Fractal Geometry in Agricultural Cash Price Dynamics

Abstract: Agricultural prices are determined by natural and socio-economic factors that are known to be self-similar at different time scales and to follow non-periodic cyclical patterns. These properties are most easily understood using Mandelbrot's fractal geometry, in which a jagged time series is treated as a jagged coastline or any other natural phenomenon. The fractal market hypothesis provides the theory needed to explain why fractal structure exists in agricultural prices. Empirical evidence confirms theoretical predictions.

Key Words: Fractals; Leptokurtosis; Long-Term Memory; R/S Analysis; Self-Similarity; Stable Distribution

Hyun J. Jin, Research Assistant Professor
The Department of Ag-Business & Applied Economics
The North Dakota State University
Fargo, ND 58105-5636.
Phone: (701) 231-8993
Fax: (701) 231-7400
Email: hjin@ndsuext.nodak.edu

Darren L. Frechette, Assistant Professor
The Department of Agricultural Economics & Rural Sociology
The Pennsylvania State University
University Park, PA 16802

This Paper is for the Annual Meeting of the American Agricultural Economics Association in Long Beach 2002.

Fractal Geometry in Agricultural Cash Price Dynamics

Abstract: Agricultural prices are determined by natural and socio-economic factors that are known to be self-similar at different time scales and to follow non-periodic cyclical patterns. These properties are most easily understood using Mandelbrot's fractal geometry, in which a jagged time series is treated as a jagged coastline or any other natural phenomenon. The fractal market hypothesis provides the theory needed to explain why fractal structure exists in agricultural prices. Empirical evidence confirms theoretical predictions.

Key Words: Fractals; Leptokurtosis; Long-Term Memory; R/S Analysis; Self-Similarity; Stable Distribution

Introduction

Agricultural economists face a variety of challenges when analyzing agricultural commodity markets because agricultural prices and production depend so much on biological and socio-economic factors. These factors are highly variable from year to year and difficult to predict. They have often been modeled as random walk processes [Mercer and Smith (1959); Cheng and Deets (1971); and Cogley (1990)], but other models with trends and cycles have also been suggested.

One such model is *fractal geometry*, proposed in the 1960s by Benoit Mandelbrot. Fractal geometry helps scientists describe natural systems in terms of a few simple rules. Although there is no all-encompassing and final definition of fractality, it can be summarized as follows: A *fractal* is a set of systematic rules that governs the properties of an object or phenomenon through time and space. It is self-similar in that smaller pieces of a system are related to the whole, and it has fractional dimension. Fractals represent a new way to study the irregularity and roughness of shapes that have not been described successfully by classical paradigms based on Euclidean geometry.

Not just shapes, but economic time series too can be treated as fractals. Economic variables are governed frequently by complex nonlinear dynamic processes. These processes can lead to persistent long-term dependence and non-periodic cyclical patterns with abrupt changes in trend. The resulting time series manifest fractal structures over a range of time scales. Their defining characteristics are self-similarity, long-term memory, and fractional dimension, all of which can be detected using two well-established models – fractional Brownian motion and the stable distribution. Some characteristics of economic time series such as long-term memory and abrupt large changes have been explained and detected using the two models already [Mandelbrot

(1963b); Falconer (1990); Peters (1994)], but a comprehensive study of agricultural cash prices has not been performed.

Prices cannot be fractal if the efficient market hypothesis is strictly true. The semi-strong form of the efficient market hypothesis says that current prices reflect all public information, including market fundamentals and price history. Thus, only new information causes a movement of prices. In an efficient market, one market participant cannot have an advantage over others and reap excess profits because prices reflect all known information, and the large number of investors ensures that the prices are fair. In the financial markets, therefore, prices have usually been assumed to follow some variant of a random walk. However, the random walk version of the efficient market hypothesis has been proven to fit real market data poorly in many cases.[1] An alternative hypothesis is needed.

Efforts to discover a suitable alternative to the efficient market hypothesis have been spent analyzing the empirical non-normality, fat-tails, and high-peakedness behavior of financial data,[2] resulting in the *fractal market hypothesis*, explained most clearly by Peters (1994, 1996). The fractal market hypothesis requires that market participants maintain heterogeneous investment horizons. News ripples through a market as groups of investors react in turn, causing self-similarity at different time scales and long-term persistent effects. Sudden crashes occur when investors with different horizons experience reductions in liquidity simultaneously. Sudden booms result when the opposite occurs. The fractal market hypothesis is a refinement of the efficient market hypothesis that has resulted in a much broader and richer understanding of market behavior by incorporating heterogeneity among investors.

