_i' + p_i' z_i(p^\circ,a^\circ), C(z(p^\circ,a^\circ))]$$

$$+ \sum_{i=1}^{S} F[a_i^\circ + p_i^\circ z_i(p^\circ, a^\circ), C(z(p^\circ,a^\circ))] - \sum_{i=1}^{S} F[a_i^\circ + p_i^\circ z_i(p', a')), C(z(p',a'))] \geq 0;$$

7. if $a + pz \succ_m a + pz(p,a)$ then $C(z) \geq C(z(p,a))$.

Property 3.1 is the same result as 2.1 for this formulation. Properties 3.2 and 3.3 show that in any state of nature the producer always prefers either a higher initial wealth or a higher commodity price. Properties 3.4 (i) and (ii) are easily interpreted in terms of randomization of payment schedules. Suppose that the states of Nature (i = 1,2,...,S) actually refer to weather states. If demand conditions for the commodity depend upon random factors other than weather, returns from producing a given level of output in state i may themselves be random. To illustrate, suppose that if state i occurs and the producer produces z_i, the producer's return is $y_i' = p_i'z_i + a_i'$ with probability μ and $y_i^o = p_i^o z_i + a_i^o$ with probability $1 - \mu$. Result 3.4 (i) says that the producer always prefers to receive the expected value of the downpayment $\mu a_i' + (1 - \mu)a_i^o$ $(i = 1,2,\ldots,S)$ for a given p_i to facing the additional uncertainty that the randomization of the downpayment introduces. Property 3.4 (ii) gives a sufficient condition for the producer to prefer facing $\mu p_i' + (1 - \mu)p_i^o$ (for given a_i) rather than facing the additional uncertainty that weather-state contingent randomization of the output price brings. (Randomization of returns is discussed further in the next section.) Property 3.5 is a smoothness condition. Property 3.6 is essentially the same as property 2.4 except stated in terms of prices and initial wealths. Property 3.7 is another manifestation of 2.5.

If $\mathcal{U}(p,a)$ is differentiable it also manifests a generalization of Hotelling's Lemma:

$$z_i(p,a) = [\partial\mathcal{U}(p,a)/\partial p_i]/[\partial\mathcal{U}(p,a)/\partial a_i].$$

We now examine the monotonicity properties of $z(p,a)$ in the state-contingent initial wealth (fixed payment) and price vectors. In the absence of risk aversion, differences in initial wealth have no impact on output allocation decisions. But differences in initial wealth can affect output allocation decisions for risk-averse producers. In the present framework, this is particularly interesting because it implies that changes in both prices and the fixed payment may cause changes in the state-contingent output vector.

Result 4: If there is no price uncertainty, F is differentiable in y, and the effort-cost function, $C(z)$, is symmetric then: (i) $z_i(p,a) > z_j(p,a)$ if and only if $a_i \le a_j$; and (ii) $pz_i(p,a) + a_i < pz_j(p,a) + a_j$ if and only if $a_i < a_j$.

Result 4 is particularly easy to understand: if $C(z)$ is symmetric there are in effect no technically good or bad states of nature because, at least in terms of costs, state-contingent outputs are interchangeable. Hence, the only way to encourage higher output in one state over another, given fixed prices, is to give the producer a greater marginal incentive to increase state-contingent output. Because the farmer is risk-averse (marginal utility of income is decreasing), providing a greater marginal incentive for a higher state-contingent output for fixed p implies decreasing the initial wealth of the producer. However, as is shown in part (ii) of the result, the extra output only partially offsets the initial wealth variation.

Absent risk aversion, producers equate price and marginal cost in each state. So, if costs are also symmetric, more is produced in states where prices are high. Result 4 shows that this 'substitution effect' between states might be offset by a wealth effect for risk-averse individuals. A standard result in uncertainty models is that the substitution effect predominates if the coefficient of relative risk aversion (or if base wealth is zero, the coefficient of proportional risk aversion) is less than 1. This result holds here, with appropriate modifications. For fixed C(z), the elementary utility function F yields a function F(y, C(z)) which behaves as a von Neumann-Morgenstern utility function in $y = a + pz$. Thus, a coefficient of partial risk aversion may be defined as

$$\mathcal{R}^p(pz) = - pz\, F_{11} / F_1.$$

Although the response of effort to price differences between states is ambiguous in the absence of information on $\mathcal{R}^p$, a simple stochastic dominance argument shows that differences in effort will never completely offset the effects of price variation so that revenue is always higher in high-price states

Result 5: If there is no wealth uncertainty, F is twice differentiable in y, and the effort-cost function, C(z), is symmetric: (i) if $\mathcal{R}^p < 1$ then $(p_i - p_j)(z_i(p,a) - z_j(p,a)) \geq 0$; and (ii) $y_i(p,a) < y_j(p,a)$ if and only if $p_i < p_j$.

Results 4 and 5 yield information on producer's output vectors when prices and wealth vary over the set of states of the world, that is the vector p (and a) is not equal to some scalar p. This is different from the notion of supply response most commonly analyzed in the literature on uncertainty and stabilization, which focuses on upward or downward shifts in the entire trajectory of state-contingent prices. This issue is addressed in the next section.

The symmetric effort-cost case also allows an analysis of producer risk attitudes in terms of S-concavity. Because S-concavity of $\mathcal{U}$ in p or a implies that the producer prefers the relevant variable to be stabilized at the mean, it may be of more interest than concavity results presented in Result 3.4.

Result 6: If $\mathcal{U}(p,a)$ is continuously differentiable, and the effort-cost function, C(z), is symmetric: (i) $\mathcal{U}(p,a)$ is Schur-concave in a if there is no price uncertainty; and (ii) $\mathcal{U}(p,a)$ is Schur-concave in p if there is no wealth uncertainty and $p_i > p_j$ implies

$$F_1(a + p_i z_i(p,a),\ C(z))z_i(p,a) < F_1(a + p_j z_j(p,a),\ C(z))z_j(p,a).$$

Result 6 (i) gives conditions under which differences in base wealth across states will reduce welfare. Because the producer is risk averse, differences in wealth across states will, *ceteris paribus,* reduce welfare relative to the case where the same mean wealth is available in every state.

