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Indirect Certainty Equivalents and Comparative Statics for the Firm Facing Price and Production Uncertainty

by

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Indirect Certainty Equivalents and Comparative Statics for the Firm Facing Price and Production Uncertainty

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Since its introduction, the Sandmovian model of a risk-averse firm maximizing the expected utility of net returns has been the focus of most economic theorizing on production decisions under price or production uncertainty (Sandmo, 1971). Because this model has an inherently nonlinear objective function, relatively little attention has been devoted to characterizing the indirect objective functions, and the associated economic implications, for the various versions of the Sandmovian model. Epstein (1978) and Pope (1980) used dual methods to examine risk-averse firm behavior under price uncertainty under differing assumptions about the timing of production decisions. But since that time, a folk wisdom seems to have emerged that little, if any, economic information can be inferred from such indirect objective functions.

Recently, however, Chambers and Quiggin (2000) have clarified the duality between cost functions and their underlying stochastic technologies by showing that standard duality arguments (Shephard, 1970; Färe, 1988) link state-contingent technologies to their dual cost functions. This note shows that it is similarly straightforward to characterize producer decisionmaking under uncertainty by using indirect objective functions. The characterization is for the class of producers with continuous and nondecreasing preferences over stochastic incomes who face *both* price and production uncertainty. Our general results are independent of both risk preferences and any notion of probability. However, imposing such additional structure on behavior allows us to refine our results in an informative fashion.

In what follows, we first introduce our notation and model. Then we specify the indirect objective function, which we term the indirect certainty equivalent, and develop its properties. We then consider the consequences of particular structural restrictions, such as constant absolute, constant relative risk aversion, constant risk aversion, and several different characterizations of risk aversion for the indirect certainty equivalent. The final section closes.

1 Model and Notation

We are interested in a multiple-output firm facing a stochastic technology. Uncertainty is modelled by 'Nature' making a choice from a finite set of states $\Omega = \{1, 2, \dots, S\}$. The state-

contingent production technology, following Chambers and Quiggin (2000), is modelled by a continuous input correspondence, $X : \mathbb{R}_+^{M \times S} \rightarrow \mathbb{R}_+^N$, which maps matrices of state-contingent outputs, \mathbf{z} , into inputs capable of producing them

$$X(\mathbf{z}) = \{\mathbf{x} \in \mathbb{R}_+^N : \mathbf{x} \text{ can produce } \mathbf{z}\} \quad \mathbf{z} \in \mathbb{R}_+^{M \times S}.$$

The vector $\mathbf{z}_s \in \mathbb{R}_+^M$ denotes the vector of *ex post* or realized outputs in state s . In addition to continuity, the input correspondence satisfies:¹

X.1 $X(\mathbf{0}_{M \times S}) = \mathbb{R}_+^N$, and $\mathbf{0}_N \in X(\mathbf{z})$ for $\mathbf{z} \geq \mathbf{0}_{M \times S}$ and $\mathbf{z} \neq \mathbf{0}_{M \times S}$.

X.2 $\mathbf{z}' \geq \mathbf{z} \Rightarrow X(\mathbf{z}) \subseteq X(\mathbf{z}')$.

X.3 if $|\text{vec } \mathbf{z}^k| \rightarrow \infty$ as $k \rightarrow \infty$, then $\bigcap_{k \rightarrow \infty} X(\mathbf{z}^k) = \emptyset$.

X.4 $\lambda X(\mathbf{z}^0) + (1 - \lambda) X(\mathbf{z}^1) \subseteq X(\lambda \mathbf{z}^0 + (1 - \lambda) \mathbf{z}^1) \quad 0 < \lambda < 1$,