Economists have applied the fractal market hypothesis to financial markets, but little attention has been given to commodity cash prices. One notable exception is Barkoulas, Labys, and Onochie (1997), who tested for long-term memory across a variety of commodity prices using a fractional integration model. Results of their study suggested that commodity cash prices are fractal. However, the data set in their study is limited to 408 observations, and they tested only one property of fractals, long-term dependence. Therefore, a more comprehensive test with a larger number of observations is needed.

The main objective of this study is to analyze the fractal structure underlying the time series of major agricultural commodity cash prices in the United States. Fractal analysis provides a better tool for understanding and forecasting future cash prices, explaining fluctuations in commodity prices, and adding insight into the nonlinear dynamics of cash prices. This study provides specific contributions in three areas. First, the market structure underlying the fractal geometry in cash price series will be explained. Second, the existence of long-term memory, whether persistent or anti-persistent, will be tested. Last, implications of the stable distribution will be examined.

Fractals and Economic Time Series

The introduction of fractal geometry is one of the defining events in contemporary science. Using fractals, scientists are able to describe natural systems in terms of a few simple rules. Using fractal geometry, one can create the image of natural objects such as snow crystals, trees, coastlines, clouds, or human brains with just a few equations. Euclidean geometry fails in providing simple rules for many natural shapes because natural objects are so irregular that they cannot be described by small numbers of geometric shapes. Natural objects are not simply rough-wrought versions of Euclidean polyhedra. Thus, mathematicians have had difficulty in determining – or even defining – the length, area, and dimension of natural objects, which share some properties with artificial constructs such as the Sierpinski gasket.

The Sierpinski gasket is a triangle that has an infinite number of smaller triangles within it, a finite object with an infinite number of objects within itself. It is the limit of the sequence pictured in Figure 1. Describing the gasket in terms of Euclidean geometry is an overwhelming task, since infinitely many shapes are needed to describe all the triangles in the gasket. The gasket is not a line, and at the same time it is not quite a plane because it has holes within it, which means that the Sierpinski gasket is neither one-dimensional nor two-dimensional. Its dimension is between one and two: it has fractional dimension. Physical scientists have proved that many natural objects have fractional dimension, and the dimension of the Sierpinski gasket is 1.58.

Another main underlying property of fractals is self-similarity. If an object can be subdivided into arbitrarily small pieces and each of the subsections is a small replica of the whole, then the object is said to be self-similar. Subsections of a fractal structure are similar in some sense to the entire structure. The Sierpinski gasket is self-similar because the shape is made up of subsets that look like the whole. No matter how small the pieces are formed by breaking down the structure, each of the pieces contains no less detail than the whole. The shape is fractal because it exhibits fractional dimension and self-similarity.

In nature, disordered structures and random processes are self-similar on certain length and time scales. So-called *random fractals* are commonplace in nature. They can be found in sunspot numbers, precipitation records, and S&P 500 returns. Although economic time series are influenced by human activities, they too can be described as random fractals. Consider a time series of prices that looks like a jagged line. The plot is neither straight, nor does it fill a plane. It is thus not one-dimensional, and at the same time it is not two-dimensional. Its dimension is between one and two: its dimension is fractional. Further, if we draw graphs of daily, weekly, and monthly returns and they look somewhat the same, then they exhibit self-similarity in time. We may therefore think of economic time series as fractals.

Fractal Structure of Commodity Cash Prices

Most market data do not follow random walk processes. They exhibit long-term memory, leptokurticity, and nonperiodic cycles, implying that they may be fractal.[3] Long-term memory means that observations are not independent. Each observation is affected by all the events that preceded it. If a series exhibits long-term memory, then there is persistent temporal dependence even between distant observations. A time series that exhibits persistent dependence has a tendency to move away from its mean in one direction for a long time before it changes direction, and once it changes direction, it moves in the other direction for another long time.

Leptokurticity means that the distributions of time series are non-normal; they have high peakedness and fat tails. Peters (1994, 1996) explained that leptokurticity can be described as a *black noise* process, which is characterized by long-term persistency with sudden catastrophic changes of trend. Black noise exhibits correlation among distant observations and abrupt discontinuous moves up and down that cause the frequency distribution to have high peaks and fat tails.

Studies of nonlinear dynamic models such as Brock (1986) and Mackey (1989) and of extended cyclical analysis such as Burton (1993) and Yang and Brorsen (1991) have explained the complex, irregular movements and embodied fractal behavior in cash prices. They also found that demand and supply forces are important for cash price determination. Random fractal shocks to the supply and demand for agricultural commodities could induce fractal structure in cash prices. Dependence or reversals in cash prices can occur because of supply lags in annual planting periods, perennial gestation periods, or inventory holdings. Business cycles can induce similar fluctuations.