Result 6 (ii) gives conditions under which differences in prices across states will reduce welfare. The condition certainly holds if the same z_i is produced in each state. Thus, the condition is also satisfied for any technology sufficiently close to this case. Thus, the less flexible the technology (i.e.,the closer to fixed proportions), the more likely price uncertainty is to be welfare reducing.

Now consider the general case when the cost function is symmetric. By Result 5(i), if the producer is very risk-averse ($\mathcal{R}^p$ is greater than one), z_i will not increase with p_i. And, more generally, the more risk-averse is the individual, the more slowly will z_i increase with p_i. Also, the more risk-averse is the individual, the more rapidly *ex post* marginal utility of revenue decreases with more revenue. Hence, as would be expected, the higher is the coefficient of risk aversion, the more likely price uncertainty is to be welfare reducing.

7. The Special Case of Additively Separable Utility

To this point, the analysis has used a very general utility structure. To illustrate the power of the state-contingent production model, the utility structure is now specialized to the form assumed in Newbery and Stiglitz's seminal work on price stabilization and production under risk,

$$F(y_1, g(x)) = u(y_1) - g(x).$$

Without loss of generality cardinalize units so that $u(0) = 0$.

A Dual Relationship

Our first result in this section helps establish a duality between the effort-cost function $c(y)$ and $U(\pi)$. For arbitrary y, the definition of the π-indirect expected utility function implies that under additive separability

$$U(\pi) \geq S^{-1}\sum_{l=1}^{S}\pi_l u(y_l) - c(y),$$

whence

$$c(y) \geq S^{-1}\sum_{l=1}^{S}\pi_l u(y_l) - U(\pi).$$

Moreover, because

$$c(y(\pi)) = S^{-1} \sum_{l=1}^{S}\pi_l u(y_l(\pi)) - U(\pi),$$

it follows that

$$\max_{\pi\in\Pi} \{ S^{-1}\sum_{l=1}^{S}\pi_l u(y_l) - U(\pi)\},$$

has a well-defined solution given by $c(y(\pi))$. The dual effort-cost function, $c^*(y)$, is defined:

$$c^*(y) = \max_{\pi \in \Pi} \{S^{-1} \sum_{l=1}^{S} \pi_l u(y_l) - U(\pi)\}.$$

Denoting

$$\pi(y) = \operatorname{argmax}_{\pi \in \Pi} \{ S^{-1} \sum_{l=1}^{S} \pi_l u(y_l) - U(\pi)\},$$

the properties of $c^*(y)$ and $\pi(y)$ are summarized in the following result.

Result 7: When $F(y_l, g(x)) = u(y_l) - g(x)$, $c^*(y)$ and $\pi(y)$ satisfy:

1. $c^*(0_S) \geq - U(\pi)$;
2. $c^*(y') \geq c^*(y) \geq c^*(0_S)$ for $y' \geq y$;
3. $\mu c^*(y') + (1 - \mu) c^*(y^o) \geq c^*(\mu y' + (1 - \mu) y^o) \quad 0 < \mu < 1$
4. for $y \in \mathbb{R}^S_{++}$ $c^*(y)$ is continuous;
5. if $y' \succ_S y^o$ then $c^*(y') \geq c^*(y^o)$;
6. $\sum_{l=1}^{S} (\pi_l(y') - \pi_l(y^o))[u(y_l') - u(y_l^o)] \geq 0$
7. $c^*(y(\pi)) = c(y(\pi))$.

Results 2 and 7 establish a duality between $U(\pi)$ and $c(y)$ for the additively separable case. Either is recapturable from the other given knowledge of the other and $u(y)$. Thus, as with other duality results, it is a matter of indifference as to whether analysis proceeds in primal terms (that is the state-contingent revenues) or in dual terms (that is in terms of the probabilities).

The properties of $c(y)$ as listed in Result 1 are a subset of those listed in Result 7 (1 - 6). Unless these additional properties are imposed upon $c(y)$, the function $c^*(y)$ recaptured from the dual program will not be the original $c(y)$. However, an obvious consequence of Result 7 is

$$U(\pi) = \underset{y}{\text{Max}} \{S^{-1} \sum_{i=1}^{S} \pi_i u(y_i) - c^*(y)\}.$$

Regardless of whether $c^*(y)$ is the original effort-cost function, using it in the producer's maximization problem generates the same economic choices as $c(y)$. Hence, $c^*(y)$ is observationally equivalent to $c(y)$. Consequently, no generality is lost in imposing properties 7.1 - 7.6 upon $c(y)$.

Much as indirect-utility minimization in consumer theory offers an algorithm for recapturing the relative consumer prices that will rationalize an observed vector of consumer demands, the dual relationship between $U(\pi)$ and $c(y)$ thus offers an algorithm for recapturing subjective probalities from the simplex Π that will rationalize any observed set of state-contingent revenues.

Additive Separability and $\mathcal{U}(p,a)$

We now turn our attention to $\mathcal{U}(p,a)$. Consider again the case of scalar output where the payment schedule satisfies:

$$y_i = a_i + p_i z_i$$

where p_i, a_i, $z_i \in \mathbb{R}_+$ $(i = 1,2,\ldots,S)$. This is properly interpreted as the case where output prices, output, and fixed payments (beginning wealth levels) are all state-contingent. For the remainder of the paper, assume that both $u(y)$ and $C(z)$ are at least twice differentiable.