Individual producers face stochastic output prices, $\mathbf{p} \in \mathbb{R}_{++}^{M \times S}$, and non-stochastic input prices, $\mathbf{w} \in \mathbb{R}_{++}^N$. Their preferences are defined over *ex post* income, $\mathbf{y} \in \mathbb{R}^S$, which is the sum of their holding of a financial asset² with state-contingent returns $\mathbf{q} \in \mathbb{R}^S$ and their flow profit from production. Hence, their returns in state s are

$$y_s = q_s + \mathbf{p}_s \mathbf{z}_s - \mathbf{w} \mathbf{x}.$$

Their evaluation of these *ex post* incomes are given by a continuous and nondecreasing certainty equivalent function, $e : \mathbb{R}^S \rightarrow \mathbb{R}$ with the property that

$$e(\mu \mathbf{1}) = \mu, \mu \in \mathbb{R}.$$

Following Quiggin and Chambers (1998), e is said to exhibit constant absolute risk aversion if

$$e(\mathbf{y} + \delta \mathbf{1}) = e(\mathbf{y}) + \delta, \quad \delta \in \mathbb{R}.$$

¹These properties are discussed in detail in Chambers and Quiggin (2000, Chapter 2). Note, in particular, that they correspond to standard properties placed on input correspondences associated with nonstochastic technologies (Färe, 1988).

²This financial asset can always be interpreted as the producer's portfolio of purely financial assets. Hence, there is no loss of generality in restricting it to be a single financial asset and for the holding of that asset to be normalized to one.

Preferences exhibit constant relative risk aversion if

$$e(\mu y) = \mu e(y) \quad \mu > 0, \quad y \in \mathbb{R}_+^S,$$

and constant risk aversion (Safra and Segal, 1998) if they satisfy both constant relative risk aversion and constant absolute risk aversion, i.e.,

$$e(\mu(y + \delta \mathbf{1})) = \mu e(y) + \mu \delta, \quad \mu > 0, \delta \in \mathbb{R}.$$

Although we have defined notions of constant absolute and constant relative risk aversion, we have not yet defined a concept of risk aversion. For continuous certainty equivalents of the type considered here a number of alternatives are available. For example, following Yaari (1969), Quiggin and Chambers (1998) define preferences to be risk averse if there exists a set of probabilities which leads the individual to uniformly prefer the sure thing to non-degenerate lotteries using those probabilities. This definition of risk aversion essentially requires that indifference curves be suitably convex in the neighborhood of the certain income vector, but does not impose any strong curvature properties upon preferences. A stronger form of risk aversion is given by requiring e to be quasi-concave (Debreu, 1959; Malinvaud, 1970). An even stronger form of risk aversion requires e to be concave. We state the following fact (Chambers and Quiggin, 2000) as a lemma for later use:

Lemma 1 *If e is quasi-concave and satisfies constant absolute risk aversion*

$$e(\mu y) \geq \mu e(y) \quad 0 < \mu < 1.$$

Thus, the combination of constant absolute risk aversion and quasi-concavity requires that the certainty equivalent be sub-homogeneous. Therefore, an immediate consequence of Lemma 1 is that preferences cannot exhibit strict quasi-concavity and constant risk aversion (Chambers and Quiggin, 2000).³

³If preferences are strictly quasi-concave and satisfy constant absolute risk aversion, then

$$e(\mu y) > \mu e(y).$$

By standard duality theorems (Färe, 1988), there is a cost function dual to $X(\mathbf{z})$ and defined

$$c(\mathbf{w}, \mathbf{z}) = \min \{ \mathbf{w}\mathbf{x} : \mathbf{x} \in X(\mathbf{z}) \}$$

if $X(\mathbf{z})$ is nonempty and ∞ otherwise. The cost function satisfies:

C.1. $c(\mathbf{w}, \mathbf{z})$ is positively linearly homogeneous, non-decreasing, concave, and continuous in \mathbf{w} ;

C.2. Shephard's Lemma;

C.3. $c(\mathbf{w}, \mathbf{z}) \geq 0$, $c(\mathbf{w}, 0_{M \times S}) = 0$, and $c(\mathbf{w}, \mathbf{z}) > 0$ for $\mathbf{z} \geq 0_{M \times S}$, $\mathbf{z} \neq 0_{M \times S}$;

C.4. $\|\text{vec} \mathbf{z}^k\| \rightarrow \infty$ as $k \rightarrow \infty \Rightarrow c(\mathbf{w}, \mathbf{z}^k) \rightarrow \infty$ as $k \rightarrow \infty$;

C.5. $c(\mathbf{w}, \mathbf{z})$ is convex and continuous in \mathbf{z} .