According to studies of commodity prices such as Hall, Brorsen, and Irwin (1989), Williams and Wright (1991), and Chambers and Bailey (1996), the time series of commodity cash prices have two common features. First, they display considerable positive autocorrelation; periods with high prices tend to follow periods with high prices and low prices to follow low prices. Second, they have spikes – periods when the price jumps abruptly to a very high level or low level relative to its long-run average. Thus, a hypothesis that there are fractal processes in the time series of agricultural commodity cash prices receives some empirical support from the data. Next we turn to theory.

Theoretical Support for Fractal Properties

Theoretical support for fractality in commodity prices is based on agent-level heterogeneity. If all information had the same impact on all market participants, there would be no liquidity. However, market participants are not homogeneous. New information is not available to all investors simultaneously and investors do not have equal perceptual abilities. The sequential revelation of information modeled by Copeland (1976) suggests that individuals shift their demand curves sequentially as new information is revealed to them. Heiner (1983) proposed that the response of each individual investor to new information should depend on environmental uncertainty and the agent's ability to understand and react to market events. Kaen and Rosenman (1986) tested Heiner's proposition and found that agents or groups of agents will switch behavior at different times and the resulting asynchronous switching will cause persistent demand movements and, eventually, sudden directional changes.

The importance of information depends also on the trading horizons of market participants. Market participants who trade simultaneously in a market may still have different trading horizons. They have information from both technical analysis and market fundamentals.

Short-term inventory holders may act primarily on technical analysis, while longer-term market positions held by hedgers and industry managers are more likely to depend on fundamentals. Likewise, the information important for each trading horizon can differ. Market participants with different trading horizons and different information sets are the sources of each other's liquidity and the stability of markets.

For example, a daily trader who confronts a regime of high volatility in his trading horizon will often find that the trading activities of other market participants with longer- and shorter-term trading horizons will lessen the volatility he faces. In this fashion, whenever there are numerous market participants who have different trading horizons than those who are in crisis, the market will be stabilized. All market participants share a similar risk level once an adjustment is made for the scale of trading horizons, and the shared risk causes the frequency distributions of price series at different trading horizons to look similar. The result is self-similarity.

Self-similarity may break down if a market becomes unstable at many time scales simultaneously. Investors with long trading horizons leave the market or become short-term traders, and the market becomes asymmetric with respect to trading horizons. This situation might occur when market participants with long-term trading horizons feel that fundamental information is unreliable or if they anticipate an economic or political crisis. Then, trading horizons are shortened and an extremely high level of short-term volatility can exist. However, if market participants with different trading horizons symmetrically coexist, a panic at one horizon can be absorbed by other trading horizons. On the other hand, if the entire market has the same trading horizon, the market becomes entirely unstable – liquidity weakens and a panic may follow. During panics, the market often skips over prices, and prices change abruptly. Fat tails appear in the frequency distribution of prices and self-similarity becomes weaker [Peters (1994)].

The fractal market hypothesis refines the Efficient Market Hypothesis by incorporating heterogeneity of investment horizons. The resulting price distributions exhibit short- and long-term dependence, self-similarity, sudden spurts of volatility, and abrupt changes of trend. These are the same properties discovered in empirical cash price series, and they are the same properties embodied by fractals. They can be detected and measured statistically.

Statistical Tools for Detecting Fractals

Two well-established generalizations of Brownian motion are known to be fractal. One is *fractional Brownian motion* (FBM) and the other is the *stable distribution*.[4] Mandelbrot and Van Ness (1968) introduced FBM, which has normally distributed but non-independent increments. The two most widely used empirical methods for studying FBM are rescaled range (R/S) and fractional integration models. R/S tests can detect long-term dependence in time series and, using different time scales, can detect self-similarity. Fractional integration models describe both long-term memory and short-term memory directly, whereas R/S analysis describes long-term memory indirectly. The purpose of this study is to detect long-term memory, rather than trying to forecast future values, so a fractional integration model is inappropriate; R/S analysis is the proper way to test for FBM.

The stable distribution was developed by Paul Levy in the 1920s by relaxing the finite variance condition of Brownian motion. It has infinite (undefined) variance, and it is discontinuous except for the special case of Brownian motion. In the 1960s, it was studied extensively by Fama (1965a, 1965b) and Mandelbrot (1963a), and it has since become one of the most important fractal models [Falconer (1990)]. The stable distribution has four parameters, which can be estimated through the characteristic function of the stable distribution.[5] The parameter estimates can confirm whether the distribution of a random variable corresponds to a Normal distribution or to a broader set of fractal distributions. Using the same estimation procedures at different time scales one can detect self-similarity of the random variable. With the two models, R/S and the stable distribution, one can test for several important properties of fractals: long-term memory, self-similarity, fractional dimension, and black noise.

Rescaled Range

Rescaled Range (R/S) analysis was first proposed by Hurst (1951) and subsequently refined and applied to economic time series in a series of studies by Mandelbrot (1963a, 1963b, and 1972). The analysis proceeds from two basic principles: dependence between periods and fractional Brownian motion. R/S analysis provides a valuable statistic called the Hurst exponent, H.

