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AGGREGABLE PRICE-TAKING FIRMS

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I. Introduction

Since the work of Stigler and Kindahl, economists have been aware that firms' prices vary. This may be due to a variety of reasons, but a prominent one is that firms choose to produce different qualities of the same generic good because there is a non-degenerate distribution of consumer tastes. For those creating and using price indexes to study the generic good, the question of an appropriate price index naturally arises. The traditional approach has been to use commonly reported indices calculated according to some noneconomic criteria. Gorman (1988) has called such indexes "mechanical". These indexes are not, and do not purport to be, exact economic indexes. Indeed, Pope and Chambers have shown that this traditional approach, rooted in econometric practice, when applied to aggregating systems of derived demands and supplies over firms facing different prices is invalid. That is, no single prices index, "mechanical" or otherwise can be used in an aggregate production system and purport to capture aggregate factor demands and product supplies. Hence, existing empirical knowledge of supply-response systems based upon aggregate models imposing homogeneity and other properties received from theory is questionable.

Negative results like those of Pope and Chambers are not new phenomena in aggregation theory. Unfortunately, the most common response to such results seems to be "... to simply ignore this aggregation problem and adopt a third approach by formulating aggregate relationships directly from the theory of the individual consumer ..." (Phlips, p. 98) or producer.

Presumably the reasoning is that aggregate outputs and demands are somehow inherently interesting enough economically to allow the researcher to ignore aggregation issues and apply an individualistic theory to describe the behavior of a fictive "representative individual". Gorman (1988) has long advocated a more constructive approach to such aggregation problems, i.e., that aggregation theory be used for more than just rationalizing existing mechanical aggregates. Aggregation theory can also be used prospectively to guide the entire process of constructing the summary measures that connect micro and macro relations. If existing aggregates are clearly inappropriate, a prospective approach would suggest finding and ultimately constructing better aggregates; a mechanical and nihilistic approach, on the other hand, would just continue to use existing aggregates and formulate meaningless economic relationships.

This paper seeks to respond to the earlier Pope and Chambers result prospectively, i.e., to initiate the search for new aggregates which can be used to formulate meaningful aggregate economic models of supply response. The problem with the traditional approach to aggregating derived-demand and supply systems is that it is only consistent with technologies exhibiting quasi-constant returns to scale for which well behaved profit functions do not exist. Quasi-constant returns is the structural restriction needed to satisfy the linear aggregation relationships typically used. Hence, an obvious approach in seeking aggregable alternatives is to consider alternative, nonlinear aggregation rules. Using a generalization of the Pope-Chambers aggregation criteria, we deduce a family of aggregable technologies. Both the firm-level technology and the aggregate technology possess profit-maximizing regularity properties.

The family of aggregable technologies represents a non-trivial and potentially useful extension of the quasi-constant returns technology. There is no "free lunch", however, and the price of the nonlinear generalization is the interpretation of aggregate response as the simple sum (or a monotonic transformation) of micro decisions. An example concludes the paper. For the case where the aggregate price index is the geometric mean of individual prices, the logarithm of the aggregate response is the weighted average of the logarithms of micro responses.

II. Notation and Assumptions

Pope and Chambers consider the existence of differentiable functions: $Y: \mathbb{R}^m_+ \times \mathbb{R} \to \mathbb{R}$; $X_k: \mathbb{R}^m_+ \times \mathbb{R} \to \mathbb{R}$ $(k = 1,\ldots,m)$; $H : \mathbb{R} \to \mathbb{R}$; $K_k : \mathbb{R} \to \mathbb{R}$ $(k = 1,\ldots,m)$; and $P : \mathbb{R}^{n+m}_+ \to \mathbb{R}$ such that

$$Y(P(p_1,\ldots,p_n;w),w) = H\left(\sum_{j=1}^{n} y_j(p_j,w)\right), \text{ and}$$