Monotonicity Results

Our first result here establishes another sufficient condition for a monotonic relationship between the fixed payment schedule and the vector of the state-contingent outputs. Earlier it was established (Result 4) that symmetry of $C(z)$ was sufficient for such a relationship. However, symmetry of the effort-cost function is a polar case, where the character of the technology severely mitigates the effects of production uncertainty. Another polar case is given by the absence of effort-cost economies of scope across states of nature. In this case, what is done to prepare for one state of nature is independent of what is done to prepare for other states of nature -- at least in effort-cost terms. We shall refer to this case as exhibiting *no effort economies of scope*. If the producer gains something by preparing for distinct states jointly, then effort economies of scope exist. The formal requirement for the presence of effort economies of scope is

$$C(z_1, 0,\ldots,0) + C(0,z_2,0,\ldots,0) + \ldots + C(0, \ldots,0, z_S) > C(z).$$

Effort economies of scope are absent when the inequality always holds as an equality implying that C(z) can be represented as having an additively separable cost structure:

$$C(z) = \sum_{i=1}^{S} \chi_i(z_i)$$

where $\chi_i(z_i)$ $(i = 1, \ldots,S)$ is nondecreasing, convex, and twice differentiable. We can then establish:

Result 8: If no effort economies of scope exist and there exists a reordering of Ω, $\Omega' = \{ [1], \ldots,[S]\}$, such that $[i] \geq [j]$ implies $\chi'_{[i]}(z) \geq \chi'_{[j]}(z)$ for all $z \in \mathbb{R}_+$, then $z_{[i]} \geq z_{[j]}$ only if $a_{[i]} - a_{[j]} \leq z_{[i]}(p_{[j]} - p_{[j]})$.

Result 8 has the following interpretation: Given the presence of naturally good and bad states, a "bad-state" state-contingent output can be higher than a "good-state" state-contingent output only if the fixed payment in the bad state is set low enough relative to the good state fixed payment to encourage extra bad-state production. And particularly, if there is no price uncertainty:

Corollary 8.1: Under the conditions of Result 8, if there is no price uncertainty then $a_{[i]} - a_{[j]} \leq 0$.

It is well known that many economic choice problems, such as labor supply, may involve backward-bending solutions in which the income effects of higher prices counteract, and outweigh, substitution effects. It has been less widely observed that, for the separable objective function, this backward-bending solution arises if and only if the coefficient of relative risk aversion is greater than 1 (John Quiggin 1991; Newbery and Stiglitz). To exclude this possiblity, our attention is initially confined to the case where u() is a constant relative risk aversion utility function. We begin with the case $a_i = 0$, $y_i = p_i z_i$.

Result 9: If $a_i = 0$ (i = 1,2,...,S) and $u(y) = Ay^R$ (0 < R < 1, A > 0), then

$$\sum_{i=1}^{S} [(p'_i)^{1-R} - (p^o_i)^{1-R}][(z_i(p',a))^{1-R} - (z_i(p^o,a))^{1-R}] \geq 0.$$

Result 9 establishes that changes in each state-contingent price are positively correlated with changes in their respective state-contingent output. This is easily seen by setting all price changes except one to zero to get

$$[(p'_i)^{1-R} - (p^o_i)^{1-R}][(z_i(p',a))^{1-R} - (z_i(p^o,a))^{1-R}] \geq 0.$$

Hence, each state-contingent supply is upward sloping in its "own" state-contingent price. An obvious corollary is

Corollary 9.1: Under the conditions of Result 9 if all prices increase proportionately, i.e, $p'_i = \mu p^o_i$ $\mu > 1$ (i = 1,2,...,S), then

$$\sum_{i=1}^{S} (p^o_i)^{1-R}[(z_i(p',a))^{1-R} - (z_i(p^o,a))^{1-R}] \geq 0.$$

If the state-contingent price trajectory shifts up proportionately, then on average the state-contingent supply response will be positive. Corollary 9.1 when combined with the first-order conditions for the producer establishes that a proportional price shift leads to an increase in producer effort in a generalized sense.

Corollary 9.2: Under the conditions of Result 9 and Corollary 9.1,

$$\sum_{i=1}^{S} C_i(z(p',a)) \, (z_i(p',a)) \geq \sum_{i=1}^{S} C_i(z(p^o,a)) \, (z_i(p^o,a)).$$

Formally Corollary 9.2 establishes that the effort-cost scale elasticity after the proportional price change exceeds the effort-cost scale elasticity before the price change.

In the special case where there are constant returns to scale in terms of effort cost, Corollary 9.2 implies that effort as measured by effort cost increases with a proportional price change. Formally,

Corollary 9.3: If $C(tz) = tC(z)$ $t > 0$, then under the conditions of Result 9 and Corollary 9.1, $C(z(p',a)) \geq C(z(p^{o},a))$.

Result 9 and its corollaries can be extended to the case of fixed base wealth simply by replacing the coefficient of relative risk aversion with the coefficient of proportional risk aversion. In general, however, the more interesting case than either of the two studied is when *both* the price and the fixed payment (initial wealth) can vary across states. Not surprisingly, generally it is impossible to disentangle the effects of simultaneous changes in both p and a because each has an income effect and a substitution effect. However, if the way in which these changes occur is restricted, very strong results are available even without restrictions upon the utility structure. Specifically, suppose that any price change or fixed payment change must leave the producer better off in the sense that

(1) $$y'_1 = a^{o}_1 + p^{o}_1 z^{o}_1 + p'_1(z'_1 - z^{o}_1).$$

Result 10: Suppose the change in a and p is restricted to the form of (1), then

$$S^{-1}\sum_{1=1}^{S} u'(a^{o}_1 + p^{o}_1 z^{o}_1 + p^{o}_1(z'_1 - z^{o}_1))[p'_1 - p^{o}_1][z_1(p',a') - z_1(p^{o},a^{o})] \geq 0.$$

By Result 10 if only one price changes, the corresponding state-contingent supply response will be positively correlated with that change.

Increases in Price and Payment Risk

So far the results of this section have been about monotonicity relationships between changes in either the state-contingent price vector or the fixed-payment vector. But equally important is the issue of how uncertain production responds to changes in risk not associated with the technology, i.e., changes in either price risk or fixed-payment risk. Newbery and Stiglitz have studied the effect of increases in multiplicative risk (either price or production) upon the organization of production. In the present model, an obvious way to study how increasing the riskiness of the fixed-payments and the state-contingent prices affects production is to recognize that the state-contingent prices and fixed payment may themselves be randomized. As noted in the discussion of Result 3, this is particularly sensible when Ω indexes states of Nature only relevant to production (e.g. weather conditions), and demand conditions depend upon random factors not indexed by Ω. Formally, the producer can then be envisioned as facing in each state of nature a conditional (on the state of nature) price and fixed-payment distribution: if state i occurs then with probability $1/K > 0$, the state-contingent price is p_{ij} and the state-contingent fixed payment is a_{ij} $(i = 1,\ldots,S)$ $(j = 1,\ldots,K)$. (The equal-probability case is considered to conserve on notation. The results generalize in a straightforward fashion.) Our previous results represent the special case of this later scheme where state-contingent price (and fixed payment) is always fixed at the mean of this distribution, e.g. $p_i = K^{-1}\sum_{j=1}^{K} p_{ij}$. So long as $p_{ij} \neq p_{ik}$ (similarly for the fixed-payment scheme) for some j and k, the randomized rewards scheme majorizes the reward scheme that we have been considering. Put simply, the randomized reward scheme is riskier in the sense of Michael Rothschild and Joseph Stiglitz than the one we have been considering.