A generalized formula for Brownian motion implies the following equation:

(1) $$E[(X_{t+\Delta t} - X_t)(X_t - X_0)] = 0.5((t+\Delta t)^{2H} - (t)^{2H} - (\Delta t)^{2H}),$$

where the increments $X_{t+\Delta t} - X_t$ are stationary, implying that the increments have probability distribution independent of *t*. The distribution functions specified by FBM do not have independent increments except in the case of Brownian motion (H = 0.5). If H = 0.5, then the value of (1) is zero, implying independent increments, but if H ≠ 0.5, then the value is non-zero and increments of $\{X_t\}$ are dependent. Specifically, if H > 0.5, (1) has positive sign and the process tends to persist in the future as it has in the past. If H < 0.5, then (1) has a negative sign and the series tends to reverse itself more frequently than a random walk. The positive effect is known as persistence, and the negative, anti-persistence.

Greene and Fielitz (1977) developed the relationship between the Hurst exponent and fractal dimension. The fractal dimension is calculated by measuring the jaggedness of a series. The fractal dimension of a random-walk process is 1.5. The fractal dimension of a straight line is

1.0, and that of a geometric plane is 2.0. Thus, the fractal dimension of a random walk would be halfway between a line and a plane. The more jagged time series are, the closer their fractal dimensions approach two [Mandelbrot (1982)]. The Hurst exponent H can be converted into a fractal dimension, D, using the formula:

(2) $D = 2 - H.$

Thus if $H = 0.5$, then $D = 1.5$.

The rescaled range is the range of partial sums of deviations of a time series from its mean, rescaled by its standard deviation. Using the rescaled range, the Hurst exponent is estimated by the following formula:

(3) $\ln (R/S)_{i,T} = \ln (a) + H \ln (T_i) + \varepsilon_{i,T},$

The steps are based on Mandelbrot and Taqqu (1979); Lo (1991); Peters (1994); and Corazza, Malliaris, and Nardelli (1997). Mandelbrot and Wallis (1969) showed that the R/S analysis is appropriate even when a distribution exhibits a great deal of skewness and kurtosis. Mandelbrot (1972) argued that R/S analysis can detect non-periodic cycles, even when the periods exceed the length of the observed time series. Mandelbrot and Taqqu (1979) proved the almost-sure convergence of the R/S statistic for stochastic processes with infinite variance. However, Lo (1991) showed that standard R/S analysis is biased toward accepting the null hypothesis of long-term dependence because it is sensitive to short-term dependence. He suggested a modified version of R/S analysis that can detect long-term dependence in the presence of short-term dependence. Details of the approach are provided in the Appendix.

The Stable Distribution

Mandelbrot (1963a, 1963b) and Fama (1963, 1965a) suggested that the distributions of stock returns are significantly different from normality and that the non-normality might be captured by the stable distribution. Subsequent studies of financial data have found distributions that are leptokurtic and incompatible with the Gaussian assumption. The stable distribution's ability to explain observed leptokurtic behavior has been tested through the stability-under-addition test using the characteristic exponent, α, of the stable distribution [Fama and Roll (1971); Fielitz and Rozelle (1983)].[6]

The estimated parameters of the stable distribution tell us about the behavior of the series: skewness, peakedness, location, and scale. There are three special cases where one can write down closed form expressions for the density of the stable distributions and verify directly that they are stable: normal, Cauchy, and Levy distributions.[7] Other than these three distributions, there are no known closed form expressions for general stable densities. Instead, the stable distribution is characterized by its characteristic functional form as follows:

(4) $F(t) = exp[\ i\,\delta t - |\gamma\, t|^{\alpha}\ (1 + j\beta\, sgn(t)\, w(t, \alpha))\ ]$,

where $w(t, \alpha) = -tan(\alpha\pi/2)$ if $\alpha = 1$, otherwise $w(t, \alpha) = (2/\pi)\, log(t)$.

The function has four parameters; α, β, δ, and γ. $\alpha \in (0, 2)$ is the characteristic exponent that accounts for the relative importance of the tails. It measures the peakedness of the distribution as well as the fatness of the tails. If $\alpha = 2$, then the distribution is Normal with finite mean and variance. When $\alpha \in (1, 2)$, the random variable has finite mean. $\beta \in [-1, 1]$ is the skewness parameter. In particular, when $\beta = 0$, the distribution is symmetric. When $\beta = +1$, the distribution is fat-tailed to the right, or skewed to the right. The degree of right skewness increases as β approaches to $+1$, and vice versa. As α approaches 2, β loses its effect and the distribution approaches the symmetric normal distribution regardless of the value of β. $\gamma \in (0, +\infty)$ is the scale parameter. It compresses or extends the distribution from the point δ. $\delta \in (-\infty, +\infty)$ is the location parameter. It shifts the distribution to the left or right.

