Primal and Dual Approaches To the Analysis of Risk Aversion

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In a classic paper, Peleg and Yaari (1975) observe that the marginal rate of substitution for risk neutral decisionmakers between state-contingent income claims is given by the relative probabilities. Thus, probabilities play the same role as prices in the traditional producer and consumer choice problems. More generally, for risk-averse individuals, the marginal rates of substitution between state-contingent incomes may be interpreted as relative 'risk-neutral' probabilities, a point developed further by Nau (2001). This analogy is particularly important in the case where there are spanning portfolios. In that case, after normalization, the induced Arrow-Debreu contingent claims prices can be interpreted as the equilibrium risk-neutral probabilities for all individuals participating in the market.

These observations suggest that preferences under uncertainty can be informatively examined in much the same fashion that one analyzes consumer preferences under certainty, that is, in terms of convex indifference sets and their supporting hyperplanes. This is not a new observation (Milne, 1995). But, to our knowledge, it has yet to be systematically exploited. This paper's goal, therefore, is to extend Peleg and Yaari's fundamental insight by analyzing decisionmaker preferences under uncertainty in this manner. In this approach, probabilities play a role analogous to those of prices in consumer and producer theory. A key consequence of this observation is the recognition that the vast literature on functional structure for consumer preferences and producer technologies can be imported, with proper modification, into the analysis of preferences under uncertainty. Indeed, the results of Lewbel and Perraudin (1995) linking the portfolio separation and demand rank literatures suggest that the process has already begun.

While the analysis of convex sets and their associated dual functions has proved extremely valuable in the analysis of consumer and producer theory and other areas, until recently little use has been made of these methods in the analysis of problems involving uncertainty. Given Arrow (1953) and Debreu's (1952) early demonstrations that problems involving uncertainty are formally identical to those under certainty once the concept of state-contingent commodities is invoked, this seems particularly surprising.

We speculate that there are a number of historical reasons. First, despite the early contributions of Arrow (1953) and Debreu (1952) and the latter contributions of Hirshleifer (1965), Yaari (1969), and Peleg and Yaari (1975), the state-contingent approach has been neglected in favor of what Hart and Holmström (1988) refer to as the 'parametrized distribution approach'. That approach focuses attention on families of probability distributions over an outcome space (usually infinite dimensional), indexed by one or more parameters. By its nature, this characterization does not lend itself to a methodology which, as typically applied, involves consideration of functions having as arguments finite-dimensional price and quantity vectors. This is perhaps best illustrated by recalling that the formal similarities between welfarist inequality measurement and the evaluation of stochastic outcomes have long been appreciated (Atkinson, 1970). Moreover, methods from convex analysis were early used to provide a formal derivation of welfarist inequality measures (Blackorby and Donaldson, 1978, 1980). But these importance of these results for preferences under uncertainty do not seem to have been fully appreciated.

Second, in a complete Arrow-Debreu equilibrium, with a finite state space, the relevant prices would be the prices of state-contingent claims. More generally, state-claim prices may be derived from the asset prices for a spanning set of securities. In many economic problems involving uncertainty however, either there are no financial assets, or only a small number exist relative to the dimension of the state space. Finally, in areas where the state-contingent approach has been used extensively (for example, general equilibrium and finance theory (Milne 1995)), the primary concern is often with questions of existence (such as the determination of asset prices) and not with welfare evaluations or comparative-static analysis, where the interplay between primal and dual measures can be exploited to great effect.

In this paper we start by representing preferences in terms of the benefit function, originally developed in the theories of inequality measurement and consumer preferences under certainty (Blackorby and Donaldson, 1980; Luenberger, 1992), and its concave conjugate, which we refer to as the expected-value function. Frequently, problems, which prove intractable in the primal representation, admit simple solutions in the conjugate, dual representation and vice versa. As we show below, a clear example of this is offered by the case of generalized linear risk tolerance, which is easily represented in terms of the conjugate, but for which no closed-form certainty equivalent exists. The second crucial tool in the analysis is the use of superdifferentials that yield simple representations of probabilities as closed, convex sets of relative prices in the spirit of Peleg and Yaari even when preferences are not smooth.

Next, we consider the notion of risk aversion, beginning with Yaari's (1969) concept of risk aversion as a quasi-concavity property, then defining risk-aversion with respect to a probability vector. This definition gives rise to dual versions of the Arrow-Pratt absolute and relative risk premiums as functions of the probabilities. If the individual is risk-averse with respect to a given probability vector, these dual risk premiums take the maximum values (zero and one) at that vector, just as the corresponding primal measures are minimized at certainty.

The power of these tools is illustrated by an analysis of the concepts of constant absolute and relative risk aversion as homotheticity properties. Homotheticity conditions of various kinds play a central role in consumer and producer theory and in their application. In particular, aggregation of consumers and producers, the computation of exact index numbers (including inequality indexes), exact welfare comparisons, and the empirical modelling of consumer demand are facilitated by the existence of appropriately homothetic preferences (Shephard, 1953; Malmquist, 1953; Gorman, 1953, 1981; Stone, 1954; Diewert, 1976a, 1976b; Muellbauer, 1976; Deaton and Muellbauer, 1980; Caves, Christensen, and Diewert, 1982a, 1982b; Diewert, 1992; Chambers, 2001). In this paper, we exploit the observation of Chambers and Färe (1998) and Quiggin and Chambers (1998) that constant absolute risk aversion corresponds to an appropriately-defined notion of translation-homotheticity, just as constant relative risk aversion corresponds to homotheticity. Just as in expected-utility theory, a notion of linear risk tolerance can then be specified which generalizes both of these concepts. Linear risk tolerance, in fact, corresponds to quasi-homothetic preferences, which are particularly tractable when formulated in conjugate terms.

The combination of constant absolute risk aversion and constant relative risk aversion yields constant risk aversion (Safra and Segal 1998). Safra and Segal (1998) demonstrate the theoretical and practical importance of constant risk aversion by showing that, in the presence of linear utility, constant risk aversion, combined with critical features of a number of important generalizations of expected utility theory, is sufficient to characterize that theory. For example, betweenness (Chew) and constant risk aversion are sufficient to characterize disappointment theory (Gul 1991) with linear utility. Important examples of preferences displaying constant risk aversion include the linear mean-standard deviation preferences, completely risk-averse preferences, Yaari's (1987) dual theory, and Weymark's (1981) generalized Gini model. We provide a complete dual characterization of constant risk aversion and show that the 'plunging' property Yaari observes for the dual theory holds more generally for preferences satisfying constant risk aversion.

In generalizing the Pratt-Arrow meaures of risk aversion to general preferences, Nau (2001) has recently introduced the concepts of the buying price, the selling price, the marignal price, and a generalized risk premium of a risky asset for differentiably smooth preferences. We generalize and extend these results in several ways. We generalize and characterize his measures for general preferences in terms of superdifferentials and directional derivatives. Among other results we show that they buying and the selling price are only equal under constant absolute risk aversion, and in that case we develop an exact and superlative index of the value of a risky asset in the presence of background risk that extends the standard Pratt-Arrow approximations. We also show that the an appropriate version of the generalized risk premium of the risky asset is convex in the risky asset which extends a basic property of the standard Pratt-Arrow risk premium.

1 Notation

We consider preferences over random variables represented as mappings from a state space Ω to an outcome space $Y \subseteq \Re$. Our focus is on the case where Ω is a finite set $\{1, ...S\}$, and the space of random variables is $Y^S \subseteq \Re^S$. The unit vector is denoted $\mathbf{1} = (1, 1, ...1)$, and $\mathcal{P} \subset \Re^S_{++}$ denotes the probability simplex. Define $\mathbf{e}_i$ as the ith row of the $S \times S$ identity matrix

$$\mathbf{e}_i = (0, ..., 1, 0, ..., 0).$$

Preferences over state-contingent incomes are given by an ordinal mapping $W : \Re^S \rightarrow \Re$. W is assumed everywhere continuous, nondecreasing, and quasi-concave in $\mathbf{y}$. Quasi-concavity ensures that the least-as-good sets of the preference mapping

$$V(w) = \{\mathbf{y} : W(\mathbf{y}) \geq w\}$$

are convex, and that the individual is averse to risk in the sense of Yaari (1969). The *benefit function*, $B : \Re \times Y^S \to \Re$, is defined for $\mathbf{g} \in \Re^S_+$ by:

$$\begin{aligned} B(w,\mathbf{y};\mathbf{g}) &= \bar{B}\left(V\left(w\right);\mathbf{y};\mathbf{g}\right) \\ &= \max\{\beta \in \Re : \mathbf{y} - \beta\mathbf{g} \in V\left(w\right)\} \end{aligned}$$

if $\mathbf{y} - \beta\mathbf{g} \in V\left(w\right)$ for some β and $-\infty$ otherwise (Blackorby and Donaldson, 1980; Luenberger, 1992).[1] The properties of $B(w,\mathbf{y};\mathbf{g})$ are well known (Luenberger, 1992; Chambers, Chung, and Färe, 1996) and are summarized for later use in the following lemma:

Lemma 1 $B(w,\mathbf{y};\mathbf{g})$ *satisfies:*

a) $B(w,\mathbf{y};\mathbf{g})$ *is nonincreasing in* w *and nondecreasing and concave in* $\mathbf{y}$*;*

b) $B(w,\mathbf{y}+\alpha\mathbf{g};\mathbf{g}) = B(w,\mathbf{y};\mathbf{g}) + \alpha$, $\alpha \in \Re$ *(the translation property);*

c) $B(w,\mathbf{y};\mathbf{g}) \geq 0 \Leftrightarrow \mathbf{y} \in V\left(w\right)$, *and* $B(w,\mathbf{y};\mathbf{g}) = 0 \Leftrightarrow W(\mathbf{y}) = w$*;*

d) $B(w,\mathbf{y};\mathbf{g})$ *is jointly continuous in* $\mathbf{y}$ *and* w *in the interior of the region* $\Re \times Y^S$ *where* $B(w,\mathbf{y};\mathbf{g})$ *is finite.*

The benefit function affords a general method for obtaining alternative representations of preferences. For example, the certainty equivalent is the particular case:

$$\begin{aligned} e(\mathbf{y}) &= \inf\{c > 0 : W(c\mathbf{1}) \geq W(\mathbf{y})\} \\ &= -B(W(\mathbf{y}),\mathbf{0},\mathbf{1}) \end{aligned}$$

The certainty equivalent trivially satisfies

$$e\left(\mu\mathbf{1}\right) = \mu, \quad \mu \in \Re.$$

The use of certainty equivalents (generalized mean values) as representations of preferences has been discussed by Chew (1982).

[1] When $\mathbf{g} = \mathbf{1}$ the benefit function corresponds to the translation function introduced by Blackorby and Donaldson (1980).