Since the elementary utility function $F(y, g(x))$ is concave in y, a Rothschild-Stiglitz (R-S) increase in the riskiness of the randomized reward scheme in any state will always reduce welfare (also see Result 3.4).[3] But the question of the supply response to increased risk remains unsettled for the present model.

These considerations lead us to consider the more general question of what happens when a producer facing a randomized reward scheme is subject to an R-S increase in risk. This includes the special case of a shift from the type of state-contingent reward scheme considered previously to a randomized payment scheme. Our next result covers the case when the state-contingent price is non random and the riskiness of the state contingent fixed payment is increased.

Result 11: Suppose that each state-contingent price is not randomized and the fixed payment scheme is made riskier in the R-S sense. A producer with nonincreasing absolute risk aversion increases expected utility by expanding each state-contingent output beyond the level optimal under the less risky fixed-payment scheme.

Increasing the riskiness of the fixed payment in each state gives the producer with nonincreasing absolute risk aversion the incentive at the margin to increase output in all states of nature. An exact analogue is not available for the state-contingent price vector. However, we can establish:

Result 12: Suppose the state-contingent fixed payment is not randomized and the state-contingent payment scheme is made riskier by an R-S increase in the riskiness of the state-contingent prices: if $u'(p_1 z_1 + a)p_1$ is convex in price the producer increases expected utility by expanding each state-contingent output beyond the level optimal under the less risky scheme. If $u'(p_1 z_1 + a)p_1$ is concave in price the producer increases expected utility by reducing each state-contingent output below the level optimal under the less risky scheme.

It follows easily from Result 12 that:

Corollary 12.1: If the producer's coefficient of relative risk aversion is constant and smaller than unity, the producer increases expected utility by reducing each state contingent output below the level optimal under the less risky scheme.

Results 11 and 12 generalize results originally due to Newbery and Stiglitz in several directions: they indicate what happens by increasing two sorts of payment risk (per unit and fixed payment); production uncertainty can be of any general form and not just multiplicative uncertainty as in Newbery and Stiglitz (multiplicative production uncertainty is equivalent to price uncertainty); and effort no longer need be a scalar variable. Each of these generalizations is an immediate byproduct of the richer formulation of the producer problem used here.

8. Conclusion

This paper develops a representation of production uncertainty which is simultaneously more realistic, more general, and more analytically tractable than the traditional production-function approach. Not only is the approach congruent with the Arrow-Debreu state-contingent model, but it is also congruent with modern axiomatic models of nonstochastic technologies (Chambers; Rolf Fare). Various indirect representations of the technology (effort-cost function, π-indirect expected utility function, and the p-indirect expected utility function) have been derived and their economic properties analyzed. In each instance, the representations generalize existing models of producer behavior. The power of the new approach has been illustrated by applying it to the additively separable utility case. Our results there include a duality between the effort-cost function and the indirect expected utility functions and generalizations of the central results on supply response in such models.

The additively separable utility model only serves as a starting point for applications of the general model. For example, the effort-cost function offers a natural method for freeing existing moral-hazard models from their reliance upon scalar "effort" and scalar output models of production uncertainty. And by disentangling the uncertain technology in a simple but informative fashion from the producer's beliefs about the likelihood of various states of nature occuring, the model at the same time promises a way to circumvent some of the more analytically difficult problems associated with moral-hazard analyses (e.g. the first-order problem) as well as offering a natural way to model differences in opinion about the state of nature. The model also offers a clear way to generalize existing models of insurance markets to situations where productive activity takes place both in the presence of moral hazard and the presence of adverse selection.

Appendix: **Proof of Results**

Result 1: Because $V^{pa}(y)$ is nonempty there exists an $\hat{x}$ such that $\hat{x} \in V^{pa}(y)$. The effort-cost minimization problem can now be restated as

$$\text{Min } \{g(x) : g(x) \leq g(\hat{x}) \text{ and } x \in V^{pa}(y)\}.$$

The continuity and monotonicity properties of g insure that the new feasible set is both closed and bounded. Therefore a minimum exists. To prove property 1 first denote

$$x(y) \in \text{argmin } \{g(x) : x \in V^{pa}(y)\}.$$

For $y' \geq y$ property V.3 implies that

$$x(y') \in V^{pa}(y)$$

where

$$\begin{aligned} c(y') &= g(x(y')) \\ &\geq \min \{g(x) : x \in V^{pa}(y)\} \\ &= c(y). \end{aligned}$$

That $c(y) \geq c(0_s)$ now follows trivially.

Convexity follows by noting that V.2 implies $(1 > \mu > 0)$

$$\mu x(y') + (1 - \mu)x(y^0) \in V^{pa}(\mu y' + (1 - \mu)\, y^0).$$

Thus,

$$\begin{aligned} \mu c(y') + (1 - \mu)c(y^0) &= \mu g(x(y') + (1 - \mu)g(x(y^0)) \\ &\geq g(\mu x(y') + (1 - \mu)x(y^0)) \\ &\geq \min \{g(x) : x \in V^{pa}(\mu y' + (1 - \mu)y^0)\} \\ &= c(\mu y' + (1 - \mu)y^0). \end{aligned}$$

The first inequality follows from the convexity of g. Convex functions defined over an open set, e.g. $y \in \mathbb{R}^S_{++}$, are continuous (Rockafellar, p. 82). The result is established.