If tails of a distribution are heavier than exponential, it is said that the distribution has heavy tails, or fat tails. The result of the fat tails is that not all moments may exist. When distributions have sufficiently long tails, the first few moments will not characterize the distribution because they diverge. Under fat tails, mean and variance may be undefined or infinite and therefore unsuited to effective description of a distribution. Cornew, Town, and Crowson (1984) produced a mathematical proof that moments may be undefined for distributions with fat tails.

McCulloch's (1986) methodology is adopted to estimate the initial values of four parameters of the stable distributions. With the quantile estimators as initial approximations to the parameters, a constrained quasi-Newton method is used to maximize the log-likelihood function for an i.i.d. stable sample of $X_1, X_2, \ldots, X_n$ as follows:

(5) $l(\omega) = \sum_i \log f(X_i | \omega),$

where $f(X_i | \omega)$ is the density function of a stable distribution, ω denotes the parameter vector by $\omega = (\alpha, \beta, \gamma, \delta)$ in a parameter space $\Omega = (0, 2] \times [-1, 1] \times (0, +\infty) \times (-\infty, +\infty)$, and the quasi-Newton method is constrained by the parameter space.

To estimate the parameters by maximizing the log-likelihood function, $f(X_i | \omega)$ must be computed for each observation i. There have been several efforts to compute stable densities, such as Holt and Crow (1975); Panton (1992); and Nolan (1997). Among them, Nolan (1997) provides an efficient numerical computation method for stable densities and distribution functions. This study adopts Nolan's methodology.

DuMouchel (1973) showed that when ω_0 is on the interior of the parameter space Ω, the maximum likelihood estimator follows the standard theory so that it is consistent and asymptotically normal. If ω is near the boundary of Ω, the finite sample behavior of the estimator is not precisely known, because the distribution of the estimator may be skewed away from the boundary. Further, in this case, the asymptotic normal distribution of the estimator tends to be a degenerate distribution at the boundary point, making it super-efficient.

Many studies using the stable distribution have not performed any empirical tests fitting the distribution to real data, and it is necessary to have some means of assessing whether the resulting fit is reasonable. One method to determine whether the data are consistent with the stable distribution is to plot a smoothed density of the data and compare it with the fitted stable density alongside the fitted normal density, as proposed by Nolan (1999). Clear multiple modes or gaps in the smoothed density are evidence that the data do not come from a stable distribution. For comparison, the data are smoothed using a Gaussian kernel [Pagan and Hong (1990) and Campbell, Lo, and MacKinlay (1997)]. The density plots indicate whether the fitted stable density matches the real data better than the fitted normal density near the mode and tails of the distribution. If the leptokurtosis and skewness are better described by the stable density than by the normal density, then the data set is clearly non-normal.

Another test is based on self-similarity because the stable distribution is fractal. If a series fits the stable distribution, then other frequencies of the series also must, and the estimated values of the four parameters of the stable distribution must be statistically indistinguishable at all frequencies. The self-similarity test embodies the theorems of Mandelbrot and Taqqu (1979) and of Falconer (1990).

Empirical Results

Data

The time series data used in this study are agricultural commodity cash prices from the Chicago Board of Trade (CBOT), Kansas City (KC), and New York Board of Trade (NYBOT). The data include broilers, cocoa, coffee, corn, large white eggs, Kansas City wheat, oats, soybeans, soybean meal, soybean oil, spring wheat, sugar #11, wheat, and wheat #1 cash prices. The series are daily, and most begin on August 28, 1992 and end on March 8, 2000. Soybean oil starts on June 2, 1969; spring wheat starts on February 1, 1983; and broilers and large white eggs begin on December 1, 1991. The number of observations is 7,766 for soybean oil; 4,325 for spring wheat; 2,068 for broilers and large white eggs; and 1,888 for other commodities. The time series are longer than those of previous studies conducted to detect long-term memory in financial data, such as Hall, Brorsen, and Irwin (1989); Pan, Liu, and Bastin (1996); and Barkoulas, Labys, and Onochie (1997).

The results of unit root tests are available from the authors. Except for broilers, large white eggs, and soybean oil, most of the series have a unit root process in levels, indicating that they are non-stationary. In practice, original price series usually have a unit-root process, while transformations such as log-returns do not have a unit root. To confirm that the transformed series are stationary, the Kwiatkowski, Phillips, Schmidt, and Shin (KPSS, 1992) test is performed. The null of stationarity is not rejected for any of the log-return series.

Empirical Results: The R/S Analysis

To estimate the Hurst exponent, log-return series of the commodity cash prices are used. It is important to use return series because R/S analysis is only appropriate for stationary series. Estimates using the original non-stationary series would be biased upward because the original series is "smoother" than the transformed stationary series.