We refer to the concave conjugate of the translation function $B(w, \mathbf{y}; \mathbf{1})$ as the *expected-value function* $E : \mathcal{P} \times \Re \to \Re$. It is defined by

$$\begin{aligned} E(\boldsymbol{\pi}, w) &= \inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\} \quad \boldsymbol{\pi} \in \mathcal{P} \\ &= \inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} : B(w, \mathbf{y}; \mathbf{1}) \geq 0\} \quad \boldsymbol{\pi} \in \mathcal{P}. \end{aligned}$$

where the second equality follows by the fact that $\mathbf{y} \in \arg\min \{\boldsymbol{\pi}\mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\}$ if and only if $\mathbf{y} + \boldsymbol{\delta}\mathbf{1} \in \arg\min \{\boldsymbol{\pi}\mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\}$ for $\delta \in \Re$ (Chambers, 2001).[2]

Because $B(w, \mathbf{y}; \mathbf{1})$ is a continuous and nondecreasing proper concave function, $E(\boldsymbol{\pi}, w)$ is concave and nondecreasing on $\mathcal{P}$ and continuous on the interior of the region of $\mathcal{P}$ where it is finite (Rockafellar, 1970). It is also continuous and nondecreasing in w. By conjugacy,

$$B(w, \mathbf{y}; \mathbf{1}) = \inf_{\boldsymbol{\pi} \in \mathcal{P}} \{\boldsymbol{\pi}\mathbf{y} - E(\boldsymbol{\pi}, w)\}.$$

1.1 Risk-neutral probabilities

An important advantage of our approach is that it does not, in general, rely on the assumption of differentiability. Normally, the admission of nondifferentiability brings little with it apart from extra theoretical rigour and generality. However recent work has shown that nondifferentiability can be particularly important for generalized expected utility models. In those models, a number of important results turn on the distinction between the concepts of second-order risk aversion that characterizes expected utility theory and Frechet-

[2]The expected-value function, thus, can be thought of as an expenditure function for the certainty equivalent $W(\mathbf{y})$ in terms of normalized state-claim prices, which in turn can be interpreted in a complete state-claim market as the individual's probabilities. By the definition of benefit function

$$\mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\mathbf{1} \in V(w).$$

Suppose the state-claim prices are given by $\mathbf{q} \in \Re^S_{++}$, then

$$\begin{aligned} \inf\{\mathbf{q}\mathbf{y} : \mathbf{y} \in V(w)\} &= \inf\{\mathbf{q}(\mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\mathbf{1})\} \\ &= \inf\{\mathbf{q}\mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\mathbf{q}\mathbf{1}\} \\ &= \mathbf{q}\mathbf{1}E\left(\frac{\mathbf{q}}{\mathbf{q}\mathbf{1}}, w\right). \end{aligned}$$

differentiable preferences (Machina, 1982) and that of first-order risk aversion that characterizes models such as rank-dependent expected utility (Epstein, Segal and Spivak). Recently, Machina (2000) has shown that Fréchet differentiable and expected-utility models cannot exhibit payoff kinks that emerge as a characteristic of an individual's preferences over lotteries while rank-dependent models cannot avoid exhibiting them. Hence, developing a representation of preferences that imposes differentiability necessarily excludes several important classes of preferences from consideration.

Because the benefit function and the expected-value function form a conjugate pair, they offer a natural method for defining and generating subjective notions of probability in terms of their superdifferentials and one-sided directional derivatives that allows the analysis of both first-order and second-order risk aversion. Moreover, because they are both cardinal representations of an ordinal preference structure, they also allow us to avoid the need to normalize the superdifferentials and the directional derivatives that representation in terms of the preference structure requires.[3]

For a proper concave function $f : \Re^S \rightarrow \Re$, its *superdifferential* at $\mathbf{x}$ is the closed, convex set:

$$\partial f(\mathbf{x}) = \left\{ \mathbf{v} \in \Re^S : f(\mathbf{x}) + \mathbf{v}(\mathbf{z} - \mathbf{x}) \geq f(\mathbf{z}) \text{ for all } \mathbf{z} \right\}. \tag{1}$$

The elements of $\partial f(\mathbf{x})$ are referred to as *supergradients.* The *one-sided directional derivative* of f in the direction of $\mathbf{z}$ is defined by

$$f'(\mathbf{x}; \mathbf{z}) = \sup_{\lambda > 0} \left\{ \frac{f(\mathbf{x} + \lambda \mathbf{z}) - f(\mathbf{x})}{\lambda} \right\}.$$

By basic results on proper concave functions for f concave (Rockafellar, 1970):

$$f'(\mathbf{x}; \mathbf{z}) = \inf_{\mathbf{v} \in \partial f(\mathbf{x})} \{\mathbf{v}\mathbf{z}\}.$$

Consequently, $f'(\mathbf{x}; \mathbf{z})$ is positively linearly homogeneous and concave in $\mathbf{z}$. Moreover,

$$f'(\mathbf{x}; \mathbf{z}) \leq -f'(-\mathbf{x}; \mathbf{z}).$$

[3]The normalization is accomplished by the choice of the reference asset. Although not explicitly stated, the existing literature seems to routinely have taken the reference asset to be the traditionally safe asset, a normalization which we adopt.

Where $f'(\mathbf{x};\mathbf{z}) = -f'(-\mathbf{x};\mathbf{z})$ for all $\mathbf{z}$, we say that f is *Gateaux differentiable* at $\mathbf{x}$.[4] When f is Gateaux differentiable at $\mathbf{x}$, the directional derivative is linear in $\mathbf{z}$. Moreover, if f is Gateaux differentiable at $\mathbf{x}$, $\partial f(\mathbf{x})$ is a singleton and corresponds to the usual gradient. If $\partial f(\mathbf{x})$ is a singleton, f is Gateaux differentiable at $\mathbf{x}$ (Rockafellar, 1970).

Because we concern ourselves with ordinal preferences over state-contingent incomes, there is no loss of generality in operating entirely in terms of certainty equivalents. The traditional way of defining probabilities is in terms of the superdifferential of the utility function along the sure-thing vector (the bisector) (Yaari, 1969). Following Nau (2001), we concern ourselves more generally with probabilities that can be defined away from the bisector.

The translation property of the benefit function (Lemma 1b) ensures that the superdifferential of B is an element of the unit simplex. In particular:

Lemma 2 *Let*

$$\mathbf{p}(e,\mathbf{y}) = \partial B(e,\mathbf{y},\mathbf{1}).$$

Then

$$\sum_{s\in\Omega} p_s(e,\mathbf{y}) = 1.$$

Proof Since

$$B(e,\mathbf{y}+\delta\mathbf{1},\mathbf{1}) = B(e,\mathbf{y},\mathbf{1}) + \delta$$

Hence if $\mathbf{v} \in \partial B(e,\mathbf{y},\mathbf{1})$ and $\mathbf{z} = \mathbf{y}+\delta\mathbf{1}$, $\mathbf{z}^* = \mathbf{y}-\delta\mathbf{1}$, the definition of the superdifferential implies

$$\begin{aligned} B(e,\mathbf{y},\mathbf{1}) + \mathbf{v}(\mathbf{z}-\mathbf{y}) &\geq B(e,\mathbf{z},\mathbf{1}) \\ B(e,\mathbf{y},\mathbf{1}) + \mathbf{v}(\mathbf{z}^*-\mathbf{y}) &\geq B(e,\mathbf{z}^*,\mathbf{1}) \end{aligned}$$

or

$$\begin{aligned} B(e,\mathbf{y},\mathbf{1}) + \delta\mathbf{v}\mathbf{1} &\geq B(e,\mathbf{y},\mathbf{1}) + \delta \\ B(e,\mathbf{y},\mathbf{1}) - \delta\mathbf{v}\mathbf{1} &\geq B(e,\mathbf{y},\mathbf{1}) - \delta \end{aligned}$$

[4] For concave functions, whose domain is $\Re^S$, Gateaux differentiability is equivalent to Fréchet differentiability (Rockafellar, 1970).

so that

$$\sum_{s\in\Omega} v_s = 1.$$

In view of Lemma 2, the elements of the vector $\mathbf{p}(e,\mathbf{y}) \subset \Re_+^S$ are referred to as *e -dependent risk-neutral probabilities*. If the benefit function is Gateaux differentiable, these probabilities are unique and given by the usual gradient. The set of *risk-neutral probabilities* $\boldsymbol{\pi}(\mathbf{y}) \subset \Re_+^S$ is defined by

$$\begin{aligned}\boldsymbol{\pi}(\mathbf{y}) &= \partial B(e(\mathbf{y}), \mathbf{y}; \mathbf{1}) \\ &= \mathbf{p}(e(\mathbf{y}), \mathbf{y}).\end{aligned}$$

When preferences are Gateaux differentiable $\pi(\mathbf{y})$ is a singleton.

By the conjugacy of the benefit function and the expected-value function (Rockafellar, 1970)

$$\boldsymbol{\pi} \in \partial B(e, \mathbf{y}; \mathbf{1}) \Longleftrightarrow \mathbf{y} \in \partial E(\boldsymbol{\pi}, e) \tag{2}$$

in the relative interior of their domains. Expression (2) is the generalization of Shepard's Lemma to potentially nondifferentiable structures. Thus, the expected-value function can be used as a dual means of obtaining virtual probabilities which correspond to

$$\mathbf{p}(e,\mathbf{y}) = \arg\inf_{\boldsymbol{\pi}\in\mathcal{P}} \{\boldsymbol{\pi}\mathbf{y} - E(\boldsymbol{\pi}, e)\}.$$

The risk-neutral probabilities have a natural price interpretation. If the individual can purchase state-claims at relative prices given by $\boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{y})$, then $\mathbf{y}$ minimises the cost of obtaining the utility level $e(\mathbf{y})$. Thus $\pi(\mathbf{y})$ is analogous to an inverse demand correspondence. The risk-neutral probabilities derived here are dual to the shadow probabilities considered by Peleg and Yaari who consider, for a given choice set C, the probabilities that would lead a risk-neutral decision-maker to choose y as the optimal element of C.

The risk-neutral probabilities associated with outcomes along the bisector are of particular interest. Because $e(e\mathbf{1}) = e$, $E(\boldsymbol{\pi}, e) \leq e$. And because preferences are quasi-concave

$$\boldsymbol{\pi} \in \partial B(e, e\mathbf{1}; \mathbf{1}) \Longleftrightarrow E(\boldsymbol{\pi}, e) = e.$$

We, thus, define the set of *subjective probabilities* $\pi(\mathbf{1}) \subset \Re^S_+$ as

$$\pi(\mathbf{1}) = \cap_e \{\partial B(e, e\mathbf{1}; \mathbf{1})\}$$

and assume that this set is non-empty. Thus, the set of subjective probabilities satisfies

$$\pi(\mathbf{1}) = \cap_e \arg\sup_{\pi \in \mathcal{P}} \{E(\pi, e) - e\}.$$

For an expected utility maximizer with subjective probabilities π,

$$\{\pi\} = \partial B(e, e\mathbf{1}; \mathbf{1}) \quad \forall e$$

The elements of $\pi(\mathbf{1})$ will share some, but not all, of the properties of subjective probabilities in expected-utility theory, and more generally, of the probabilities associated with probabilistically sophisticated beliefs in the sense of Machina and Schmeidler (1992). Savage (1954) presents a set of axioms which imply both the existence of well-defined subjective probabilities and an expected-utility representation. Machina and Schmeidler (1992) drop Savage's sure-thing principle, and strengthen Savage's requirements for the consistency of comparative probability judgements to derive conditions under which preferences will preserve first-order stochastic dominance with respect to a unique probability distribution π. Since the definition of W employed here does not require satisfaction of the sure-thing principle, we will focus on the more general Machina-Schmeidler concept.