$$X_k(P(p_1,\ldots,p_n;w),w) = K\left(\sum_{j=1}^{n} x_{kj}(p_j,w)\right)$$

$(k = 1,\ldots,m)$. Here $p_i \in \mathbb{R}_+$ $(i = 1,\ldots,n)$ is a price specific to individual i; $w \in \mathbb{R}^m_+$ is a vector of prices common across individuals; $y_i(p_i,w)$ is a net output for individual i (the price of this net output is p_i); $x_{kj}(p_j,w)$ is the net output k for individual j (the price of net output k is w_k); P is a price-index which summarizes the distribution of p_i across firms (note it can depend on w); Y is the aggregate net output of the y_i; X_k is the aggregate net output of the x_{kj}; and H and K_k $(k = 1,\ldots,n)$ are the rules by which the aggregates are constructed from the micro net outputs. Pope and Chambers show (Result 5) that there do not exist functions satisfying these equalities which are also consistent with $y_j(p_j,w)$ and $x_{kj}(p_j,w)$ $(k = 1,\ldots,m)$ $(j = 1,\ldots,n)$ being derived from profit maximizing behavior by individual agents.

This paper generalizes Pope and Chambers and seeks functions satisfying

$$Y(P(p_1,\dots,p_n;w),w) = H\left(\sum_{j=1}^{n} h_j(y_j(p_j,w))\right), \text{ and}$$

$$X_k(P(p_1,\dots,p_n;w),w) = K_k\left(\sum_{j=1}^{n} t_{kj}(x_{kj}(p_j,w))\right) \tag{1}$$

$(k = 1,\dots,m)$. We impose two separate assumptions.

Assumption 1: P is continuous and strongly monotonic in each element of p, Y is continuous and strongly monotonic in P, X_k $(k = 1,\dots,m)$ is continuous and strongly monotonic in P, H is continuous and strongly monotonic, K_k $(k = 1,..,m)$ is continuous and strongly monotonic, h_j $(j = 1,\dots,n)$ is strongly monotonic and continuous, t_{kj} $(j = 1,\dots,n)$ $(k = 1,\dots,m)$ is strongly monotonic and continuous.

Assumption 2: Each vector $[y_j(p_j,w),\ x_j(p_j,w)] \in \mathbb{R}^{m+1}$ $(j = 1,\dots,n)$ satisfies

$$(y_j(p_j,w),\ x_j(p_j,w)) = \underset{y,x}{\text{argmax}}\ \{p_j y + wx : (x,y) \in T_j\},$$

where $T_j \subseteq \mathbb{R}^{m+1}$ is the production possibilities set for firm j. T_j is nonempty, compact, and strictly convex. Each $y_j(p_j,w)$ is strictly monotonic in p_j. Each firm-level restricted profit function $c_j : \mathbb{R}^m_+ \times \mathbb{R} \to \mathbb{R}$ $(j = 1,\dots,n)$

$$c_j(w,y_j) = \text{Max}\ \{wx : (x,y_j) \in T_j\}.$$

satisfies Hotelling's lemma, i.e.,

$$x_j(w,y_j) = \text{argmax}\ \{wx : (x,y_j) \in T_j\}$$

is unique and equals the gradient of $c_j(w,y_j)$ in w. $x_j(w,y_j)$ is differentiable $(j = 1,\dots,n)$ and $x_j(w,y_j)$ is strictly monotonic in y_j $(j = 1,\dots,n)$.

Assumption 1 gives the regularity conditions which are imposed upon the *aggregation structure* (1). Assumption 2 requires that net outputs result from firm-level profit maximization. Notice in particular that by well-known results

$$x_j(p_j, w) = x_j(w, y_j(p_j, w)).$$

3. Results

Our first result establishes the conditions necessary for the existence of an aggregation structure (1) satisfying Assumption 1. Assumption 2 is not yet imposed. (Proofs are in an Appendix.)

Result 1: Aggregation structure (1) is satisfied under Assumption 1 if and only if there exist functions

$$P(p_1, \ldots, p_n; w) = F\left(\sum_{j=1}^{n} h_j(y_j(p_j, w)); w\right)$$

$$t_{kj}(x_{kj}) = \beta_{kj}(w) + \upsilon_k(w)\ h_j(y_j(p_j, w))$$

$(j = 1, \ldots, n)$ $(k = 1, \ldots, m)$ where F is continuous and strictly monotonic in its first argument.