Result 2: By definition

$$U(\pi) \geq \sum_{l=1}^{S} \pi_l F(y_l, c(y))$$

set $y = 0_S$ to obtain property 1. Convexity is established by

$$\mu U(\pi') + (1 - \mu)U(\pi^0) = \mu \sum_{l=1}^{S} \pi'_l F(y_l(\pi'), c(y(\pi')))$$

$$+ (1 - \mu) \sum_{l=1}^{S} \pi^0_l F(y_l(\pi^0), c(y(\pi^0)))$$

$$\geq \mu \sum_{l=1}^{S} \pi'_l F(y_l(\hat{\pi}), c(y(\hat{\pi})))$$

$$+ (1 - \mu) \sum_{l=1}^{S} \pi^0_l F(y_l(\hat{\pi}), c(y(\hat{\pi})))$$

$$= \sum_{l=1}^{S} (\mu\pi'_l + (1 - \mu)\pi^0_l) F(y_l(\hat{\pi}), c(y(\hat{\pi})))$$

$$= U(\mu\pi' + (1 - \mu)\pi^0)$$

for $\hat{\pi} = \mu\pi' + (1 - \mu)\pi^0$ and $0 < \mu < 1$. The inequality follows by the optimality of $y(\pi')$ and $y(\pi^0)$ for π' and π^0, respectively. Continuity follows from convexity because Π is an open set (Rockafellar). By the definition of $y(\pi)$

$$\sum_{l=1}^{S} \pi'_l F(y_l(\pi'), c(y(\pi'))) \geq \sum_{l=1}^{S} \pi'_l F(y_l(\pi^0), c(y(\pi^0))),$$

$$\sum_{l=1}^{S} \pi^0_l F(y_l(\pi^0), c(y(\pi^0))) \geq \sum_{l=1}^{S} \pi^0_l F(y_l(\pi'), c(y(\pi'))).$$

Adding these inequalities and rearranging establishes 4. To establish 5 suppose the contrary, that is, $y \succ_S y(\pi)$ and $c(y) < c(y(\pi))$. The strict concavity of F in y_l and its nonincreasingness in c(y) then imply

$$\sum_{l=1}^{S} \pi_l F(y_l, c(y)) \geq \sum_{l=1}^{S} \pi_l F(y_l(\pi), c(y(\pi)))$$

contradicting the fact that $y(\pi)$ is an optimizer.

Corollary 2.1: In the equal probability case $y \succ_S y(\pi)$ and $y \succ_m y(\pi)$ are equivalent if $\sum_l y_l = \sum_l y_l(\pi)$.

Result 3: Except for 3.2 - 3.4, the proof of Result 3 is virtually identical to Result 2. To prove, 3.2 consider increasing any element of a from a_l to a_l'. If the producer chooses exactly the same set of state-contingent outputs as before, expected utility increases by the strict monotonicity of F(). Hence, the optimal response to changing a cannot lead to a fall in expected utility. Result 3.3 is proved analogously. The following chain of inequalities proves 3.4:

$$\begin{aligned}\mathcal{U}(p,\mu a + (1-\mu)a') &\geq S^{-1} \sum_{l=1}^{S} F(\mu[a_l + p_l z_l] + (1-\mu)[a_l' + p_l z_l'], \\ &\qquad C(\mu z + (1-\mu)z')) \\ &\geq S^{-1} \sum_{l=1}^{S} F(\mu(a_l + p_l z_l) + (1-\mu)(a_l' + p_l z_l a'), \\ &\qquad \mu C(z) + (1-\mu)C(z')) \\ &\geq S^{-1} \sum_{l=1}^{S} \mu F(a_l + p_l z_l, C(z)) + (1-\mu)F(a_l' + p_l z_l', C(z')) \\ &= \mu\mathcal{U}(p,a) + (1-\mu)\mathcal{U}(p,a'),\end{aligned}$$

where $z = z(p,a)$ and $z' = z(p,a')$. The first inequality follows by the definition of $\mathcal{U}(p,a)$ as the maximum, the second inequality follows by the convexity of C(z) and the fact that F() is nonincreasing in g. The third inequality follows by the concavity of F in y and g. The last equality is definitional. Result 3.4(ii) is proved similarly.

Result 4: (i) The proof is by contradiction. Because there is no price uncertainty, without loss of generality choose units so that the price equals one. Suppose the effort-cost function is symmetric and choose i and j such that in the optimum the producer chooses $(z_i - z_j) > 0$. Also suppose contrary to the result that $(a_i - a_j) > 0$. Reallocating z_j to the ith state and z_i to the jth state, respectively, causes no change in cost if C(z) is symmetric. Define

$$\nu(z_j + a_i) = F(z_j + a_i, C(z)).$$

$\nu(z_j + a_i)$ is strictly increasing and strictly concave. Thus reallocating z_j and z_i in this manner allows us to operate in terms of ν instead of F because C(z) is unchanged. This reallocation changes the producer's expected utility by the amount

$$D = S^{-1}\{\nu(z_j + a_i) - \nu(z_i + a_i) + \nu(z_i + a_j) - \nu(z_j + a_j)\}.$$

The strict concavity of $\nu()$ implies

(a) $$D > S^{-1}\{ \nu'(z_j + a_i)(z_j - z_i) + \nu'(z_i + a_j)(z_i - z_j)\}$$

and

(b) $$D > S^{-1}\{ \nu'(z_j + a_i)(a_i - a_j) + \nu'(z_i + a_j)(a_j - a_i)\}.$$

By the presumption that in the optimum $(z_i - z_j) > 0$ (a) requires that

(c) $$\nu'(z_i + a_j) - \nu'(z_j + a_i) < 0$$

otherwise the reallocation increases the producer's expected utility contradicting the presumption that the original allocation was an optimum. By the presumption that $(a_i - a_j) > 0$, (b) requires that

(d) $$\nu'(z_i + a_j) - \nu'(z_j - a_i) > 0$$

otherwise the reallocation increases the producer's expected utility. But (d) and (c) are contradictory. This completes the proof of (i).

The proof of 4(ii) is also by contradiction. First, we require a technical lemma

Lemma: If C(z) is symmetric, $a + pz \succ_m a + pz(p,a)$, and $z \succ_m z(p,a)$, then $C(z) = C(z(p,a))$.