The features shared by the superdifferential $\pi(\mathbf{1})$ and the Savage-Machina-Schmeidler concept of subjective probability relate to acceptable betting odds for a risk-averse decision-maker. The definition of the superdifferential of a concave function implies that, beginning with a non-stochastic income $e\mathbf{1}$, welfare will never be improved by acceptance of a bet $\varepsilon \in \Re^S$ which is fair with respect to probabilities $\pi \in \pi(\mathbf{1})$ in the sense that $\pi\varepsilon = 0$. Conversely, if $\pi \notin \pi(\mathbf{1})$, there exists some e and ε with $\pi\varepsilon = 0$ such that $W(e\mathbf{1} + \varepsilon) > W(e\mathbf{1})$. Moreover, for individuals who are risk-averse and probabilistically sophisticated with subjective probabilities π in the sense of Machina and Schmeidler (1992), it must be true that $\pi \in \pi(\mathbf{1})$. If, in addition, preferences are smooth, $\pi(\mathbf{1})$ is a singleton containing the unique probability vector π for which an individual with non-stochastic initial income $e\mathbf{1}$ will reject all fair and unfavorable bets (those with $\pi\varepsilon \leq 0$), but will accept all sufficiently small favorable bets. To state the latter condition more precisely, for any e and any ε with $\pi\varepsilon > 0$, there exists $k > 0$ such that $W(e\mathbf{1}+k\varepsilon) > W(e\mathbf{1})$. Thus, in the case when W is smooth in a neighborhood of the vector $\{e\mathbf{1} : e \in \Re\}$, the unique element $\pi \in \pi(\mathbf{1})$ defines the acceptable betting odds.

The differences between the superdifferential, $\pi(\mathbf{1})$, and the Machina-Schmeidler definition reflect the fact that the superdifferential is a local characterization of preferences for a decision-maker who is assumed to be risk-averse. By contrast, the Machina-Schmeidler definition yields a global stochastic dominance ordering, and the decision-maker is not necessarily risk averse.

Taking the second point first, the Machina-Schmeidler definition of probabilistic sophistication implies that subjective probabilities are unique, but when preferences are not smooth, in a neighborhood of the vector $\{e\mathbf{1} : e \in \Re\}$, $\pi(\mathbf{1})$ will have more than one element. Consider for example, the case when $S = 2$ and the individual has risk-averse rank-dependent expected utility preferences which preserve first-order stochastic dominance with respect to the unique probability vector $\pi = (\frac{1}{2}, \frac{1}{2})$. The general form of preferences is:

$$W(y_1, y_2) = \begin{cases} q\left(\frac{1}{2}\right) u(y_1) + \left(1 - q\left(\frac{1}{2}\right)\right) u(y_2) & y_1 \leq y_2 \\ q\left(\frac{1}{2}\right) u(y_2) + \left(1 - q\left(\frac{1}{2}\right)\right) u(y_1) & y_2 \leq y_1 \end{cases}$$

where q is the probability weighting function and u is the utility function as in Quiggin (1993). Preferences are risk-averse if u is concave and $q\left(\frac{1}{2}\right) \geq \left(1 - q\left(\frac{1}{2}\right)\right)$.[5]

Now consider bets (a, b) with payoff $a > 0$ received in one state and $-b < 0$ received in the other, and suppose that the individual is free to accept or reject the bet (ka, kb) for any $k > 0$ at initial wealth $e\mathbf{1}$. For small k, the change in welfare associated with increasing the level of the bet is

$$\frac{\partial W}{\partial k} \approx u'(e)\left[-bq\left(\frac{1}{2}\right) + a\left(1 - q\left(\frac{1}{2}\right)\right)\right]$$

which is negative if

$$\frac{a}{b} \leq \frac{q\left(\frac{1}{2}\right)}{\left(1 - q\left(\frac{1}{2}\right)\right)}$$

Hence

$$\pi(\mathbf{1}) = \left\{(\pi, 1-\pi) : \left(1 - q\left(\frac{1}{2}\right)\right) \leq \pi \leq q\left(\frac{1}{2}\right)\right\}$$

[5]These conditions are sufficient, but not necessary for the individual to reject all fair bets (Cohen, Chateauneuf and Meilijson). Sufficiency is all that is required for this illustrative example.

A second, and more fundamental distinction between the existence of a set of *subjective probabilities* $\pi(\mathbf{1})$ as defined here and probabilistic sophistication in the sense of Machina and Schmeidler (1992) is that the characterization of $\pi(\mathbf{1})$ depends solely on preferences in a neighborhood of the vector $\{e\mathbf{1} : e \in \Re\}$. Consider an individual with smooth preferences facing an Ellsberg urn problem, with balls of three colours (say 30 red, and 60 either green or yellow). Such an individual might accept all sufficiently small favorable bets at probabilities $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, but might display a preference for bets on the unambiguous outcome red when the payoffs are large. In this case, $\pi(\mathbf{1}) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$, but the individual is not probabilistically sophisticated.

Finally, it should be observed, following Grant and Karni, that if preferences are state-dependent, it is not generally possible to identify subjective probabilities from behavioral observations alone. The identification of $\pi(\mathbf{1})$ with subjective probabilities rests implicitly on an assumption that preferences in the neighborhood of $e\mathbf{1}$ are state-independent.

In summary, our usage of the terms 'risk-neutral probabilities' and 'subjective probabilities' has been adopted to maximize consistency with the literature. However, we are concerned solely with the formal properties of probability vectors as elements of the superdifferential. Nothing in the analysis that follows depends on whether the elements of $\pi(\mathbf{y})$ are 'really' probabilities.

2 Risk aversion

Early writers on the expected-utility model, such as Friedman and Savage, noted that, in this model, risk-aversion was equivalent to concavity of the utility function. The classic work of Arrow and Pratt introduced and integrated two approaches to the analysis of risk-aversion. The first was a behavioral approach, based on the concept of the risk premium which, in the simplest case, is the difference (or ratio) between the expected value of a risky prospect and the certainty equivalent of that prospect. The second was an index number approach in which risk-aversion was characterized by coefficients of absolute (and relative) risk-aversion derived from the first and second derivatives of utility functions at a given value y. Arrow and Pratt related the two approaches in a number of ways. First, they showed that an individual would have non-negative risk premiums for all risky prospects if and only if the coefficients of risk-aversion were positive for all $\mathbf{y}$. These properties in turn were equivalent to concavity of the utility function. Second, they used the coefficients of risk-aversion to derive approximations to the risk premium for prospects in a neighborhood of $\mathbf{y}$. Finally, they characterized the property of constant absolute (relative) risk aversion both behaviorally, by the requirement that a change in base wealth should not change the absolute (relative) risk premium and, in index number terms, by the requirement that the coefficient of absolute (relative) risk-aversion should be the same for all $\mathbf{y}$.

The Arrow-Pratt index-number approach has been extended to generalized expected utility models through consideration of local utility functions (Machina 1982), conditions on probability transformations in rank-dependent models (Chew, Karni and Safra) and matrix analogs of the Arrow-Pratt coefficients, applicable to state-dependent utility models (Nau, 2001). In these generalized models, the Arrow-Pratt results must be modified. As Machina observes, the most natural notion of risk-aversion for models with smooth preferences, namely that all local utility functions should be concave, is stronger than the requirement for a positive risk premium.

In this paper, we focus on the most basic concept of risk-aversion. Following Yaari (1969), risk aversion may simply be regarded as the property that preferences are convex over the state-contingent outcome space Y^S or equivalently, that W satisfies the quasi-concavity assumption imposed above. This concept does not require any specification of objective probabilities. Even if objective probabilities are known, quasi-concavity need not imply a preference for outcomes of the form $k\mathbf{1}$ among the set of state-contingent income vectors with a given expectation at the known objective probabilities. As Nau (2001) observes, risk-averse preferences may be characterized by state-dependent utility or by the existence of undiversifiable background risk, such that, if objective probabilities are given by $\boldsymbol{\pi}$, it need not be true that $\boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{1})$.

In this paper, however, we will maximize comparability with the literature on primal measures of risk aversion by considering the existence of a probability vector $\boldsymbol{\pi}^0$ such that for all $e, \mathbf{y}$ with $e(\mathbf{y}) = e$, $\boldsymbol{\pi}^0\mathbf{y} \geq e$ with a corresponding risk premium

$$\boldsymbol{\pi}^0\mathbf{y} - e(\mathbf{y}) \geq 0.$$

Dually, we can define:

Definition 1 *An individual is risk-averse with respect to probabilities* $\boldsymbol{\pi}^0$ *if* $\boldsymbol{\pi}^0 \in \pi(\mathbf{1})$.

The definition implies that an individual is risk-averse with respect to π^0 if, from an initial position of certainty represented by some $e\mathbf{1}$, the individual will reject all bets $\mathbf{z}$ that are fair in the sense that $\boldsymbol{\pi}^0\mathbf{z} = 0$ and, *a fortiori*, all bets that are unfavorable in the sense that $\boldsymbol{\pi}^0\mathbf{z} < 0$. In the case where $\pi(\mathbf{1})$ is empty, there exists no probability vector with this property for all e. However, it may still be possible to interpret preferences as risk-averse with respect to some set of state-dependent preferences (Grant and Karni, 2000).

The preceding discussion implies:

Lemma 3 *An individual is risk-averse with respect to probabilities* π^0 *if and only if:*

$$E\left(\boldsymbol{\pi}^0, e\right) = e \quad \forall e.$$

Consider the two polar cases of nondecreasing, quasi-concave preferences risk-averse with respect to a given set of probabilities, $\boldsymbol{\pi}^0$. They are risk-neutral preferences,

$$e^{\pi^0 N}(\mathbf{y}) = \sum_{s\in\Omega} \pi_s^0 y_s$$

and maximin preferences (corresponding to complete aversion to risk)

$$e^M(\mathbf{y}) = \min\{y_1, y_2, ..., y_S\}.$$

For any other members of the class of nondecreasing, quasi-concave preferences risk averse with respect to $\boldsymbol{\pi}^0$, with corresponding at-least-as-good sets $V^{\boldsymbol{\pi}^0}$, it is true that

$$V^M(e) \subset V^{\boldsymbol{\pi}^0}(e) \subset V^{\boldsymbol{\pi}^0 N}(e).$$

Our general definition of an increase in risk aversion between individuals reflects this basic property that an increase in risk aversion is reflected by a shrinking of the at-least-as-good sets.