Consistency in aggregation requires the price index to be additively separable (completely (strictly) separable in the partition $\{1, \ldots, n\}$ of the p subscripts) in the p_i. Pope and Chambers show that traditional price indexes, such as the Laspeyres, are only consistent with additive separability under extreme restrictions on the firm-level net outputs. Their specific results can be applied here in a straightforward manner. Result 1 only reflects the structural restrictions associated with imposing the nonlinear adding up restrictions required by (1) on separate equations. If Assumption 2 is also imposed, it follows immediately that the restricted profit function must satisfy

$$c_j(w,y_j) = \sum_{k=1}^{m} w_k q_{kj}\left[\beta_{kj}(w) + \upsilon_k(w)h_j(y_j)\right]$$

where $q_{kj} : \mathbb{R} \to \mathbb{R}$ is strongly monotonic, $\beta_{kj}(w)$ and $\upsilon_k(w)$ are homogeneous of degree zero (q_{kj} is the inverse of t_{kj}). Moreover, by standard properties of restricted profit functions (McFadden), this form must be convex in w.

If the restricted profit function assumes this form, each element of $x_j(w,y_j) \in \mathbb{R}^m$ can be thought of as being a strongly monotonic transformation of a derived netput associated with a technology quasi-homothetic in x_j (in the sense that the asymmetric transformation function for y is homothetic). This representation clarifies why Pope and Chambers found aggregation impossible using traditional aggregation procedures. These procedures require each h_j and t_{kj} to be the identity mapping. Aggregation thus requires the restricted profit function to assume the form:

$$\begin{aligned} c_j(w,y_j) &= \sum_{k=1}^{m} w_k\beta_{kj}(w) + \upsilon_k(w)y_j \\ &= \beta_j(w) + \upsilon(w)\ y_j \end{aligned}$$

which only differs from a constant returns to scale technology by a fixed net profit $\beta_j(w)$ unique to each firm. This technology does not satisfy Assumption 2 because the dual T_j is not compact.

An important special case of an aggregable technology satisfying Result 1 and Assumption 2 is the technology where each t_{kj} is the identity mapping but $h_j(y_j)$ is not. In this case, the technology is of the quasi-homothetic form

$$\begin{aligned} c_j(w,y_j) &= \sum_{k=1}^{m} w_k\beta_{kj}(w) + \upsilon_k(w)h_j(y_j) \\ &= \beta_j(w) + \nu(w)\ h_j(y_j). \end{aligned} \qquad (2)$$

Our second result establishes that (2) is the only technology consistent with Assumptions 1 and 2.

Result 2: Aggregation structure (1) is satisfied under Assumptions 1 and 2 if and only if the firm-level restricted profit functions satisfy expression (2) where $\beta_j(tw) = t\beta_j(w)$ and $\nu(tw) = t\nu(w)$ for $t > 0$, and (2) is convex in w.

Remark 1: To make the structure in (2) consistent with convexity, one might think to impose convexity upon $\beta_j(w)$ and $\nu(w)$. However, unless h_j's range ($j = 1,\ldots,n$) only includes the nonnegative reals, this latter restriction will not insure convexity of (2) in w.

Remark 2: By Result 2, a "representative" producer with a restricted profit function

$$c(w, Y) = \beta(w) + \nu(w)\, H^{-1}(Y)$$

where $\beta(w) = \sum_{j=1}^{m} \beta_j(w)$ and $H^{-1}(Y) = \sum_{j=1}^{m} h_j(y_j)$ rationalizes the net outputs

$$X_k(w,Y) = \sum_{j=1}^{m} x_{kj}(w,y_j),$$

in a manner consistent with individual restricted profit maximization. ($H^{-1}(\ )$ is the inverse image of $H(\ )$ in (1) where existence is guaranteed by Assumption 1.) Also by Result 2 if each $h_j : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly concave function then firm-level profit functions of the form