Table 1 summarizes the Hurst exponent estimates obtained from the classical and modified R/S analyses at different time scales. Most of the estimates fall on the positive, persistent long-term memory side, greater than 0.5, although some estimates are not statistically distinguishable from 0.5. Broilers and eggs fall on the negative, anti-persistent long-term memory side, less than

0.5. The pattern of estimates supports the premise that most cash price series exhibit persistent long-term memory, a key characteristic of fractals.

There is one more property of fractals to be examined: fractional dimension. Fractional dimension measures the jaggedness of a series, and the estimates of H can be converted into a fractal dimension. The results are displayed in Table 2. The fractal dimensions of most series are clearly less than 1.5. They are closer to a straight line than a random walk is. Measured in this way, all the series have fractional dimension, another important property of fractals.

Empirical Results: The Stable Distribution

The parameters, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\delta}$, are estimated using the quantile method. Initial estimates are used as an approximation to maximize the log-likelihood function corresponding to (5). To calculate the stable density for the price distributions, Nolan's (1997) method is used. Further, to generate confidence intervals, a covariance matrix is calculated [DuMouchel (1973)].

The results of fitting the stable distribution to the cash price series are reported in Table 3. The $\hat{\alpha}$ values are not close to 2; they lie between 1.260 and 1.775, implying the distributions are not normal. The values of $\hat{\alpha}$ for eggs and oats are less than 1.0, which indicates that the shapes of their distributions are quite different from those of other commodities. The results show that none of the series fit well to the normal distribution, but their distributions bear out a more generalized one, the stable distribution.

If the series are fractal, they must also exhibit self-similarity. To check for self-similarity, the estimation was repeated for weekly frequency data (n = 5), and bi-weekly frequency data (n = 10). The values of $\hat{\alpha}$ are similar at all three time scales n = 1, n = 5, and n = 10. Within a 95% confidence interval they are statistically indistinguishable, and therefore the hypothesis of self-similarity cannot be rejected.

Daily eggs prices exhibit anti-persistency. An anti-persistent process is mean-reverting, however not every mean-reverting process is anti-persistent. Mean-reversion implies that the mean is state-dependent and varies with time, thus there is no fixed or finite mean. The $\hat{\alpha}$ for daily eggs series suggests that the mean cannot be used as a location parameter in the distribution of daily eggs series. Since $\hat{\alpha}$ is less than 1.0, the mean cannot be defined.

Plotting the smoothed density, the fitted stable density, and the fitted normal density together helps determine how well the stable distribution describes the data. One such plot is displayed in Figure 2, and the others are available from the authors. In the figure, there are no clear multiple modes or gaps. The density plots admit two interpretations. First, none of the series seem to have a high degree of asymmetry or skewness. Second, a normal distribution cannot explain the leptokurticity of the series. The plot suggests that the normal distribution cannot capture the behavior of the agricultural cash prices as well as the stable distribution can.

Implications and Conclusions

Most economic price series have two properties – long-term dependence and leptokurticity – that are among the representative characteristics of fractals. Long-term memory can be detected and quantified through using R/S analysis, and leptokurticity can be modeled using the stable distribution. The time series of major agricultural commodity cash prices in the U.S. have been analyzed to detect fractal structure. Empirical results indicate evidence of long-term memory, fractional dimension, and self-similarity, suggesting that the price series have fractal structure. The following implications can be drawn from the results.

First, the empirical results suggest that the Efficient Market Hypothesis is inappropriate for the analysis of agricultural commodity markets. The fractal market hypothesis may be a suitable alternative for the Efficient Market Hypothesis in this case. Second, self-similarity allows the behavior of low frequencies such as annual or monthly to be predicted by looking at high frequencies such as daily or weekly. Third, since there is long-term dependence in the agricultural cash prices, it is possible to forecast more accurately by including long-term memory in univariate time series models such as ARIMA. Fractional integration models have been developed, such as fractionally integrated ARMA (ARFIMA) and fractionally integrated GARCH (FIGARCH). These models can improve price forecasting because the models consider both short-term and long-term dependence at the same time. Geweke and Porter-Hudak (1983) confirmed that ARFIMA models provide substantially more reliable out-of-sample forecasts than do more conventional procedures such as ARIMA.

Last, without being proven, it has frequently been assumed in studies of commodity and financial markets that price distributions are Gaussian with finite mean and variance. This study has shown that agricultural cash price series do not bear out the Gaussian assumption; instead, they are more appropriately fitted to the stable distribution. It was also found that most of the cash price series have α less than 2.0 but greater than 1.0, which suggests that variance is undefined or infinite. Therefore, sample variances are inappropriate measures of dispersion. The eggs and oats series have α less than 1.0, indicating that there is no stable mean for them, and thus sample mean is not a good representative of location for those series. Therefore, using first and second moments might lead to misleading results in an economic analysis if we confirm that a series has fractal structure. If so, then alternative measures of location and dispersion are needed. Estimates from the stable distribution may be suitable for such purposes.