Definition 2 *A is more risk-averse than B if for all e*

$$V^A(e) \subseteq V^B(e).$$

There are several immediate consequences of these definitions. We summarize them in the following theorem and corollary:

Theorem 4 *The following are equivalent (a) A is more risk averse than B; (b)* $B^A(e, \mathbf{y}; \mathbf{1}) \leq B^B(e, \mathbf{y}; \mathbf{1})$ *for all* $\mathbf{y}$ *and* e*; (c)* $E^A(\boldsymbol{\pi}, e) \geq E^B(\boldsymbol{\pi}, e)$ *for all* π *and* e*;(d) for all* $\mathbf{y}$, $e^A(\mathbf{y}) \leq e^B(\mathbf{y})$. *Moreover, if A is more risk-averse than B, and B is risk-averse with respect to probabilities* $\boldsymbol{\pi}^0$*, so is A.*

Proof *(a)⇒(c) is immediate. (c)⇒(b) follows by applying* $E^A(\boldsymbol{\pi}, e) \geq E^B(\boldsymbol{\pi}, e)$ *for all* π *and* e *in the conjugacy mapping. (b)⇒(d) follows because* $e(\mathbf{y})$ *is determined by*

$$\max\{e : B(e, \mathbf{y}; \mathbf{1}) \geq 0\}.$$

(d)⇒(a) is immediate from the definition of V. The second part of the theorem is trivial.

Corollary 5 *If both A and B are expected utility maximizers then A is more risk-averse than B if and only if* u^A *is a concave transformation of* u^B.

Corollary 5 follows from Arrow and Pratt, who show that A will have lower certainty equivalents than B if and only if u^A is a concave transformation of u^B. As part (d) of Theorem 4 shows, this is equivalent to our definition of more risk averse.

2.1 Dual Measures of risk aversion

We consider an absolute and a relative measure of risk aversion. The *dual absolute risk premium* is defined

$$a(\boldsymbol{\pi}, e) = E(\boldsymbol{\pi}, e) - e,$$

and the *dual relative risk premium* (defined only for $e > 0$) by

$$r(\boldsymbol{\pi}, e) = \frac{E(\boldsymbol{\pi}, e)}{e}.$$

Notice that $a(\boldsymbol{\pi}, e) \leq 0$ and $r(\boldsymbol{\pi}, e) \leq 1$. Moreover, because E is concave in $\boldsymbol{\pi}$, so are a and r. These two measures are directly related in the case $e > 0$ by

$$a(\boldsymbol{\pi}, e) = e(r(\boldsymbol{\pi}, e) - 1).$$

We observe

Lemma 6 *The following conditions are equivalent: (1) A is more risk-averse than B; (2)* $a^A(\boldsymbol{\pi},e) \geq a^B(\boldsymbol{\pi},e) \quad \forall \pi, e$*; and (3) for all* $e > 0$ $r^A(\boldsymbol{\pi},e) \geq r^B(\boldsymbol{\pi},e) \quad \forall \pi$.

Lemma 7 *An individual is risk-averse with respect to probabilities* $\boldsymbol{\pi}^0$ *if and only ij* $a(\boldsymbol{\pi}^0,e) = 0$ *and* $r(\boldsymbol{\pi}^0,e) = 1$.

The polar cases of risk neutrality and complete aversion to risk illustrate the properties of these two measures. If preferences are risk-neutral with respect to π^0:

$$\begin{aligned} E(\boldsymbol{\pi},e) &= \inf_{\mathbf{y}} \left\{ \boldsymbol{\pi}\mathbf{y} - \sum_s \pi_s^0 y_s + e \right\} \\ &= e + \inf_{\mathbf{y}} \left\{ \boldsymbol{\pi}\mathbf{y} - \sum_s \pi_s^0 y_s \right\} \\ &= \begin{cases} -\infty & \boldsymbol{\pi} \neq \boldsymbol{\pi}^0 \\ e & \boldsymbol{\pi} = \boldsymbol{\pi}^0 \end{cases}. \end{aligned}$$

Thus,

$$a(\boldsymbol{\pi},e) = \begin{cases} -\infty & \boldsymbol{\pi} \neq \boldsymbol{\pi}^0 \\ 0 & \boldsymbol{\pi} = \boldsymbol{\pi}^0 \end{cases},$$

and

$$r(\boldsymbol{\pi},e) = \begin{cases} -\infty & \boldsymbol{\pi} \neq \boldsymbol{\pi}^0 \\ 1 & \boldsymbol{\pi} = \boldsymbol{\pi}^0 \end{cases}$$

It follows immediately that

Lemma 8 *An individual is risk-averse with respect to probabilities* π^0 *if and only if he is more risk-averse than an individual with preferences that are risk-neutral with respect to* π^0

For completely risk averse preferences preferences

$$e(\mathbf{y}) = \min\{y_1, y_2, ..., y_S\},$$

whence

$$\begin{aligned} E(\boldsymbol{\pi},e) &= \inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} - \min\{y_1, y_2, ..., y_S\}\} + e \\ &= e \end{aligned}$$

because $\boldsymbol{\pi}\mathbf{y} - \min\{y_1, y_2, ..., y_S\} \geq 0$. Hence,

$$\begin{aligned} a(\boldsymbol{\pi}, e) &= 0, \\ r(\boldsymbol{\pi}, e) &= 1. \end{aligned}$$

Preferences exhibit *constant absolute risk aversion* if for all $\boldsymbol{\pi}$

$$a(\boldsymbol{\pi}, e) = a(\boldsymbol{\pi}, e') \quad \text{all } e, e'.$$

Preferences exhibit *constant relative risk aversion* if for all $\boldsymbol{\pi}$

$$r(\boldsymbol{\pi}, e) = r(\boldsymbol{\pi}, e') \quad \text{all } e, e' > 0.$$

Our next result shows that these notions of constant absolute risk aversion and constant relative risk aversion are equivalent to the more familiar notions. It also characterizes the risk-neutral probabilities for both classes of preferences.

Theorem 9 *Preferences exhibit constant absolute risk aversion if and only if*

$$E(\boldsymbol{\pi}, e) = \hat{a}(\boldsymbol{\pi}) + e,$$

where $\hat{a}(\boldsymbol{\pi}) \leq 0$ a nondecreasing proper concave function that is continuous on the interior of the region of $\mathcal{P}$ where it is finite,

$$B(e, \mathbf{y}; \mathbf{1}) = B(0, \mathbf{y}; \mathbf{1}) - e,$$

and

$$\boldsymbol{\pi}(\mathbf{y} + \beta\mathbf{1}) = \boldsymbol{\pi}(\mathbf{y}), \quad \beta \in \Re.$$

Preferences exhibit constant relative risk aversion if and only if

$$E(\boldsymbol{\pi}, e) = \hat{r}(\boldsymbol{\pi})\, e$$

where $\hat{r}(\boldsymbol{\pi}) \leq 1$ is proper concave function that is continuous on the region of $\mathcal{P}$ where it is finite,

$$B(e, \mathbf{y}; \mathbf{1}) = eB\left(1, \frac{\mathbf{y}}{e}; 1\right),$$

and

$$\boldsymbol{\pi}(\mu\mathbf{y}) = \boldsymbol{\pi}(\mathbf{y}), \quad \mu > 0.$$

Proof *The proof is for the case of CARA. The proof for CRRA is exactly parallel. By constant absolute risk aversion*

$$a(\boldsymbol{\pi},e) = \hat{a}(\boldsymbol{\pi}),$$

with $\hat{a}(\boldsymbol{\pi}) \leq 0$ *a nondecreasing proper concave function that is continuous on the region of* $\mathcal{P}$ *where it is finite by the properties of the expected-value function. Hence,*

$$E(\boldsymbol{\pi},e) = \hat{a}(\boldsymbol{\pi}) + e.$$

By conjugacy,

$$\begin{aligned} B(e,\mathbf{y};\mathbf{1}) &= \inf_{\boldsymbol{\pi}} \{\boldsymbol{\pi}\mathbf{y} - \hat{a}(\boldsymbol{\pi})\} - e \\ &= B(0,\mathbf{y};\mathbf{1}) - e, \end{aligned}$$

where $B(0,\mathbf{y};\mathbf{1})$ *is the concave conjugate of* $\hat{a}(\boldsymbol{\pi})$ *satisfying the properties in Lemma 1. Because*

$$B(e,\mathbf{y};\mathbf{1}) = B(0,\mathbf{y};\mathbf{1}) - e,$$

it follows that

$$\mathbf{p}(e,\mathbf{y}) = \mathbf{p}(0,\mathbf{y})$$

for all $\mathbf{y}$. *By the translation property*

$$\mathbf{p}(0,\mathbf{y}+\beta\mathbf{1}) = \partial B(0,\mathbf{y}+\beta\mathbf{1};\mathbf{1}) = \partial B(0,\mathbf{y};\mathbf{1}) = \mathbf{p}(0,\mathbf{y}).$$

Corollary 10 *If preferences exhibit constant absolute risk aversion*

$$\boldsymbol{\pi} \in \cap_e\{\partial B(e,e\mathbf{1};\mathbf{1})\} \Longleftrightarrow \hat{a}(\boldsymbol{\pi}) = 0.$$

If preferences exhibit constant relative risk aversion

$$\boldsymbol{\pi} \in \cap_e\{\partial B(e,e\mathbf{1};\mathbf{1})\} \Longleftrightarrow \hat{r}(\boldsymbol{\pi}) = 1.$$

Remark 1 *A direct consequence of Lemma 1c and Theorem 9 is that*

$$e(\mathbf{y}) = B(0,\mathbf{y};\mathbf{1})$$

in the case of constant absolute risk aversion, whence by the translation property

$$e(\mathbf{y}+\beta\mathbf{1}) = e(\mathbf{y}) + \beta,$$

which is the more traditional definition of constant absolute risk aversion (Quiggin and Chambers. 1998). Hence, $W(\mathbf{y})$ *is translation homothetic (Blackorby and Donaldson, 1980; Chambers and Färe, 1998). Similarly, in the case of constant relative risk aversion* $e(\mathbf{y})$ *is the implicit solution to*

$$B\left(1, \frac{\mathbf{y}}{e}; 1\right) = 0,$$

whence

$$e(\mu\mathbf{y}) = \mu e(\mathbf{y}) \quad \mu > 0,$$

and $W(\mathbf{y})$ *is homothetic.*

Because CRRA corresponds to homotheticity of the welfare functional and CARA corresponds to translation homotheticity of the welfare functional, it is natural to speculate that quasi-homothetic preferences, which contain both CRRA and CARA as special cases will prove useful. Quasi-homothetic preferences are characterized by the fact that their income-expansion paths are straight lines which do not necessarily emanate from the origin. In the expected-utility literature, this characteristic of linear expansion paths has come to be associated with preferences that exhibit *linear risk tolerance (*Brennan and Kraus, 1976; Milne, 1979). We, therefore, say that preferences exhibit *linear risk tolerance* if

$$E(\pi,e) = E^0(\pi) + E^1(\pi)\, e$$

where $E^0(\pi)$ and $E^1(\pi)$ are expected-value functions for least-as-good sets that are independent of the certainty equivalent. CARA is the special case of linear risk tolerance where $E^1(\pi) = \pi\mathbf{1} = 1$ for all π, while CRRA is the special case of linear risk tolerance where $E^0(\pi) = \pi\mathbf{0} = 0$ for all π.