Moreover, by standard duality theorems (Färe 1988):

$$X(\mathbf{z}) = \cap_{\mathbf{w} > 0} \{ \mathbf{x} : \mathbf{w}\mathbf{x} \geq c(\mathbf{w}, \mathbf{z}) \}.$$

2 The Indirect Certainty Equivalent

Consider the following correspondence giving feasible net return levels,

$$B(\mathbf{w}, \mathbf{p}, \mathbf{q}) = \{ \mathbf{y} : y_s \leq \mathbf{p}_s \mathbf{z}_s + q_s - c(\mathbf{w}, \mathbf{z}), \quad \mathbf{z} \in \mathbb{R}_+^{M \times S}, s \in \Omega \}.$$

By the properties of the cost function B is continuous, and $B(\mathbf{w}, \mathbf{p}, \mathbf{q})$ is a closed set. Where needed we shall strengthen these properties to include boundedness from above to ensure the existence of well-defined maxima for the producer's problem. Moreover,

$$B(\mu \mathbf{w}, \mu \mathbf{p}, \mu \mathbf{q}) = \mu B(\mathbf{w}, \mathbf{p}, \mathbf{q}), \quad \mu > 0.$$

Our main interest is in the firm's input and output choices. Therefore, we examine these choices conditional on its holding of its financial asset.⁴ The *indirect certainty equivalent*⁵ is defined

⁴To determine the interaction between their optimal portfolio choice and their production decisions, one can always use the indirect certainty equivalent derived below to characterize optimal portfolio choice. Chambers and Quiggin (1997, 2000) examine these joint choices for the case of a single product firm facing, respectively, expected-utility and generalized Schur concave preferences.

⁵Strictly speaking this is an indirect certainty equivalent conditioned on \mathbf{q} .

$$\begin{aligned}
I(\mathbf{w}, \mathbf{p}, \mathbf{q}) &= \max_{\mathbf{r}, c} \{e(\mathbf{y}) : \mathbf{y} \in B(\mathbf{w}, \mathbf{p}, \mathbf{q})\} \\
&= \max_{\mathbf{z}} \{e(\mathbf{p}\mathbf{z} + \mathbf{q} - c(\mathbf{w}, \mathbf{z}) \mathbf{1})\}
\end{aligned} \tag{1}$$

if $B(\mathbf{w}, \mathbf{p}, \mathbf{q})$ is nonempty and $-\infty$ otherwise. Both definitions of $I(\mathbf{w}, \mathbf{p}, \mathbf{q})$ prove convenient in what follows. We have (all proofs are in the appendix):

Proposition 2 *$I(\mathbf{w}, \mathbf{p}, \mathbf{q})$ is continuous in $(\mathbf{w}, \mathbf{p}, \mathbf{q})$, nondecreasing in \mathbf{p} and \mathbf{q} , and non-increasing and quasi-convex in \mathbf{w} .*

Denote

$$\mathbf{z}(\mathbf{w}, \mathbf{p}, \mathbf{q}) \in \arg \max_{\mathbf{z}} \{e(\mathbf{p}\mathbf{z} + \mathbf{q} - c(\mathbf{w}, \mathbf{z}) \mathbf{1})\}.$$

By the theorem of the maximum (Berge, p.116), the elements of $\mathbf{z}(\mathbf{w}, \mathbf{p}, \mathbf{q})$ are upper semi-continuous. Moreover, upon applying Shephard's lemma (Färe, 1988) to $c(\mathbf{w}, \mathbf{z}(\mathbf{w}, \mathbf{p}, \mathbf{q}))$ in the case of a unique cost minimizing solution we obtain

$$\mathbf{x}(\mathbf{w}, \mathbf{p}, \mathbf{q}) = \nabla_{\mathbf{w}} c(\mathbf{w}, \mathbf{z}(\mathbf{w}, \mathbf{p}, \mathbf{q}))$$

where $\mathbf{x}(\mathbf{w}, \mathbf{p}, \mathbf{q})$ is the vector of optimal input demands and ∇ denotes the gradient with respect to the subscripted vector or element of the vector as appropriate. Hence, we obtain the following generalization of Hotelling's lemma for the generalized Sandmövian model as a straightforward consequence of standard arguments in optimization theory.