Proof: If C(z) is symmetric then Result 1.2 implies C(z) is Schur-convex (Marshall and Olkin). Hence, if $z \succ_m z(p,a)$ then $C(z) \leq C(z(p,a))$. But if $a + pz \succ_m a + pz(p,a)$, Result

3.7 implies $C(z) \geq C(z(p,a))$ establishing the lemma.

To proceed with the proof now suppose that $a_i < a_j$ but that at the optimum $y_j > y_i$. In state i the producer produces $z_i = y_i - a_i$ and in state j the producer produces $z_j = y_j - a_j$. Now consider the alternative production vector given by $z'_i = y_j - a_i$ and $z'_j = y_i - a_j$. This new production vector (resp. return vector) is majorized by the optimal production vector (resp. optimal return vector). Hence, the Lemma implies that it is equally costly to the optimal. But the strict concavity of F() in y implies expected utility is higher with the new vector than the optimal vector yielding a contradiction.

Result 5: To prove (i) we first show that if $p_j < p_i$ it cannot be true that $z_i = z_j$. Suppose to the contrary that $p_j < p_i$ and $z_i = z_j$ in the optimum. A shift to $z_j - \delta$ and $z_i + \delta$ results in a new schedule that is majorized by the old schedule. Because C(z) is S-convex (see the proof of the Lemma), costs cannot increase for the new schedule. If costs remain the same with the new schedule, the change in the objective function is given by

$$\delta\ [p_i F_1(a + p_i z_i,\ C(z)) - p_j F_1(a + p_j z_j,\ C(z))]$$

which is positive for $\delta > 0$, if $\mathcal{R}^p < 1$. Thus, this reallocation must result in a strictly greater expected utility (remember cost cannot increase) contradicting the optimality of the original allocation. A similar argument establishes that z_j can never be strictly greater than z_i. This establishes (i).

To prove (ii) suppose to the contrary that $p_j < p_i$ but that $y_j > y_i$ in the optimum. There always exists a $\varepsilon > 0$ such that the revenue vector that results by substituting $y'_j = y_j - p_j\varepsilon$ and $y'_i = y + p_i\varepsilon$ where the original production vector majorizes the new production vector. Because C(z) is S-convex, the new production vector is less costly. But even if costs were to remain the same with the new production vector instead of decrease, the new revenue vector second-order stochastically dominates the original production vector and hence will be preferred to the original by all risk averters again contradicting the optimality of the original vector.

Result 6: Theorem 3.A.4 in Albert Marshall and Ingram Olkin yields the following conditions for an arbitrary continuously differentiable function ϕ: $I \to \mathbb{R}$ to be S-concave: (i) ϕ is symmetric, (ii) $\phi_{(i)}(z) = \partial\phi/\partial z_i$ is increasing in i for all $z \in \mathcal{D}$ (that is arranged in descending order). Assuming C(z) is symmetric, it follows immediately that in the absence of price uncertainty $\mathcal{U}(p,a)$ is symmetric in a and in the absence of wealth uncertainty $\mathcal{U}(p,a)$ is symmetric in p. Hence, we only need to verify that (ii) above holds under the conditions stated in the result. For $\mathcal{U}(p,a)$ continuously differentiable, the envelope theorem in the absence of price uncertainty implies

$$\partial\mathcal{U}/\partial a_i = S^{-1}F_1(a_i + pz_i(p,a), C(z)).$$

Similarly, the envelope theorem in the case of a certain fixed payment yields

$$\partial\mathcal{U}/\partial p_i = S^{-1}F_1(a + p_i z_i(p,a), C(z))z_i(p,a).$$

Rearranging the a_i and the p_i in descending order as required gives the result after making use of Result 4 (i) and (ii).

Result 7: Properties 1, 3, 4, and 7 are all proved analogously to methods used in Result 2. Separate proofs are not provided. Consider $y' \geq y$. By the fact that u is nondecreasing in y_i.

$$\begin{aligned} c^*(y) &= \sum_{i=1}^{S} \pi_i(y)\, u(y_i) - U(\pi(y)) \\ &\leq \sum_{i=1}^{S} \pi_i(y)\, u(y'_i) - U(\pi(y)) \\ &\leq \sum_{i=1}^{S} \pi_i(y')\, u(y'_i) - U(\pi(y')) \\ &= c^*(y') \end{aligned}$$

which establishes 2. By definition

$$c^{*}(y') \geq \sum_{l=1}^{S} \pi_l(y^0)\, u(y'_l) - U(\pi(y^0))$$

Subtracting the definition of $c^{*}(y^0)$ from the above yields

$$c^{*}(y') - c^{*}(y^0) \geq \sum_{l=1}^{S} \pi_l(y^0)\, u(y'_l) - \sum_{l=1}^{S} \pi_l(y^0) u(y^0_l).$$

If $y' \succ_S y$ the right hand side of this expression is positive thus establishing 5.

$$c(y(\pi)) \geq \sum_{l=1}^{S} \hat{\pi}_l u(y_l(\pi)) - U(\hat{\pi})$$

for all $\hat{\pi} \in \Pi$. Thus, $c(y(\pi))$ is an upper bound for

$$\sum_{l=1}^{S} \hat{\pi}_l u(y_l(\pi)) - U(\hat{\pi})$$

over $\hat{\pi} \in \Pi$. Because

$$c(y(\pi)) = \sum_{l=1}^{S} \pi_l u(y_l(\pi)) - U(\pi)$$

the upper bound is an achievable least upper bound over $\hat{\pi} \in \Pi$, where

$$c(y(\pi)) = \max_{\hat{\pi} \in \Pi} \sum_{l=1}^{S} \hat{\pi}_l u(y_l(\pi)) - U(\hat{\pi})$$
$$= c^{*}(y(\pi)).$$

Result 8: If there are no effort economies of scope

$$C(z) = \sum_{l=1}^{S} \chi_l(z_l)$$

By the presumptions of the result a reordering of Ω, $\Omega' = \{[1], \ldots, [S]\}$, exists such that $[i] \geq [j]$ implies $\chi'_{[i]}(z) \geq \chi'_{[j]}(z)$ for all $z \in \mathbb{R}_+$. Also suppose that in the optimum $z_{[i]} \geq z_{[j]}$ for some $[i] \geq [j]$. The producer's first-order conditions require

$$S^{-1}u'(a_{[1]} + p_{[1]}z_{[1]}) - \chi'_{[1]}(z_{[1]}) = S^{-1}u'(a_{[J]} + p_{[J]}z_{[J]}) - \chi'_{[J]}(z_{[J]})$$
$$\geq S^{-1}u'(a_{[J]}+p_{[J]}\, z_{[1]}) - \chi'_{[J]}(z_{[1]}).$$