The associated risk premiums are given by:

$$\begin{aligned} a(\pi,e) &= E^0(\pi) + e\left(E^1(\pi) - 1\right) \\ r(\pi,e) &= \frac{E^0(\pi)}{e} + E^1(\pi). \end{aligned}$$

Because e can take all positive values, for two individuals with linear risk tolerance, i is more risk-averse than j if and only if

$$E_i^0(\boldsymbol{\pi}) \geq E_j^0(\boldsymbol{\pi})$$
$$E_i^1(\boldsymbol{\pi}) \geq E_j^1(\boldsymbol{\pi})$$

Both CRRA and CARA preferences are particularly tractable analytically in either their dual or their primal formulation. Preferences exhibiting linear risk tolerance are much simpler when expressed in terms of the expected-value function. As is well known, quasi-homothetic preferences generally do not have a closed form certainty equivalent. The manifestation of this in terms of the benefit function is derived directly from composition rules due to Chambers, Chung, and Färe(1996):

Lemma 11 *Preferences exhibit linear risk tolerance if and only if*

$$B(e, \mathbf{y}; 1) = \sup\left\{\min\left\{B^0(\mathbf{y}^0; 1), eB^1\left(\frac{\mathbf{y}^1}{e}; 1\right)\right\} : \mathbf{y}^0 + \mathbf{y}^1 = \mathbf{y}\right\},$$

where B^0 is the benefit function conjugate to E^0 and B^1 is the benefit function conjugate to E^1.

Because preferences with linear risk tolerance generally do not have closed form welfare functionals, they have received only limited attention in the literature on primal representation of preferences over stochastic incomes. Indeed, even using the more general concept of a superdifferential, it will generally be difficult, and intuitively uninformative to attempt a primal evaluation of the risk-neutral and subjective probabilities. However, one special case that has received attention because of its convenient ability to represent market outcomes in terms of a representative consumer are affinely homothetic preferences (Milne, 1979), which are the special case of linear risk tolerance given by

$$E^0(\boldsymbol{\pi}) = \boldsymbol{\pi}\mathbf{v} \quad \mathbf{v} \in \Re_+^S.$$

Perhaps the best known member of the affinely homothetic class of preferences is the Stone-Geary utility structure which underlies the linear-expenditure system.

Another special case of the quasi-homothetic family, which does not appear to have received much attention in the portfolio-selection or uncertainty literature, is the class of preferences translation homothetic in a direction other than that given by the certainty vector. This class, which has played a role in the empirical modelling of labor demand and consumer preferences (Blackorby, Boyce, Russell, 1978; Dickinson, 1980) is defined by

$$E^1(\boldsymbol{\pi}) = \boldsymbol{\pi}\mathbf{u},$$

where $\mathbf{u} \in \Re^S$. Intuitively, this is the class of preferences for which real income effects are independent of the economic environment. CARA corresponds to the special case where real income effects are constant and the same for all states of nature.

Preferences satisfying constant relative risk aversion, constant absolute risk aversion, and linear risk-tolerance can all be characterized in terms of the notion of demand rank, which corresponds to the dimension of the function spaced spanned by the individual's Engel curves (Lewbel, 1991). By our results and Theorem 1 of Lewbel (1991), constant relative risk aversion corresponds to a rank-one demand system, while constant absolute risk aversion, and linear risk-tolerance each correspond to rank-two demand systems. Further, using the general results of Lewbel and Perraudin (1995), this establishes that each of these preference structures satisfy the conditions for complete-market portfolio separation associated with the theory of mutual funds. Constant relative risk aversion implies that preferences can be represented indirectly in terms of a single mutual fund, and the corresponding holdings of the respective state-claims per unit of real income are given by the gradient of $\hat{r}(\boldsymbol{\pi})$. Constant absolute risk aversion is associated with preferences that can be represented indirectly in terms of two mutual funds, one of which is degenerate and corresponds to the traditionally safe asset. Only the holding of the degenerate mutual fund is affected by the level of real wealth, and it is this characteristic of constant absolute risk aversion which yields the well-known result that changes in real wealth do not affect the individual's holding of the risky asset. Linear risk tolerance is the generalization of two-mutual fund preferences which makes neither mutual fund degenerate.

In the literature on expected utility preferences, it is well known that only risk-neutral preferences can jointly exhibit constant absolute risk aversion and constant relative risk aversion. Safra and Segal (1998) have recently investigated these type of preferences, which they refer to as constant risk aversion, in the case of an infinite dimensional state space. Preferences with constant risk aversion are interesting not only because they encompass both the important polar cases of risk neutrality and maximal risk aversion, but also because a number of widely-used representations of risk preferences display this property, including Yaari's (1987) dual model, and preferences linear in the mean and standard deviation. Safra and Segal characterize the requirements for constant risk aversion in a number of models of choice under uncertainty.

Quiggin and Chambers (1998) provide necessary and sufficient conditions for general preferences defined over a finite state space to exhibit constant risk aversion, and Quiggin and Chambers (2000) have shown that strictly quasi-concave preferences cannot exhibit constant risk aversion. Here we extend their results to obtain explicit functional representations of constant risk averse preferences defined over a finite state space.

Theorem 12 *Preferences exhibit constant risk aversion if and only if either*

$$E(\pi, e) = e \quad \forall \pi \in \mathcal{P}$$

or

$$E(\pi, e) = \begin{cases} e & \pi \in \hat{\mathcal{P}} \subset \mathcal{P} \\ -\infty & \pi \notin \hat{\mathcal{P}} \end{cases}.$$

Proof *Preferences exhibit constant absolute risk aversion if and only if*

$$E(\pi, e) = \hat{a}(\pi) + e$$

where $\hat{a}(\pi) \leq 0$ is a proper concave benefit function. To satisfy constant relative risk aversion, it, therefore, follows that

$$\mu \hat{a}(\pi) = \hat{a}(\pi) \quad \mu > 0.$$

There are two possibilities either $\hat{a}(\pi) = 0$ or $\hat{a}(\pi) = -\infty$. If $\hat{a}(\pi) = -\infty$ for all π, preferences are not well defined, and we therefore rule that case out. This establishes necessity. Sufficiency follows trivially.

To proceed, let $\Omega^* \subseteq \Omega$ be the subset of the state indexes satisfying

$$y_i = \min\{y_1, y_2, ..., y_S\},$$

then applying the conjugacy mapping establishes:

Corollary 13 *Preferences exhibit constant risk aversion if and only if either*

$$B(e, \mathbf{y}; \mathbf{1}) = \min\{y_1, y_2, ..., y_S\} - e,$$

with

$$\boldsymbol{\pi}(\mathbf{y}) = \mathcal{P}.$$

if $\mathbf{y} = e\mathbf{1}$ *and if* $\mathbf{y} \neq e\mathbf{1}$,

$$\boldsymbol{\pi}(\mathbf{y}) = conv\{\mathbf{e}_s : s \in \Omega^*\}.$$

or

$$B(e, \mathbf{y}; \mathbf{1}) = \inf\left\{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \hat{\mathcal{P}}\right\} - e$$

with

$$\boldsymbol{\pi}(\mathbf{y}) = \arg\inf\left\{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \hat{\mathcal{P}}\right\}.$$

Remark 2 *A direct observation from Corollary 13 is that the only quasi-concave certainty equivalents consistent with constant risk aversion are the concave certainty equivalents*

$$e(\mathbf{y}) = \min\{y_1, y_2, ..., y_S\}$$

and

$$e(\mathbf{y}) = \inf\left\{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \hat{\mathcal{P}}\right\}.$$

We conclude, therefore, that the only types of quasi-concave preferences that are consistent with constant risk aversion are ones that are piecewise linear or ones that are completely averse to risk. Because true piecewise linearity (i.e., $\hat{\mathcal{P}}$ is not a singleton) is not consistent with the additively separable expected-utility form, the well-known result that the only expected utility preferences which are consistent with constant risk aversion are the ones that are linear (that is, risk neutral) follows as a trivial corollary of these results.

We also have

Corollary 14 *Preferences exhibit constant risk aversion if and only if either* $\partial E(\boldsymbol{\pi}, e) = e\mathbf{1}$ *or* $\partial E(\boldsymbol{\pi}, e)$ *is undefined.*

This corollary is a generalization of Yaari's (1987) observation that preferences in the dual model display 'plunging' behavior. Either the individual will reject a given risk entirely, or they will accept an amount of the risk that is either unbounded or fixed by the constraints of the choice problem. This is a general characteristic of constant risk aversion.

Definition 3 *Preferences display decreasing absolute risk aversion if for all* π*,* $a(\boldsymbol{\pi}, e)$ *is decreasing in* e*.*

Definition 4 *Preferences display decreasing relative risk aversion if for all* $\boldsymbol{\pi}$*,* $r(\boldsymbol{\pi}, e)$ *is decreasing in* $e > 0$*.*

The following corollary is a trivial consequence of the definitions and Theorem 9:

Corollary 15 *If preferences exhibit constant absolute risk aversion, they exhibit nondecreasing relative risk aversion. If preferences exhibit constant relative risk aversion, they exhibit nonincreasing absolute risk aversion.*

$E(\boldsymbol{\pi}, e)$ also offers a convenient method to characterize many existing concepts. For example, for the class of state-dependent, linear in probability preferences, Karni (1986) has defined the *reference set* as "...the optimal distribution of wealth across states of nature that is chosen by a risk-averse decision maker facing fair insurance" at the probabilities $\boldsymbol{\pi}$. For given $\boldsymbol{\pi}$, using (2), the reference set is completely characterized by $E(\boldsymbol{\pi}, e)$ as

$$RS(\boldsymbol{\pi}) = \cup_e \{\mathbf{y} \in \partial E(\boldsymbol{\pi}, e)\},$$

and has the equivalent interpretation in our framework as the optimal distribution of "real wealth" across states of nature as chosen by a risk-averse decisionmaker. Karni (1986) also defines a generalization of the Arrow-Pratt risk premium for a given outcome $\mathbf{y}$ as the "...maximum reduction in actuarial value that the decision maker is willing to accept to attain a point on the reference set rather than bear actuarially neutral risk" for $\boldsymbol{\pi}$. In our terms, this is computed simply using $E(\boldsymbol{\pi}, e)$ as

$$\boldsymbol{\pi}\mathbf{y} - E(\boldsymbol{\pi}, e(\mathbf{y})).$$

From this observation and results in Karni (1986), it thus follows immediately that if A and B have the same reference set, then A is more risk averse than B in Karni's (1986) sense if and only if

$$E^A\left(\pi, e^A\left(\mathbf{y}+\mathbf{x}\right)\right) \leq E^B\left(\pi, e^B\left(\mathbf{y}+\mathbf{x}\right)\right)$$

for all $\mathbf{y} \in RS\left(\pi\right)$, $\pi \in \mathcal{P}$, and $\mathbf{x}$ such that $\sum_s \pi_s x_s = 0$.