$$\begin{aligned}
\pi_j(p_j,w) &= \max_{y,x} \{p_j y + wx:\ (x,y) \in T_j\} \\
&= \max_{y} \{p_j y + \beta_j(w) + \nu(w)h_j(y)\} \\
&= \beta_j(w) + \nu(w) \max_{y} \left\{\frac{p_j}{\nu(w)}\, y + h_j(y)\right\} \\
&= \beta_j(w) + \nu(w)\ \mu_j\left(\frac{p_j}{\nu(w)}\right).
\end{aligned}$$

are consistent with aggregation structure (1). Assuming further that $H^{-1}(Y)$ is strictly concave (thus guaranteeing a unique solution) implies the existence of a "representative" profit function:

$$\pi(P,w) = \underset{Y}{\text{Max}} \{PY + \nu(w)H^{-1}(Y) + \beta(w)\}$$

$$= \beta(w) + \nu(w)\mu\left(\frac{P}{\nu(w)}\right)$$

where

$$\mu\left(\frac{P}{\nu(w)}\right) = \underset{Y}{\text{Max}} \left\{\frac{PY}{\nu(w)} + H^{-1}(Y)\right\} .$$

Using Hotelling's lemma, expressions (1), and Assumption 1 implies that the price index defined implicitly by

$$\mu\left(\frac{P}{\nu(w)}\right) - \frac{P}{\nu(w)}\,\mu'\left(\frac{P}{\nu(w)}\right) = \sum_{j=1}^{n} \mu_j\left(\frac{p_j}{\nu(w)}\right) - \frac{p_j}{\nu(w)}\,\mu'_j\left(\frac{p_j}{\nu(w)}\right) \qquad (3)$$

is the price index that permits the aggregate system

$$X_k(w,Y) = \sum_{j=1}^{n} x_{kj}(w,y_j(p_j,w))$$

$$H^{-1}(Y) = \sum_{j=1}^{n} h_j(y_j(p_j,w))$$

to be rationalized by a "representative producer" with profit function $\beta(w) + \nu(w)\mu\left(\frac{P}{\nu(w)}\right)$. To be consistent with Assumption 1, the left-hand side of (3) must be strictly monotonic in P. This condition is satisfied if $\mu(P/v(w))$ is strictly convex. Thus having a representative producer at this level of aggregation requires imposing further curvature conditions on $h_j(y_j)$ $(j = 1,\ldots,n)$ and $H(\;)$ beyond those in Assumption 1.

An Example: The procedure for deriving individual profit functions, a "representative-producer" profit function, and a price index are now illustrated for the aggregation rule:

$$Y = \exp\left(\sum_{j=1}^{n} \alpha_j \ell n\ y_j\right) \tag{4}$$

In applying Result 2 and Remark 2, one should notice that inevitably there is some degree of arbitrariness in specifying $H^{-1}(y)$ and $h_j(y_j)$. To see this note that both

$$H^{-1}(Y) = \beta \ell n Y$$

$$h_j(y_j) = \beta_j \ell n\ y_j$$

and

$$H^{-1}(Y) = \ell n\ Y$$

$$h_j(y_j) = \alpha_j \ell n\ y_j$$

are consistent with (4) so long as $\alpha_j = \beta_j/\beta$ $(j = 1,\ldots, n)$. In what follows we shall presume the second representation.

Applying the results in Remark 2 yields

$$\pi_j(p_j, w) = \beta(w) + \nu(w)\ \alpha_j \left[\ell n\left(\alpha_j \frac{\nu(w)}{-p_j}\right) - 1\right]$$

and

$$\pi(P_1 w) = \beta(w) + \nu(w) \left[\ell n\left(\frac{\nu(w)}{-P}\right) - 1\right].$$

Expression (3) now implies after some straightforward manipulation the following Cobb-Douglas price index.

$$P = \nu(w)^{\left[1 - \sum_{i=1}^{n} \alpha_j\right]} \prod_{i=1}^{n} \alpha_j^{\alpha_j} \prod_{i=1}^{n} p_j^{\alpha_j}.$$

This example illustrates, for example, that aggregate Y as the geometric mean of the individual y_i and P as the geometric mean of the p_i would be consistent aggregates capable of supporting a "representative producer" interpretation of the data.