Proposition 3 *If c is differentiable in \mathbf{w} at $\mathbf{z}(\mathbf{w}, \mathbf{p}, \mathbf{q})$ and I is differentiable with $\nabla_{\mathbf{q}} I(\mathbf{w}, \mathbf{p}, \mathbf{q}) > 0^S$, then*

$$\begin{aligned}
\mathbf{z}_s(\mathbf{w}, \mathbf{p}, \mathbf{q}) &= \frac{\nabla_{\mathbf{p}_s} I(\mathbf{w}, \mathbf{p}, \mathbf{q})}{\nabla_{\mathbf{q}_s} I(\mathbf{w}, \mathbf{p}, \mathbf{q})}, \quad s \in \Omega \\
\mathbf{x}(\mathbf{w}, \mathbf{p}, \mathbf{q}) &= -\frac{\nabla_{\mathbf{w}} I(\mathbf{w}, \mathbf{p}, \mathbf{q})}{\nabla_{\mathbf{q}} I(\mathbf{w}, \mathbf{p}, \mathbf{q}) \mathbf{1}}.
\end{aligned}$$

Because

$$\mathbf{x}(\mathbf{w}, \mathbf{p}, \mathbf{q}) = \nabla_{\mathbf{w}} c(\mathbf{w}, \mathbf{z}(\mathbf{w}, \mathbf{p}, \mathbf{q})),$$

a standard comparative-static decomposition of price effects exists for the optimal input demands. Hence, in the smooth case, the compensated input demands are downward sloping

and symmetric as a consequence of the concavity of $c(\mathbf{w}, \mathbf{z})$. This observation extends the comparative-static results of Pope (1978) and generalizes them to the case of both price and production uncertainty.

To this point, all of our results have been obtained independent of any probability measure. However, in many instances, researchers may not be as interested in the *ex post* supplies as they are in an expected supply expression.⁶ Presuming the existence of such a probability measure, which is known to the researcher and given by π_s , $s \in \Omega$, then it is a straightforward consequence of Proposition 3 that

$$E_{\pi} \mathbf{z}(\mathbf{w}, \mathbf{p}, \mathbf{q}) = \sum_{s \in \Omega} \pi_s \frac{\nabla_{\mathbf{p}_s} I(\mathbf{w}, \mathbf{p}, \mathbf{q})}{\nabla_{\mathbf{q}_s} I(\mathbf{w}, \mathbf{p}, \mathbf{q})}. \quad (2)$$

3 Restrictions on preferences and the form of the indirect certainty equivalent

Proposition 4 *If preferences satisfy constant absolute risk aversion*

$$I(\mathbf{w}, \mathbf{p}, \mathbf{q}) = I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}),$$

where

$$I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}) = \max \{e(\mathbf{p}\mathbf{z} + \mathbf{q}) - c(\mathbf{w}, \mathbf{z})\},$$

and I^A is continuous in $(\mathbf{w}, \mathbf{p}, \mathbf{q})$, nondecreasing in \mathbf{p} and \mathbf{q} , nonincreasing and convex in \mathbf{w} , and

$$I^A(\mathbf{w}, \mathbf{p}, \mathbf{q} + \delta \mathbf{1}) = I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}) + \delta, \quad \delta \in \mathbb{R}.$$

It is immediate from Proposition 4 that optimal input demands, as well as optimal state-contingent outputs, are independent of any non-stochastic changes in wealth if preferences exhibit constant absolute risk aversion. This reiterates the classic finding from portfolio analysis that the amount of a risky asset purchased is independent of the individual's wealth under constant absolute risk aversion. Moreover, Proposition 4 also establishes that optimal

⁶For example, econometricians studying supply and input demand response under uncertainty may not possess enough degrees of freedom to estimate the *ex post* supplies accurately.

input demands are always downward sloping in their own prices and for suitably smooth cases that the sub-Hessian matrix of input demands is negative semi-definite in input prices. We also have:

Corollary 5 *If preferences satisfy constant absolute risk aversion and $I^A(\mathbf{w}, \mathbf{p}, \mathbf{q})$ is differentiable in input prices*

$$e(\mathbf{p}\mathbf{z}(\mathbf{w}, \mathbf{p}, \mathbf{q}) + \mathbf{q}) = I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}) - \mathbf{w}\nabla_{\mathbf{w}}I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}).$$

Notice the similarity between Corollary 5 and Lau's (1978) early results on normalized profit functions. Under CARA, the producer's problem reduces to one that is isomorphic to normalized profit maximization. Hence, one intuitively expects to find results on optimal input demand behavior that exactly parallel those results.