3 Comparative Statics and $E\left(\pi, e\right)$

$E\left(\pi, e\right)$ is a particularly convenient representation of preferences in cases where the risk-neutral probabilities are exogenous to the individual. Such a case obviously pertains when the individual is a small participant in a complete contingent claims markets. In that case, the risk-neutral probabilities can be treated as the normalized Arrow-Debreu contingent-claim prices. In the case of complete contingent claims, the importance of linear risk tolerance in obtaining several standard results in finance theory including results on the existence of aggregate consumers and the two-fund separation theorem has already been recognized (Milne, 1995; DeTemple and Gottardi). In this section, we show how standard results in consumer theory can be coupled with our results on dual measures of aversion to risk to obtain results for general preferences.

In what follows, for simplicity, we assume that $E\left(\pi, e\right)$ is smoothly differentiable,[6] and we restrict attention to $\mathbf{y} \in \Re^S_+$. As a convenient reference point, we also consider the special case of expected utility, where the certainty equivalent satisfies

$$u\left(e\left(\mathbf{y}\right)\right) = \sum_{s \in \Omega} \hat{\pi}_s u\left(y_s\right).$$

Here $\hat{\pi}$ is the vector of subjective probabilities and u is a smooth increasing concave function. For this case,

$$E\left(\pi, e\right) = \hat{E}\left(\pi, u\left(e\right)\right),$$

where $\hat{E}\left(\pi, u\right)$ is the expected-value function for the expected-utility functional.

[6] As pointed out above, there are many instances (CARA and linear risk tolerance) where the expected-value function will be smooth even though the primal representation is not.

Consider the case of a pure-exchange economy where the individual's endowment of the contingent claims is given by the vector $\mathbf{y}^*$. Her equilibrium welfare level, $e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi})$, is determined as the implicit solution to her budget constraint:[7]

$$E(\boldsymbol{\pi}, e) = \boldsymbol{\pi}\mathbf{y}^*. \tag{3}$$

Using (2), her corresponding equilibrium holding of the contingent claims is given by

$$\mathbf{y}(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi}) = \nabla_{\boldsymbol{\pi}} E(\boldsymbol{\pi}, e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi})), \tag{4}$$

where ∇ represents the usual gradient with respect to the subscripted vector.

Expressions (3) and (4) offer a platform from which to conduct a variety of comparative-static experiments. Suppose, for example, that one is interested in calculating the individual's response to a change in her endowment vector by the small amount $\triangle\mathbf{y}^*$. The effect on her equilibrium certainty equivalent is

$$\frac{\boldsymbol{\pi}\triangle\mathbf{y}^*}{E_e(\boldsymbol{\pi}, e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi}))},$$

while the associated change in her holding of the contingent claims is given by

$$\pi\triangle\mathbf{y}^* \frac{\nabla_{\pi e} E(\boldsymbol{\pi}, e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi}))}{E_e(\boldsymbol{\pi}, e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi}))}.$$

This change in the contingent claims corresponds to movements along Karni's (1986) reference set for the probabilities $\boldsymbol{\pi}$ and to movements along Nau's (2001) wealth expansion path.

If the market's evaluation of her wealth change, $\pi\triangle\mathbf{y}^*$, is positive, then several observations follow immediately. $E_e(\boldsymbol{\pi}, e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi}))$ is the reciprocal of the marginal utility of income, and for small enough changes in the certainty equivalent it can be interpreted as the compensating variation of the change in $e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi})$ induced by the wealth change. Hence, it is positive (this also follows from the properties of the expected-value function). For the expected-utility model,

$$E_e(\boldsymbol{\pi}, e) = \hat{E}_u(\boldsymbol{\pi}, u(e))\, u'(e).$$

[7] In more familiar terms, $e^*(\pi\mathbf{y}^*, \pi)$ is the individual's indirect utility function.

The magnitude of the change in her certainty equivalent, therefore, depends on whether her preferences exhibit increasing aversion to risk or not. If preferences exhibit decreasing relative risk aversion, it must be locally true that

$$E_e(\pi, e) \leq r(\pi, e) \leq 1,$$

while if they exhibit decreasing absolute risk aversion, it must be locally true that

$$E_e(\pi, e) \leq 1.$$

Hence, decreasing relative risk aversion implies decreasing absolute risk aversion, but not the contrary. From these observations we conclude the following for a positive wealth change: If preferences exhibit decreasing absolute risk aversion, the certainty equivalent increases by more than the associated increase in wealth; if preferences exhibit decreasing relative risk aversion, the percentage change in the certainty equivalent is no less than the percentage change in wealth If preferences exhibit increasing relative risk aversion but decreasing absolute risk aversion, the certainty equivalent increases more than the change in wealth, but the percentage change in the certainty equivalent is less than the percentage change in wealth. If preferences exhibit increasing absolute risk aversion, they must also exhibit increasing relative risk aversion. Hence, if preferences exhibit increasing absolute risk aversion both the absolute and the percentage change in the certainty equivalent must be less than the corresponding wealth change.

For the expected-utility model, the notions of decreasing risk aversion have particularly simple interpretations. Notice that by standard results in consumer theory, $\hat{E}_u(\pi, u)$ is the reciprocal of the marginal utility of income. Hence, under expected utility, decreasing absolute risk aversion requires that the marginal utility of the certainty equivalent be less than the marginal utility of income to the individual. Decreasing relative risk aversion, on the other hand, requires that[8]

$$\frac{u(e)\,\hat{E}_u(\pi, u(e))}{\hat{E}(\pi, u(e))}\frac{u'(e)\,e}{u(e)} \leq 1,$$

so that the elasticity of utility at the certainty equivalent is less $\left(\frac{u(e)\hat{E}_u(\pi,u(e))}{\hat{E}(\pi,u(e))}\right)^{-1}$. By standard results from producer theory, this latter expression is the elasticity of scale of the

[8]Here for the sake of simplicity, we have taken $u > 0$.

expected utility functional at the solution to the expected-value minimization problem. We leave it to the reader to rephrase the general consequences of decreasing risk aversion for the certainty equivalent in these terms for the expected-utility model.

The induced change in the sth contingent claim is positive if and only if the sth element of the vector, $\nabla_{\pi e}E(\boldsymbol{\pi},e^*(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}))$, is positive. This vector, in turn, measures the change in the compensated demands induced by a change in real income, as measured by the certainty we equivalent. Hence, on the basis of well-known results in consumer theory, we conclude that demand for the contingent claim rises as a result of the wealth change if and only if it is a normal good. For general preferences, the existence of inferior goods is well recognized, and hence general preferences over stochastic incomes can exhibit instances where contingent claims can fall as a result of an increase in income. Compare this general result with what emerges under expected utility. There for an interior solution to the expected-value problem,

$$\pi_s - \hat{E}_u(\pi,u(e))\,\hat{\pi}_s u'(y_s)\,, \quad s \in \Omega.$$

Differentiating with respect to e and rearranging gives

$$\frac{\nabla_{\pi e}E(\boldsymbol{\pi},e^*(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}))}{E_e(\boldsymbol{\pi},e^*(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}))} = \frac{\nabla_{\pi u}\hat{E}_u(\boldsymbol{\pi},u(e))}{\hat{E}_u(\boldsymbol{\pi},u(e))} = -\left(\frac{\hat{E}_{uu}(\boldsymbol{\pi},u(e))}{\hat{E}_u(\boldsymbol{\pi},u(e))^2}\right)\left[\frac{u'(y_s)}{u''(y_s)}\right].$$

Because the expected-utility function is strictly concave in $\mathbf{y}$, its conjugate expected-value function is convex in u. Hence, it follows immediately that the induced change in the sth contingent claim is of the same sign as the wealth change and just proportional to the individual's risk tolerance at that contingent claim. Put another way, under expected-utility all contingent claims are normal goods, and the relative adjustments in contingent claims are determined by their relative risk tolerances.

If preferences exhibit linear risk tolerance, the change in the holding of her contingent claims is given by

$$\frac{\boldsymbol{\pi}\triangle\mathbf{y}^*}{E^1(\boldsymbol{\pi})}\nabla_{\pi}E^1(\boldsymbol{\pi})\,,$$

which is positive in all its components. Hence, when the individual preferences are characterized by linear risk tolerance, all contingent claim demands rise. In the special case where preferences exhibit constant absolute risk aversion then all contingent claims holdings rise by the same small amount $\boldsymbol{\pi}\triangle\mathbf{y}^*$. This is the analogue of the well-known result that a wealth change does not change an individual with CARA preference purchases of the risky asset in the portfolio problem. More generally, in the case where preferences are translation homothetic but do not exhibit CARA, the relative impacts of an income change on contingent claims, measured by $\frac{u_i}{u_j}$ for all i and j, are independent of real income and relative prices.

The welfare effect of a price change, $\triangle\boldsymbol{\pi}$, on the individual can be computed as

$$\frac{(\mathbf{y}^* - \nabla_{\boldsymbol{\pi}} E(\boldsymbol{\pi}, e))}{E_e(\boldsymbol{\pi}, e)} \triangle\boldsymbol{\pi},$$

and its sign hinges upon whether the price change turns the terms-of-trade in favor of the individual or against the individual. Similarly, it also straightforward to show that the response of the individual's holdings of the contingent claims to a price change can be decomposed into a compensated demand effect and a real-income effect. The compensated demand effect is given by the Hessian matrix of E in the probabilities. It follows immediately that in the case of linear risk tolerance, and its special cases of CARA and CRRA, that each demand responds negatively to a change in its own price.