4. Conclusion

This paper generalizes the Pope-Chambers aggregation problem to show that aggregable price-taking technologies exist. Both firm-level technologies and the "representative" technology must be quasi-homothetic.

Appendix: Proof of Results

Result 1: The aggregation structure requires:

$$Y(P(p_1,\ldots,p_n;w),w) = H\left(\sum_{i=1}^{n} h_j(y_j(p_j,w))\right), \text{ and}$$

$$X_k(P(p_1,\ldots,p_n;w),w) = K_k\left(\sum_{j=1}^{n} t_{kj}(x_{kj}(p_j,w))\right) \tag{1}$$

$(k = 1,\ldots,m)$. Use the fact that Y is continuous and strongly monotonic in P to invert the first equation to get

$$P(p_1,\ldots,p_n;w) = Y^{-1}\left(H\left(\sum_{j=1}^{n} h_j(y_j(p_j,w))\right);\ w\right)$$

$$= F\left(\sum_{j=1}^{n} h_j(y_j(p_j,w));\ w\right)$$

which establishes the first equation in Result 1. The properties of F follow from Assumption 1. Insert the result into the remaining equations in (1)

$$X_k\left(F\left(\sum_{j=1}^{n} h_j(y_j(p_j,w));\ w\right),w\right) = K_k\left(\sum_{j=1}^{n} t_{kj}(x_{kj}(p_j,w))\right).$$

$(k = 1,\ldots,m)$. Use the monotonicity and continuity properties of K_k to invert:

$$\sum_{j=1}^{n} t_{kj}(x_{kj}(p_j,w)) = K_k^{-1}\left[X_k\left(F\left(\sum_{j=1}^{n} h_j(y_j(p_j,w));\ w\right),w\right)\right]$$

$$= J_k\left[\sum_{j=1}^{n} h_j(y_j(p_j,w));\ w\right] \tag{a}$$

$k = 1,\ldots,n$. Pick an arbitrary reference vector $\bar{p} \in \mathbb{R}_+^n$ and set all $p_j = \bar{p}_j$ $(j \neq i)$ in (a)

$$t_{ki}(x_{ki}(p_i,w)) + \sum_{j\neq i}^{n} t_{kj}(x_{kj}(\bar{p}_j,w)) = J_k\left[h_i(y_i(p_i,w)) + \sum_{j\neq i}^{n} h_j(y_j(\bar{p}_j,w));\ w\right].$$

$k = 1,\ldots,m$, $i = 1,\ldots n$. Substituting these results into (a) yields a vector- valued Pexider equation in the $h_i(y_i)$ with known solution (Aczél, Theorem 8.2, p.348)

$$t_{kj}(x_{kj}) = \beta_{kj}(w) + \nu_k(w)h_j(y_j)$$

$(j = 1,\ldots,n)$ $(k = 1,\ldots,m)$. Sufficiency is straightforward.

Result 2: By Result 1

$$t_{kj}(x_{kj}) = \beta_{kj}(w) + \nu_k(w)h_j(y_j)$$

$(k = 1,\ldots,m)$ $(j = 1,\ldots,n)$. By Assumption 2 this implies

$$t_{kj}(c_j^k(w,y_j)) = \beta_{kj}(w) + \nu_k(w)h_j(y_j) \qquad \text{(a)}$$

$(k = 1,\ldots,m)$ $(j = 1,\ldots,n)$ where $c_j^k(w, y_j) = \dfrac{\partial c_j(w,y_j)}{\partial w_k}$ $(k = 1,\ldots,m)$.