However, this study has some limitations. In the modified R/S analysis, choosing the truncation lag q is troublesome. When q is large relative to sample size, the finite-sample distribution of the estimator can be radically different from its asymptotic limit. On the other hand, when q is too small some substantial autocorrelation beyond q is not captured in the weighted sum. Thus, special care should be taken when choosing the truncation lag, and further research is needed to provide a more sophisticated rule for choosing the lag q for any data generating assumption. Although we can use the δ and γ of the stable distribution to substitute for the mean and standard deviation in case of infinite or undefined moments, the validity of this approach requires further study.

The value of this study lies in several areas. Previous work with commodity price distributions and the fractal market hypothesis provide empirical and theoretical justifications for exploring fractality in agricultural commodity prices. A long data set of prices for several agricultural commodities was constructed and a comprehensive look at fractality was undertaken. Two different models used to detect fractality in financial data were applied to the price series. Results from both models confirmed that the price series are fractal. Long-term dependence and self-similarity were indicated, which may allow future studies to predict behavior of a frequency even when the frequency affords few observations. The location and scale parameters of the stable distribution were proposed as suitable candidates to replace undefined mean and variance in future work on market risk and returns. Ultimately this work may lead to a better understanding of commodity price distributions and the behavior of market participants.

Table 1.

Results of Estimating the Hurst Exponent, H.

Variables	**$H_{1,0}$**	**$H_{1,q}$**	**$H_{5.0}$**	**$H_{5,q}$**	**$H_{10,0}$**	**$H_{10,q}$**
Broilers	0.5229	0.4732	0.4655	0.4308*	0.4351*	0.3637*
Cocoa	0.5398	0.5236	0.5984*	0.5856*	0.6185*	0.6120*
Coffee	0.5995*	0.5994*	0.6518*	0.6347*	0.6612*	0.6485*
Corn	0.6465*	0.6384*	0.6841*	0.6592*	0.6946*	0.6614*
Large White Eggs	0.3723*	0.3171*	0.4062*	0.3793*	0.4119*	0.3890*
Kansas City Wheat	0.5801*	0.5741*	0.6242*	0.6016	0.6554*	0.6232*
Oats	0.5530	0.5271	0.5704	0.5703	0.5826*	0.5753
Soybeans	0.5858*	0.5646*	0.6217*	0.5834*	0.6224*	0.5917*
Soybean Meal	0.5983*	0.5742*	0.6299*	0.6003*	0.6377*	0.6322*
Soybean Oil	0.5736*	0.5579*	0.5656	0.5474	0.5693	0.5408
Spring Wheat	0.5389	0.5203	0.5446	0.5279	0.5902*	0.5619
Sugar #11	0.5413	0.5399*	0.6003*	0.5832*	0.5856*	0.5702*
Wheat	0.5835*	0.5540*	0.6424*	0.6275*	0.6292*	0.6173*
Wheat #1	0.6282*	0.6204*	0.6678*	0.6503*	0.6942*	0.6688*

The first and second columns are $\hat{H}$ for the time scale $n = 1$, assuming respectively short-term independence and dependence. When assuming short-term independence, the exponents are estimated using the classical R/S analysis. Otherwise, they are estimated using the modified R/S analysis. The third and fourth columns are for $\hat{H}$ when the time scale is $n = 5$, weekly frequency, and the fifth and last columns are for $\hat{H}$ when time scale is $n = 10$, bi-weekly frequency, also assuming respectively independence and dependence.

* indicates that the value's difference from 0.5 is statistically significant at the 5% level.

Table 2.

Fractional Dimension, D.

Variables	Daily		Weekly		Bi-Weekly	
Broilers	1.4561	1.4861	1.3932	1.4285	1.3277	1.3695
Cocoa	1.4429	1.4585	1.3864	1.4053	1.3237	1.3449
Coffee	1.3950	1.3952	1.3633	1.3795	1.3284	1.3439
Corn	1.3634	1.3825	1.3846	1.4064	1.3182	1.3522
Large White Eggs	1.6455	1.6233	1.4882	1.4984	1.3846	1.4147
Kansas City Wheat	1.4201	1.4202	1.3850	1.3983	1.3417	1.3497
Oats	1.4521	1.4525	1.4348	1.4350	1.3982	1.3993
Soybeans	1.4311	1.4392	1.4076	1.4201	1.3874	1.4004
Soybean Meal	1.4200	1.4281	1.4543	1.4616	1.3744	1.3986
Soybean Oil	1.4467	1.4577	1.4146	1.4344	1.4398	1.4404
Spring Wheat	1.4569	1.4591	1.3900	1.4032	1.3885	1.3910
Sugar #11	1.4422	1.4661	1.3940	1.4269	1.3481	1.4034
Wheat	1.4223	1.4326	1.3225	1.3311	1.2693	1.2905
Wheat #1 Cash	1.3786	1.3850	1.3784	1.3945	1.3829	1.3930