Proposition 6 *If the certainty equivalent satisfies constant relative risk aversion*

$$I(\mu\mathbf{w}, \mu\mathbf{p}, \mu\mathbf{q}) = \mu I(\mathbf{w}, \mathbf{p}, \mathbf{q}).$$

An immediate consequence of Proposition 6 is that optimal state-contingent revenues and, thus, optimal input demands are homogeneous of degree zero in prices and the producer's portfolio. Combining Propositions 4 and 6, we obtain

Corollary 7 *If preferences exhibit constant risk aversion*

$$I(\mathbf{w}, \mathbf{p}, \mathbf{q}) = I^A(\mathbf{w}, \mathbf{p}, \mathbf{q})$$

with I^A continuous and positively linearly homogeneous in $(\mathbf{w}, \mathbf{p}, \mathbf{q})$, nondecreasing in \mathbf{p} and \mathbf{q} , nonincreasing and convex in \mathbf{w} , and

$$I^A(\mathbf{w}, \mathbf{p}, \mathbf{q} + \delta \mathbf{1}) = I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}) + \delta, \quad \delta \in \mathbb{R}.$$

Under expected utility preferences, it is well-known that constant risk aversion is only possible if the individual is risk neutral. In that case, $I^A(\mathbf{w}, \mathbf{p}, \mathbf{q})$ would correspond to the expected profit function plus the expected value of the producer's portfolio. However,

it is also well-known that other risk-averse preference functionals can exhibit constant risk aversion. For example, both maximin preferences

$$e(y) = \min \{y_1, \dots, y_S\}$$

and linear mean-standard deviation preferences satisfy constant risk aversion. Corollary 7 establishes that in these cases the optimal input demands for a risk averter would behave very similarly to the derived demands for a risk-neutral individual. In fact, in our framework, the only thing that distinguishes these input demands from their risk-neutral counterparts is their dependence upon the producer's asset portfolio.

Proposition 8 *If e is quasi-concave, $I(w, p, q)$ is quasi-concave in q .*

Proposition 8 establishes that the indirect certainty equivalent inherits the producer's basic risk aversion. Because the set

$$\{q : I(w, p, q) \geq i\}$$

is convex, under quasi-concavity there will always exist a supporting hyperplane⁷ to

$$\{q : I(w, p, q) \geq I(w, p, \mu 1)\}$$

such that

$$\hat{q} \in \{q : I(w, p, q) \geq I(w, p, \mu 1)\} \implies \sum_s p_s \hat{q}_s \geq \mu \sum_s p_s.$$

Thus,

$$\hat{\pi}_s = \frac{p_s}{\sum_k p_k}, \quad s \in \Omega$$

can be interpreted as a set of probabilities for which the producer exhibits aversion to risk in the financial asset in the sense of Quiggin and Chambers (1998). For these probabilities, the producer always weakly prefers the non-stochastic portfolio, $\mu 1$, to any portfolio with the same expected value. When $I(w, p, q)$ is smoothly differentiable, these probabilities are unique and given by

$$\hat{\pi}_s = \frac{\nabla_{q_s} I(w, p, \mu 1)}{\nabla_q I(w, p, \mu 1) 1}, \quad s \in \Omega.$$

⁷There can be more than one.