The inequality follows by the concavity of u and the convexity of χ and the presumption that $z_{[1]} \geq z_{[J]}$. Hence,

$$S^{-1}u'(a_{[1]} + p_{[1]}z_{[1]}) - \chi'_{[1]}(z_{[1]}) \geq S^{-1}u'(a_{[J]}+ p_{[J]}\, z_{[1]}) - \chi'_{[J]}(z_{[1]}).$$

where the last inequality follows by the definition of Ω'. The fact that u() is strictly concave then requires that $a_{[1]} + p_{[1]}z_{[1]} \leq a_{[J]} + p_{[J]}\, z_{[1]}$.

Result 9: Result 9 follows directly by applying 3.6 to this utility structure.

Result 10: Apply 3.6 to establish that

$$\sum_{i=1}^{S} u(a_i^o + p_i^o z_i^o + p_i'(z_i' - z_i^o)) - u(a_i^o + p_i^o z_i^o + p_i^o(z_i' - z_i^o)) \geq 0.$$

The expression in the result must be larger than the left-hand side here by the strict concavity of u(). The result is demonstrated.

Result 11: For the incentive scheme where the farmer receives the fixed payment a_i in state i, the farmer's optimum, $\hat{z}$, is characterized by

$$S^{-1}u'(p_i\hat{z}_i + a_i)p_i = C_i(\hat{z})$$

i = 1,2,...,S. An R-S increase in risk for the fixed payment schedule can be represented by the addition of a random variable ε_i to a_i such that $E(\varepsilon_i | a_i) = 0$ (i = 1, 2,...,S).

If the producer exhibits nonincreasing absolute risk aversion u'() is a convex function.

Hence, it follows immediately that

$$S^{-1}E_\varepsilon u'(p_i\hat{z}_i + a_i + \varepsilon_i)p_i \geq S^{-1}u'(p_i\hat{z}_i + a_i)p_i$$

(i = 1,...,S) where E_ε denotes the expectation over ε_i. For each state of nature expected marginal utility under the riskier reward scheme exceeds marginal effort cost at $\hat{z}$ thus establishing the result.

Result 12: For the incentive scheme where the farmer receives a deterministic payment p_i in state i, the farmer's optimum, $\hat{z}$, is characterized by

$$S^{-1}u'(p_i\hat{z}_i + a_i)p_i = C_i(\hat{z})$$

i = 1,2,...,S. A R-S increase in risk can be represented by the introduction of another random variable ε_i such that $E(\varepsilon_i | p_i) = 0$. Now proceed exactly as in Result 11 to establish the result under the conditions stated.

Footnotes

1. The resulting choice set is very restricted. Jack Meyer (1987) shows that the choice set in the standard firm problem (Sandmo; Feder) may be regarded as a line in mean-standard deviation space. This result is generalized by Michael Ormiston and John Quiggin (1991).

2. Here it is assumed that Ω indexes all possible sources of uncertainty including both production and payment uncertainty. This assumption is relaxed in sections 6 and 7 below.

3. By contrast, since producers may vary their output across states, the effects of differences in prices between states is ambiguous. That the producer may prefer some price variation across states is well documented from the price instability literature (Newbery and Stiglitz). Result 3.4 (ii) yields a sufficient condition for the producer to prefer a fixed price p to a state-contingent price vector with mean p.

References

Arrow, Kenneth (1953). "Le Role des valeurs boursieres pour la repartition la meilleure des risques." Cahiers du Seminaire d'Econometrie. CNRS, Paris.

Chambers, Robert G. (1988) Applied Production Economics: A Dual Approach. Cambridge University Press: Cambridge.

Debreu, Gerard (1959). Theory of Value: An Axiomatic Analysis of Economic Equilibrium. Yale University Press: New Haven.

Fare, Rolf (1988). Fundamentals of Production Theory. Springer-Verlag: Berlin.

Feder, Gershon (1977), 'The impact of uncertainty in a class of objective functions,' Journal of Economic Theory 16, 504-12.

Hartman, Richard (1976), 'Factor demand with output price uncertainty,' American Economic Review 66(4), 675-84.

Ishii, Yasunori (1977). "On the Theory of the Firm under Price Uncertainty: Note" American Economic Review 67: 768-9.

Leland, Hayne (1972), 'Theory of the Firm facing Uncertain Demand,' American Economic Review 62(3), 278-91.

Lewis, Tracy and D.E.M. Sappington (1988). "Regulating a Monopolist with Unknown Demand" American Economic Review 78: 986-998.

Marshall, Albert and Olkin, Ingram (1979), Inequalities:theory of majorization and its applications, Academic Press, N.Y.

Newbery, David and Stiglitz, Joseph (1981), The Theory of Commodity Price Stabilisation - A Study in the Economics of Risk, Oxford University Press, Oxford.

Ormiston, Michael. and Quiggin, John (1992), Two-parameter decision models and expected utility maximization: A generalization, Working Paper, University of Maryland, College Park.

Pauly, Mark (1974). "Overinsurance and the Public Provision of Insurance." Quarterly Journal of Economics LXXXVIII: 44-62.

Quiggin, John (1992), 'Reconciling contradictory predictions on supply response under stabilization', forthcoming in Australian Journal of Agricultural Economics.

Radner, Roy (1968). "Competitive Equilibrium under Uncertainty." Econometrica 36: 31-54.

Raviv, Arthur (1979). "The Design of an Optimal Insurance Policy." American Economic Review 69: 84 - 96.

Riley, John G. and William F. Samuelson (1980). "Optimal Auctions." American Economic Review 71: 381-91.

Rockafellar, R. Tyrell (1970). Convex Analysis. Princeton University Press: Princeton, NJ.

Rothschild, Michael and Stiglitz, Joseph (1970), 'Increasing risk: I. A definition,' Journal of Economic Theory 2(4), 225-243.