It is particularly easy in this framework to assess how an individual's welfare is affected by an increase or decrease in the riskiness of the contingent claims prices. For example, suppose that $\boldsymbol{\pi}^o \in \cap_e \{\partial B(e, e\mathbf{1}; \mathbf{1})\}$, and that for those subjective probabilities the contingent claims prices undergo the simple mean preserving change

$$d\pi_1 = -\frac{\pi_2^o}{\pi_1^o} d\pi_2.$$

If $\pi_2 > \pi_1$ and $d\pi_2$, this corresponds to a simple mean preserving spread. Welfare, therefore, only rises if[9]

$$\frac{(y_2 - E_2)}{\pi_2^o} - \frac{(y_1 - E_1)}{\pi_1^o} \geq 0$$

[9]A sufficient condition for this inequality to always be satisfied is that

$$\boldsymbol{\pi}\mathbf{y}^* - E(\boldsymbol{\pi}, e),$$

which is referred to as the *balance-of-trade function* in the literature on international trade, be generalized Schur convex for $\boldsymbol{\pi}^o$ in the sense of Chambers and Quiggin (1997).

4 Primal characterization of asset demands

The problem of primal characterizations the demand for risky assets is of considerable interest. Moreover, since all economic decisions under uncertainty share with portfolio choices the property that they may regarded as choices of state-contingent income or consumption vectors, general results on asset demands may be extended to apply to a wide range of problems such as consumption-savings choices (Sandmo 1970), output decisions for owner-operated firms (Sandmo 1971), and labor supply decisions (Block and Heineke 1973).

Various measures of willingness to pay for risky assets will be useful in what follows. Following Luenberger (1996), we define the compensating and equivalent benefits (in units of g) by

$$\begin{aligned} CB(\mathbf{y}^0,\mathbf{y}^1) &= B(e^0,\mathbf{y}^1;\mathbf{g}) - B\left(e^0,\mathbf{y}^0;\mathbf{g}\right) \\ EB(\mathbf{y}^0,\mathbf{y}^1) &= B(e^1,\mathbf{y}^1;\mathbf{g}) - B(e^1,\mathbf{y}^0;\mathbf{g}). \end{aligned}$$

The compensating and equivalent benefits can be recognized as generalizations of Allais's measure of disposable surplus (Luenberger, 1996; Chambers, 2001). In words, they are, respectively, the units of the reference risky asset, g, that can be subtracted from $\mathbf{y}^1$ to leave the individual just indifferent to $\mathbf{y}^0$, and the units of the reference bundle that must be added to $\mathbf{y}^0$ to make him indifferent to $\mathbf{y}^1$. When preferences are strictly increasing in all state-contingent incomes, these measures reduce to

$$\begin{aligned} CB(\mathbf{y}^0,\mathbf{y}^1) &= B(e^0,\mathbf{y}^1;\mathbf{g}) \\ EB(\mathbf{y}^0,\mathbf{y}^1) &= -B(e^1,\mathbf{y}^0;\mathbf{g}). \end{aligned}$$

because $B(e^i,\mathbf{y}^i;\mathbf{g}) = 0$. Chambers and Färe (1998) establish

Lemma 16 *$CB = EB$ globally if and only if preferences are translation homothetic (in the direction of* g*).*

Special cases of the compensating benefit and equivalent benefit which have received attention in the literature on preferences over stochastic outcomes (Nau, 2001) are the *buying price* of the asset z (at y)

$$P_b(\mathbf{z},\mathbf{y}) = B\left(e\left(\mathbf{y}\right),\mathbf{y}+\mathbf{z},\mathbf{1}\right),$$

and the *selling price* of the asset

$$P_s(\mathbf{z}, \mathbf{y}) = -B(e(\mathbf{y} + \mathbf{z}), \mathbf{y}, \mathbb{1}).$$

In words, the buying price and the selling price of the asset are just the number of units of the traditionally safe asset (that is the one identified by the 45^o line) that the producer is willing to give or accept for the asset.[10] By the properties of the benefit function, it follows immediately that the buying price of an asset is positive if and only if the selling price of the asset is positive and if and only if $e(\mathbf{y} + \mathbf{z}) \geq e(\mathbf{y})$. By the definition of the superdifferential

$$\boldsymbol{\pi}\mathbf{z} \geq P_b(\mathbf{z}, \mathbf{y}) \quad \forall \boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{y})$$

and

$$\boldsymbol{\pi}\mathbf{z} \geq P_s(\mathbf{z}, \mathbf{y}) \quad \forall \boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{y} + \mathbf{z}).$$

Because the buying price and selling price of the asset are expressed in units of the certainty vector, it is not surprising that their relationship to one another is closely related to other measures of risk which are normalized in the same fashion. However, because of the presence of real-wealth effects, the buying and the selling price generally diverge from one another, although as Nau (2001) shows for small enough asset vectors they are approximately equal. More precisely, we have as a direct consequence of Lemma 16 and results in Quiggin and Chambers (1998) that the buying price and the selling price of the asset always equal one another only when these real wealth effects are completely neutral. More formally:[11]

Theorem 17 *The buying price and the selling price of an asset are always equal if and only if preferences are characterized by constant absolute risk aversion.*

Proof *If preferences satisfy constant absolute risk aversion, by Theorem 9*

$$B(e, \mathbf{y}; \mathbb{1}) = B(0, \mathbf{y}; \mathbb{1}) - e$$

[10] It is straightforward to define a buying price and a selling price denominated in terms of a risky asset rather than the traditionally safe asset. However, since such measures do not appear to have been recognized in the literature, we leave their consideration to a future paper.

[11] For the sake of completeness, we provide a direct proof.

whence

$$\begin{aligned} P_b(\mathbf{z},\mathbf{y}) &= B(0,\mathbf{y}+\mathbf{z},\mathbf{1}) - e(\mathbf{y}) \\ &= e(\mathbf{y}+\mathbf{z}) - B(0,\mathbf{y},\mathbf{1}) \\ &= -B(e(\mathbf{y}+\mathbf{z}),\mathbf{y};\mathbf{1}) \\ &= P_s(\mathbf{z},\mathbf{y}). \end{aligned}$$

Conversely, if for all $\mathbf{y}$ *and* $\mathbf{z}$

$$-B(e(\mathbf{y}+\mathbf{z}),\mathbf{y};\mathbf{1}) = B(e(\mathbf{y}),\mathbf{y}+\mathbf{z};\mathbf{1})$$

setting $\mathbf{y} = \mathbf{0}$ *to obtain*

$$-B(e(\mathbf{z}),\mathbf{0};\mathbf{1}) = B(0;\ \mathbf{z};\mathbf{1}).$$

Nau (2001) defines the *marginal price of a financial asset* in the smooth case as the two-sided directional derivative of the preference structure, $\boldsymbol{\pi}(\mathbf{y})\,\mathbf{z}$ (in the smooth case $\boldsymbol{\pi}(\mathbf{y})$ is a singleton). Because we allow for nondifferentiable preferences, we thus define two marginal prices of a financial asset using the more general notion of one-sided directional derivatives. We have

$$\begin{aligned} P_m^+(\mathbf{z},\mathbf{y}) &= B'(e(\mathbf{y}),\mathbf{y},\mathbf{1};\mathbf{z}) \\ P_m^-(\mathbf{z},\mathbf{y}) &= -B'(e(\mathbf{y}),\mathbf{y},\mathbf{1};-\mathbf{z}). \end{aligned}$$

We have

$$\begin{aligned} P_m^+(\mathbf{z},\mathbf{y}) &= \inf_{\boldsymbol{\pi}\in\boldsymbol{\pi}(\mathbf{y})}\{\boldsymbol{\pi}\mathbf{z}\} \\ P_m^-(\mathbf{z},\mathbf{y}) &= \sup_{\boldsymbol{\pi}\in\boldsymbol{\pi}(\mathbf{y})}\{\boldsymbol{\pi}\mathbf{z}\}, \end{aligned}$$

whence

$$P_m^-(\mathbf{z},\mathbf{y}) \geq P_m^+(\mathbf{z},\mathbf{y}) \geq P_b(\mathbf{z},\mathbf{y}). \tag{5}$$

Expression (5) generalizes Nau's Proposition 1 to the nondifferentiable case. As Nau notes, the fact that the marginal price of the asset always exceeds the selling price of the asset is a straightforward consequence of the convexity of the individual's at-least-as-good sets reflected here by the concavity of the benefit function and the properties of the superdifferential.

Following Nau (2001), an asset $\mathbf{z}$ is *neutral if* $P_m^+(\mathbf{z},\mathbf{y}) = 0$. By (5), (1), and Lemma 1, an asset is neutral if and only if its buying and selling prices are negative from which one concludes that $e(\mathbf{y}) \geq e(\mathbf{y}+\mathbf{z})$. More generally, we can completely characterize the marginal price of the asset in both the nondifferentiable and differentiable cases by exploiting the basic properties of one-sided directional derivatives for concave functions.

Lemma 18 $P_m^+(\mathbf{z},\mathbf{y})$ *is a nondecreasing, concave, and positively linearly homogeneous function of* $\mathbf{z}$. $P_m^-(\mathbf{z},\mathbf{y})$ *is a nondecreasing, convex, and positively linearly homogeneous function of* $\mathbf{z}$. *If preferences are Gateaux differentiable at* $\mathbf{y}$

$$P_m(\mathbf{z},\mathbf{y}) = P_m^-(\mathbf{z},\mathbf{y}) = P_m^+(\mathbf{z},\mathbf{y})$$

is a nondecreasing, linear function of $\mathbf{z}$.

The case where the buying and the selling price of an asset are always equal, constant absolute risk aversion, is particularly interesting not only because of its centrality to much of the literature on primal measures of risk aversion, but also because it allows us to glean some further insight into the nature of the connection between state-claim prices and the risk-neutral probabilities. We have the following extension of a result originally due to Chambers (2001):

Theorem 19 *If the buying and selling price of an asset are always equal, and* B *is generalized quadratic then*

$$P_b(\mathbf{z},\mathbf{y}) = P_s(\mathbf{z},\mathbf{y}) = \frac{1}{2}\left(\pi(\mathbf{y}) + \pi(\mathbf{y}+\mathbf{z})\right)\mathbf{z}.$$

Proof *By Theorem 17 if the buying and selling price of the asset are always equal then* $B(e,\mathbf{y};\mathbb{1}) = B(0,\mathbf{y};\mathbb{1}) - e$. *Because* $B(e(\mathbf{y}),\mathbf{y};\mathbb{1}) = 0$, *in this case*

$$\begin{aligned} P_b(\mathbf{z},\mathbf{y}) &= B(e(\mathbf{y}),\mathbf{y}+\mathbf{z};\mathbb{1}) - B(e(\mathbf{y}),\mathbf{y};\mathbb{1}) \\ &= B(0,\mathbf{y}+\mathbf{z};\mathbb{1}) - B(0,\mathbf{y};\mathbb{1}). \end{aligned}$$

Diewert's (1976) quadratic lemma applied here then yields

$$B(0, \mathbf{y}+\mathbf{z}; \mathbf{1}) - B(0, \mathbf{y}; \mathbf{1}) = \frac{1}{2}\left(\nabla_{\mathbf{y}} B(0, \mathbf{y}+\mathbf{z}; \mathbf{1}) + \nabla_{\mathbf{y}} B(0, \mathbf{y}; \mathbf{1})\right)\mathbf{z},$$

which gives the result.