Set $y_j = \hat{y}_j$ and $y_j = \bar{y}_j$ $(\hat{y}_j \neq \bar{y}_j)$ respectively to get

$$\beta_{kj}(w) + \nu_k(w)h_j(\hat{y}_j) = t_{kj}(c_j^k(w,\hat{y}_j))$$

$$\beta_{kj}(w) + \nu_k(w)h_j(\bar{y}_j) = t_{kj}(c_j(w,\bar{y}_j)).$$

$(k = 1,\ldots,m)$ $(j = 1,\ldots,n)$. Solving for $\beta_{kj}(w)$ and $\nu_k(w)$ and substituting into (a) gives

$$t_{kj}(c_j^k(w, \hat{y}_j))(h_j(\bar{y}_j) - h_j(y_j)) + t_{kj}(c_j^k(w,\bar{y}_j))(h_j(y_j) - h_j(\hat{y}_j))$$
$$= [h_j(\bar{y}_j) - h_j(\hat{y}_j)]\ t_{kj}(c_j^k(w,y_j)) \qquad \text{(b)}$$

$(k = 1,\ldots,m)$ $(j = 1,\ldots,n)$. Assumption 2, Slutsky symmetry, $(c_j^{ik}(w,y_i) = c_j^{ki}(w,y_j)$ $(k = 1,\ldots,m)$ $(i = 1,\ldots,m))$ and (b) together imply

$$\frac{t'_{ij}(c_j^i(w,y_j))}{t'_{kj}(c_j(w,y_j))} = \frac{t'_{ij}(c_j^i(w,\bar{y}_j))}{t'_{kj}(c_j(w,\bar{y}_j))} = \frac{t'_{ij}(c_j^i(w,\hat{y}_j))}{t'_{kj}(c_j(w,\hat{y}_j))} \qquad \text{(c)}$$

where superscript primes denote derivatives. Return to (a) and use Assumption 1 and Assumption 2 to obtain

$$t'_{kj}(c_j^k(w,y_j)) = \frac{\nu_k(w)h'_j(y_j)}{c_j^{ky}(w,y_j)} \tag{d}$$

where $c_j^{ky}(w,y_j) = \partial x_{kj}(w,y_j)/\partial y_j$. Expressions (c) and (d) imply

$$\frac{c_j^{ky}(w,y_j)}{c_j^{iy}(w,y_j)} = \frac{c_j^{ky}(w,\bar{y}_j)}{c_j^{iy}(w,\bar{y}_j)} = \frac{c_j^{ky}(w,\hat{y}_j)}{c_j^{iy}(w,\hat{y}_j)}$$

implying that these ratios are independent of y_j. Symmetry then implies that the ratios

$$\frac{c_j^{yk}(w,y_j)}{c_j^{yi}(w,y_j)} = \frac{c_j^{yk}(w,\bar{y}_j)}{c_j^{yi}(w,\bar{y}_j)} = \frac{c_j^{yk}(w,\hat{y}_j)}{c_j^{yi}(w,\hat{y}_j)}$$

are independent of y_j. Marginal restricted profit $c_j^y(w, y_j)$ thus must have w separable from y_j. It is well-known (Gorman, 1976) that this implies $c_j(w,y_j)$ is quasi-homothetic in x_j, i.e.,

$$c_j(w,y_j) = \beta_j(w) + \nu(w)h_j(y_j),$$

where $\beta_j(w)$ and $\nu(w)$ are chosen to satisfy Assumption 2. This demonstrates necessity. Sufficiency is straightforward.

References

Aczél, J. Lectures on Functional Equations and Their Applications. Academic Press: New York, 1966.

Gorman, W.M. "Tricks with Utility Functions," in Essays in Economic Analysis (eds., M.J. Artis and A.R. Nobay. Cambridge University Press: New York, 1976.

McFadden, D. "Cost, Revenue, and Profit Functions," in Production Economics: A Dual Approach to Theory and Applications (eds.) M. Fuss and D. McFadden. North-Holland/Elsevier: New York, 1978.

Phlips, L. Applied Consumption Analysis. North-Holland/Elsevier: New York, 1974.

Pope, R.D. and R.G. Chambers. "Price Aggregation When Price Taking Firms Prices Vary" Review of Economic Studies 56(1989): 297-310.

Stigler, G. and J. Kindahl. The Behavior of Industrial Prices. Columbia University Press: New York, 1970.

________. "Compatible Indexes," Economic Journal 96 Supplement (1986): 83-95.

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