_________ (1971), 'Increasing risk: II. Its economic consequences,' Journal of Economic Theory 3(1), 66-84.

Sandmo, Agnar (1971), 'The competitive firm under output price uncertainty,' American Economic Review 61(1), 65-73.

Shephard, Ronald W. (1970). Theory of Cost and Production Functions. Princeton University Press: Princeton, NJ.

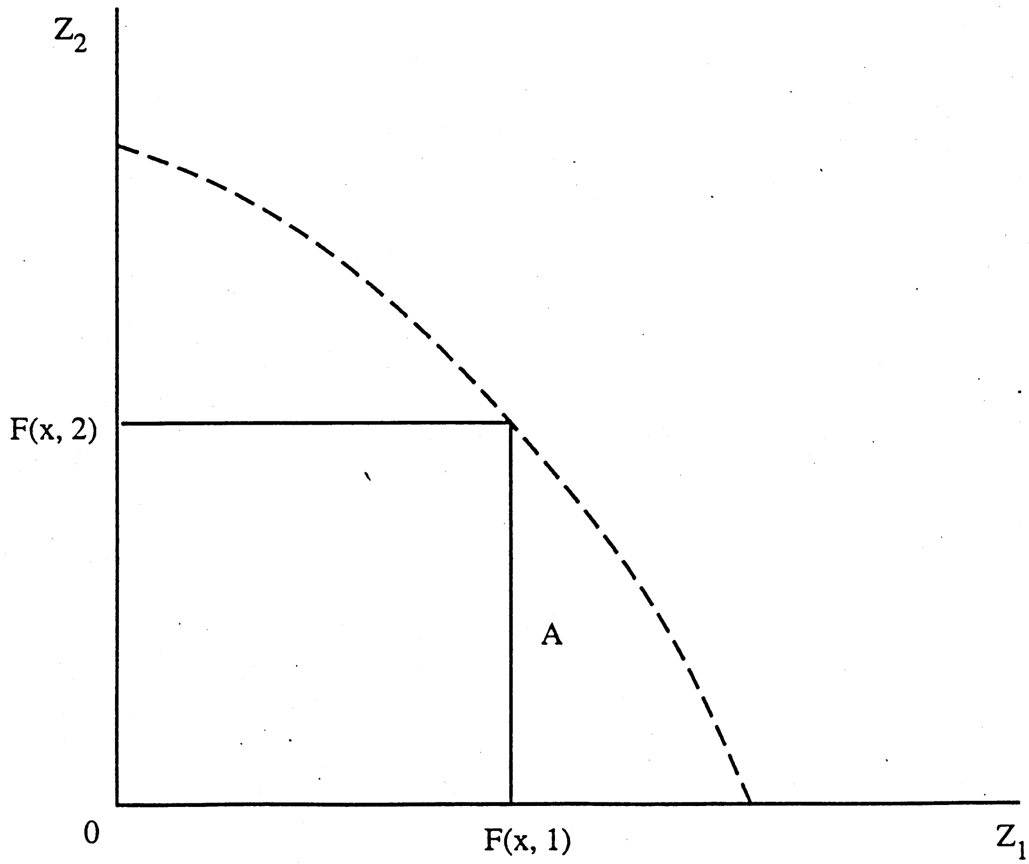

Figure 1

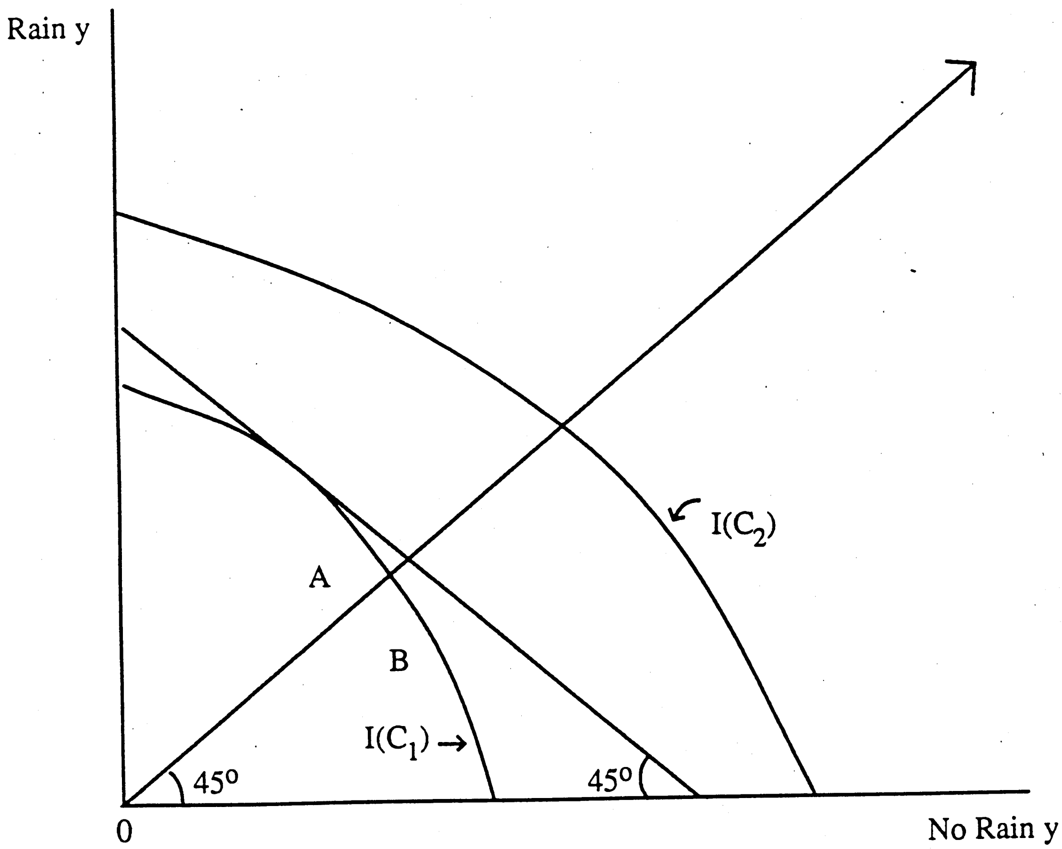

Figure 2