Proposition 9 *If e exhibits strong risk aversion, I is concave in q .*

Hence, strong risk aversion implies that the indirect certainty equivalent is increasing at a decreasing rate in and the state-contingent returns from the asset portfolio, and exhibits strong aversion to risk in the financial asset. Moreover, by Proposition 4 and Lemma 1

Corollary 10 *If e is quasi-concave and satisfies constant absolute risk aversion*

$$I(w, p, q) = I^A(w, p, q),$$

where

$$I^A(w, p, q) = \max \{e(pz + q) - c(w, z)\},$$

where I^A is continuous in (w, p, q) , nondecreasing in p , nondecreasing and quasi-concave in q , nonincreasing and convex in w , with

$$I^A(w, p, q + \delta 1) = I^A(w, p, q) + \delta, \quad \delta \in \mathbb{R},$$

and

$$I^A(w, p, \mu q) \geq \mu I^A(w, p, q) \quad 0 < \mu < 1.$$

4 Expected Utility

The most common restriction on preferences is that they be consistent with risk-averse expected utility preferences. All of our general results apply without change. We leave that to the reader. However, we note that the linear-in-probabilities nature of the expected utility model makes it trivial to deduce further consequences from standard results in duality theory. Previously, we worked in terms of certainty equivalents, but here it is more convenient to work directly in terms of the preference function.⁸

In the expected utility framework, the producer seeks to maximize

$$\sum_{s=1}^S \pi_s u_s$$

⁸To convert certainty equivalent results into this format simply apply the inverse mapping of u to the indirect certainty equivalent.

where

$$u_s = u(\mathbf{p}_s \mathbf{z}_s + q_s - c(\mathbf{w}, \mathbf{z}))$$

for some utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is assumed to be continuous, concave, and monotone increasing, with $u(0) = 0$.⁹ Define the *state-contingent utility correspondence*¹⁰ by

$$U(\mathbf{z}) = \left\{ \mathbf{u} \in \mathbb{R}_+^S : u_s \leq u(\mathbf{p}_s \mathbf{z}_s + q_s - c(\mathbf{w}, \mathbf{z})), \quad s \in \Omega \right\},$$

and the utility set, $\mathcal{U} \subset \mathbb{R}_+^S$, as the range of the state-contingent utility correspondence

$$\mathcal{U} = \cup_{\mathbf{z}} U(\mathbf{z}).$$

The *expected utility function* $W(\pi)$ is defined by

$$W(\pi) = \sup \left\{ \sum_{s=1}^S \pi_s u_s : \mathbf{u} \in \mathcal{U} \right\}.$$

By standard arguments $W(\pi)$ is convex and continuous on any open subset of the probability simplex. The convexity properties of $W(\pi)$ are the dual reflection of Blackwell's (1951) famous result that individuals with expected utility preferences always place a positive value on information.

Moreover, by standard duality theorems (Färe, 1988)

$$U^* = \cap_{\pi} \left\{ \mathbf{u} : \sum_{s=1}^S \pi_s u_s \leq W(\pi) \right\},$$

where

$$U^* = cl(conv(\mathcal{U})),$$

i.e., the closure of the convex hull of \mathcal{U} .

Denote

$$\mathbf{u}(\pi) = \arg \sup \left\{ \sum_{s=1}^S \pi_s u_s : \mathbf{u} \in \mathcal{U} \right\},$$

Any members of $\mathbf{u}(\pi)$ lie on an exposed face of U^* for the supporting hyperplane with normal π . Moreover

$$\mathbf{u}(\pi) \subset \partial W(\pi)$$

⁹More generally, the range of u could be extended to the negative reals with few changes in the arguments.

¹⁰Both the state-contingent utility correspondence and the utility set depend upon \mathbf{p}, \mathbf{q} , and \mathbf{w} . We suppress that dependence for notational convenience in this section.

where $\partial f(\mathbf{x})$ denotes the subdifferential of the function $f(\mathbf{x})$ at \mathbf{x} . Because $\mathbf{u}(\pi) \subset \partial W(\pi)$, it follows immediately that

$$(\pi' - \pi^0)(\mathbf{u}(\pi') - \mathbf{u}(\pi^0)) \geq 0,$$

for all members of the probability simplex. If we define the conjugate of $W(\pi)$ ¹¹ by

$$W^*(\mathbf{u}) = \sup_{\pi} \{\pi \mathbf{u} - W(\pi)\},$$

then by standard results (Rockafellar, 1970)

$$\mathbf{u} \in \partial W(\pi) \iff \pi \in \partial W^*(\mathbf{u}).$$

Finally, also by standard results (Rockafellar, 1970), if $W(\pi)$ is Fréchet differentiable then $\mathbf{u}(\pi)$ is a singleton, and conversely if $\mathbf{u}(\pi)$ is a singleton, then $W(\pi)$ is Fréchet differentiable.