Perhaps the easiest way to interpret Theorem 19 is to recall the interpretation of the risk-neutral probabilities as the individual's internal state-claim prices. Theorem 19 shows that in the case where wealth effects are neutral (CARA preferences make them neutral everywhere), then the individual's internal price of the asset is approximately equal to Hicks' (1945-46) many-market consumer surplus measure for the asset in terms of those state-claims prices. Moreover, in the presence of complete state-claims markets (or nearly complete state-claims markets), the individual's internal price of the asset for small enough changes will be well approximated by the market value of the asset. In a complete state-claims markets, all individuals will equate their relative internal state-claim prices to the market's, and hence the market's evaluation can be used to evaluate the value of the asset.

Moreover, because the quadratic provides a second-order flexible approximation to any smooth preference structure, an immediate consequence of Theorem 19 is that $\frac{1}{2}\left(\boldsymbol{\pi}(\mathbf{y}) + \boldsymbol{\pi}(\mathbf{y}+\mathbf{z})\right)\mathbf{z}$ represents a superlative indicator, in the sense of Diewert (1976), for the buying price of the asset under CARA preferences.Theorem 19 provides exact results for the buying price of the asset under specific restrictions on preferences. If these conditions do not hold, a more standard approach can be taken to approximating the buying price of the asset using standard second-order Taylor's series approximations for the case of differentiable preferences. Because $B(e(\mathbf{y}), \mathbf{y}; \mathbf{1}) = 0$,

$$\begin{aligned}
P_b(\mathbf{z}, \mathbf{y}) &= B(e(\mathbf{y}), \mathbf{y}+\mathbf{z}; \mathbf{1}) - B(e(\mathbf{y}), \mathbf{y}; \mathbf{1}) \\
&\approx \nabla_{\mathbf{y}} B(e(\mathbf{y}), \mathbf{y}; \mathbf{1})\,\mathbf{z} + \frac{1}{2}\mathbf{z}'\nabla_{\mathbf{yy}} B(e(\mathbf{y}), \mathbf{y}; \mathbf{1})\,\mathbf{z} \\
&= \pi(\mathbf{y})\,\mathbf{z} + \frac{1}{2}\mathbf{z}'\nabla_{\mathbf{y}} p(e(\mathbf{y}), \mathbf{y})\,\mathbf{z} \\
&= P_m(\mathbf{z}, \mathbf{y}) + \frac{1}{2}\mathbf{z}'\nabla_{\mathbf{y}} p(e(\mathbf{y}), \mathbf{y})\,\mathbf{z}.
\end{aligned}$$

Similarly, presuming the existence of a set of subjective probabilities and letting $\mu = \boldsymbol{\pi}(\mathbf{1})\,\mathbf{y}$, one can use similar methods to arrive at the standard approximation for the risk premium in terms of the benefit function.

When preferences exhibit either constant absolute risk aversion, constant relative risk aversion, or constant risk aversion, we can use Theorems 9 and 12 to strengthen these results.

Corollary 20 *If preferences exhibit constant absolute risk aversion,*

$$\begin{aligned} P_b(\mathbf{z}+\beta\mathbf{1},\mathbf{y}) &= P_b(\mathbf{z},\mathbf{y})+\beta, \\ P_b(\mathbf{z},\mathbf{y}+\beta\mathbf{1}) &= P_b(\mathbf{z},\mathbf{y}), \\ P_m^+(\mathbf{z},\mathbf{y}+\beta\mathbf{1}) &= P_m^+(\mathbf{z},\mathbf{y}), \\ P_m^-(\mathbf{z},\mathbf{y}+\beta\mathbf{1}) &= P_m^-(\mathbf{z},\mathbf{y}), \quad \beta\in\Re. \end{aligned}$$

If preferences exhibit constant relative risk aversion

$$\begin{aligned} P_b(\mu\mathbf{z},\mu\mathbf{y}) &= P_b(\mathbf{z},\mathbf{y}), \\ P_s(\mu\mathbf{z},\mu\mathbf{y}) &= P_s(\mathbf{z},\mathbf{y}), \\ P_m^+(\mathbf{z},\mu\mathbf{y}) &= P_m^+(\mathbf{z},\mathbf{y}), \\ P_m^-(\mathbf{z},\mu\mathbf{y}) &= P_m^-(\mathbf{z},\mathbf{y}), \quad \mu>0. \end{aligned}$$

If preferences exhibit constant risk aversion, then either

$$P_b(\mathbf{z},\mathbf{y}) = \min\{y_1+z_1,...,y_S+z_S\} - \min\{y_1,y_2,...,y_S\},$$

$$\begin{aligned} P_m^+(\mathbf{z},\mathbf{y}) &= \inf_{s\in\Omega^*}\{\mathbf{e}_s\mathbf{z}\}, \\ P_m^-(\mathbf{z},\mathbf{y}) &= \sup_{s\in\Omega^\circ}\{\mathbf{e}_s\mathbf{z}\}, \end{aligned}$$

or

$$P_b(\mathbf{z},\mathbf{y}) = \inf\left\{\pi(\mathbf{y}+\mathbf{z}) : \pi\in\hat{\mathcal{P}}\right\} - \inf\left\{\pi\mathbf{y} : \pi\in\hat{\mathcal{P}}\right\}$$

$$\begin{aligned} P_m^+(\mathbf{z},\mathbf{y}) &= \inf\{\pi(\mathbf{y})\mathbf{z}\}, \\ P_m^-(\mathbf{z},\mathbf{y}) &= \sup\{\pi(\mathbf{y})\mathbf{z}\}, \end{aligned}$$

where

$$\pi(\mathbf{y}) = \arg\inf_{\mathcal{P}}\left\{\pi\mathbf{y} - \inf\left\{\pi\mathbf{y} : \pi\in\hat{\mathcal{P}}\right\}\right\}.$$

Proof *The first two parts are trivial. By theorem 12, if preferences exhibit constant risk aversion then either*

$$e(\mathbf{y}) = \min\{y_1, y_2, ..., y_S\},$$

or

$$e(\mathbf{y}) = \inf\left\{\pi\mathbf{y} : \pi \in \hat{\mathcal{P}}\right\}.$$

Consider first the case of maximin preferences. Then

$$\begin{aligned} P_b(\mathbf{z},\mathbf{y}) &= B(0,\mathbf{y}+\mathbf{z};\mathbf{1}) - B(0,\mathbf{y};\mathbf{1}) \\ &= \min\{y_1+z_1, ..., y_S+z_S\} - \min\{y_1, y_2, ..., y_S\}, \end{aligned}$$

and

$$\pi(\mathbf{y}) = \mathcal{P}.$$

if $\mathbf{y} = e\mathbf{1}$. *If* $\mathbf{y} \neq e\mathbf{1}$,

$$\pi(\mathbf{y}) = conv\{\mathbf{e}_s : s \in \Omega^*\}.$$

Thus, for either $\mathbf{y} = e\mathbf{1}$ *or* $\mathbf{y} \neq e\mathbf{1}$

$$\begin{aligned} P_m^+(\mathbf{z},\mathbf{y}) &= \inf_{s\in\Omega^*}\{\mathbf{e}_s\mathbf{z}\}, \\ P_m^-(\mathbf{z},\mathbf{y}) &= \sup_{s\in\Omega^*}\{\mathbf{e}_s\mathbf{z}\}. \end{aligned}$$

For $e(\mathbf{y}) = \inf\left\{\pi\mathbf{y} : \pi \in \hat{\mathcal{P}}\right\}$*, the result follows by conjugacy.*

Associated with the marginal price of the asset and the buying price of the asset, to account for the possibility of nondifferentiable preferences, there are now two relevant notions of a buying risk premium

$$\begin{aligned} r_b^-(\mathbf{z},\mathbf{y}) &= P_m^-(\mathbf{z},\mathbf{y}) - P_b(\mathbf{z},\mathbf{y}) \geq 0 \\ r_b^+(\mathbf{z},\mathbf{y}) &= P_m^+(\mathbf{z},\mathbf{y}) - P_b(\mathbf{z},\mathbf{y}) \geq 0, \end{aligned}$$

with $r_b^-(\mathbf{z},\mathbf{y}) \geq r_b^+(\mathbf{z},\mathbf{y})$. We have:

Theorem 21 $r_b^-(\mathbf{z},\mathbf{y})$ *is convex in* $\mathbf{z}$. *If preferences are Gateaux differentiable at* $\mathbf{y}$,

$$r_b^-(\mathbf{z},\mathbf{y}) = r_b^+(\mathbf{z},\mathbf{y}) = r_b(\mathbf{z},\mathbf{y}) \quad \forall\mathbf{z},$$

and $r_b^-(\mathbf{z},\mathbf{y}) = r_b^+(\mathbf{z},\mathbf{y})$ *is convex in* $\mathbf{z}$*, and*

$$r_b(\mathbf{z},\mathbf{y}) \approx -\frac{1}{2}\mathbf{z}'\nabla_{\mathbf{y}} p(e(\mathbf{y}),\mathbf{y})\,\mathbf{z}.$$

Theorem 22 *If preferences satisfy constant absolute risk aversion*

$$\begin{aligned} r_b^+(\mathbf{z},\mathbf{y}+\beta\mathbf{1}) &= r_b^+(\mathbf{z},\mathbf{y}), \quad \beta \in \Re, \\ r_b^-(\mathbf{z},\mathbf{y}+\beta\mathbf{1}) &= r_b^-(\mathbf{z},\mathbf{y}), \quad \beta \in \Re. \end{aligned}$$

Corollary 23 If preferences satisfy constant absolute risk aversion and B *is generalized quadratic then*

$$r_b^+(\mathbf{z},\mathbf{y}) = r_b^-(\mathbf{z},\mathbf{y}) = \frac{1}{2}(\pi(\mathbf{y}) - \pi(\mathbf{y}+\mathbf{z}))\,\mathbf{z}.$$

Theorem 24 A *is more risk-averse than* B *if and only if for any* $\mathbf{z}, e$,

$$\begin{aligned} P_b^A(\mathbf{z}, e\mathbf{1}) &\leq P_b^B(\mathbf{z}, e\mathbf{1}) \\ P_s^A(-\mathbf{z}, \mathbf{z}+e\mathbf{1}) &\geq P_b^B(-\mathbf{z}, \mathbf{z}+e\mathbf{1}) \end{aligned}$$

5 Concluding comments

The observation that probabilities may be regarded as shadow prices is not new, but the implied potential for the application of convex analysis and duality theory is only just beginning to be exploited. In this paper, some of the basic building blocks of such an approach have been developed, including primal and dual measures of risk aversion and the characterization of homotheticity and quasi-homotheticity properties such as linear risk tolerance.

The methods of convex analysis and duality theory have yielded powerful tools for the analysis of producer and consumer behavior under certainty. Using a state-contingent representation, similar tools can be developed and applied for problems involving uncertainty.

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