5 Concluding Remarks

We have developed the general properties of the indirect certainty equivalent and demonstrated a generalized version of Hotelling's lemma for producers facing both price and production uncertainty independent of any assumptions about the producer's attitudes toward risk and independent of any notion of probabilities. We have also examined the structural consequences for the indirect certainty equivalent of various restrictions on producer preferences including the expected utility formulation.

In the expected-utility formulation, we showed that the expected-utility function is dual to the closure of the convex hull of the utility set. For the more general specification, however, we have not deduced any such duality correspondence. Global duality theorems seem difficult to obtain in this setting. However, local duality correspondences can be obtained straightforwardly by applying the general theorems of Epstein (1981) and Diewert (1982) to this specific case after imposing some additional structure on $B(\mathbf{w}, \mathbf{p}, \mathbf{q})$.

¹¹ $W^*(\mathbf{u}) = 0$.

6 Appendix: Proofs

Proof Proposition 2: Continuity follows by the theorem of the maximum (Berge, 1963, p.116) and the properties of B . The monotonicity properties are trivial. Let

$$\hat{z} = z(\lambda w^o + (1 - \lambda) w^1, p, q).$$

Quasi-convexity in w is established by

$$\begin{aligned} I(w, p, \lambda w^o + (1 - \lambda) w^1) &= e(p\hat{z} + q - c(\lambda w^o + (1 - \lambda) w^1, \hat{z})) \\ &\leq e(p\hat{z} + q - \lambda c(w^o, \hat{z}) - (1 - \lambda) c(w^1, \hat{z})) \\ &\leq e(p\hat{z} + q - \min\{c(w^o, \hat{z}), c(w^1, \hat{z})\}), \end{aligned}$$

for $0 < \lambda < 1$. The first inequality follows by C.1 (concavity in w).

Proof of Proposition 4: By constant absolute risk aversion

$$e(pz + q - c1) = e(pz + q) - c.$$

Continuity follows from the theorem of the maximum, and the monotonicity properties are obvious. By the concavity of $c(w, z)$ in w (C.1), $e(pz + q) - c(w, z)$ is convex in w . Hence, $I(w, p, q)$ is the pointwise supremum of a series of convex functions and thus convex in w by Theorem 5.5 of Rockafellar (1970).

Proof of Proposition 6:

$$\begin{aligned} I(\mu w, \mu p, \mu q) &= \max_z \{e(\mu pz + \mu q - c(\mu w, z) 1)\} \\ &= \max_z \{e(\mu pz + \mu q - \mu c(w, z) 1)\} \\ &= \mu \max_z \{e(pz + q - c(w, z) 1)\}. \end{aligned}$$

under constant relative risk aversion.

Proof of Proposition 8 Let

$$z^0 = z(w, p, q^0), z^1 = z(w, p, q^1)$$

$$\begin{aligned}
\hat{z} &= \lambda z^0 + (1 - \lambda) z^1, \\
y^0 &= pz^0 + q^0 - c(w, z^0) 1, \\
y^1 &= pz^1 + q^1 - c(w, z^1) 1
\end{aligned}$$

Then for $0 < \lambda < 1$,

$$\begin{aligned}
I(w, p, \lambda q^0 + (1 - \lambda) q^1) &\geq e(p\hat{z} + \lambda q^0 + (1 - \lambda) q^1 - c(w, \hat{z}) 1) \\
&\geq e(\lambda y^0 + (1 - \lambda) y^1) \\
&\geq \min \{e(y^0), e(y^1)\}.
\end{aligned}$$

The second inequality follows by C.5 (convexity), and the third by quasi-concavity of e .

Proof of Proposition 9: Replace last line of the proof of Proposition 8 by

$$\lambda e(y^0) + (1 - \lambda) e(y^1)$$

which follows by concavity of e .

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