The first and second columns are values of D, fractional dimension coefficient, that are calculated from respectively $H_{1,0}$ and $H_{1,q}$. The third and fourth columns are for weekly frequency and calculated from $H_{5,0}$ and $H_{5,q}$. The fifth and last columns are for bi-weekly frequency and are calculated from $H_{10,0}$ and $H_{10,q}$.

Table 3.

Estimated Parameters of the Stable Distribution.

Variables	Characteristic Exponent α	Skewness Parameter β	Scale Parameter γ	Location Parameter δ
Broilers	1.646(0.067)	-0.019(0.168)	0.526(0.022)	0.009(0.040)
Cocoa	1.775(0.063)	0.347(0.228)	13.985(0.574)	-1.477(1.103)
Coffee	1.260(0.068)	0.004(0.104)	1.540(0.086)	-0.772(0.115)
Corn	1.583(0.071)	-0.062(0.156)	1.999(0.093)	0.998(0.159)
Large White Eggs	0.617(0.039)	0.225(0.053)	0.095(0.008)	-0.009(0.004)
Kansas City Wheat	1.649(0.070)	0.091(0.178)	3.323(0.149)	-0.127(0.266)
Oats	0.624(0.042)	-0.103(0.057)	0.363(0.036)	0.456(0.168)
Soybeans	1.596(0.072)	-0.019(0.160)	4.303(0.200)	0.169(0.344)
Soybean Meal	1.523(0.072)	0.085(0.141)	1.488(0.072)	-0.066(0.118)
Soybean Oil	1.466(0.035)	0.073(0.064)	0.216(0.003)	-0.010(0.008)
Spring Wheat	1.421(0.047)	0.003(0.082)	2.956(0.099)	0.036(0.152)
Sugar #11	1.671(0.069)	-0.054(0.184)	0.108(0.004)	0.004(0.008)
Wheat	1.773(0.076)	0.126(0.278)	3.797(0.186)	-0.122(0.354)
Wheat #1 Cash	1.548(0.084)	0.242(0.169)	2.488(0.138)	0.406(0.233)

The values in parentheses are estimates of the 95% confidence interval deviates for the parameter estimates.

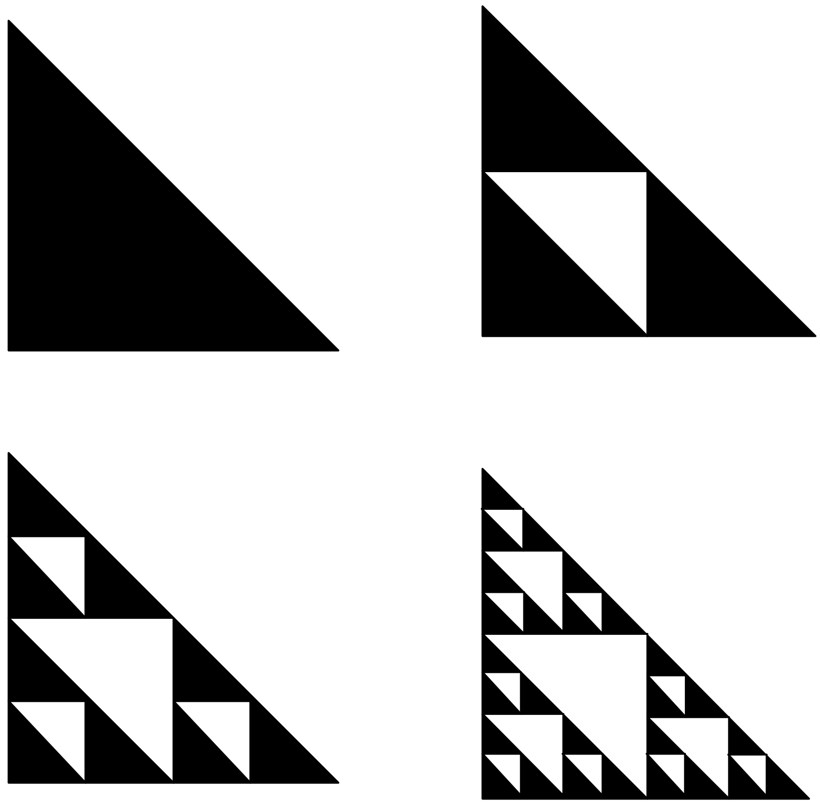
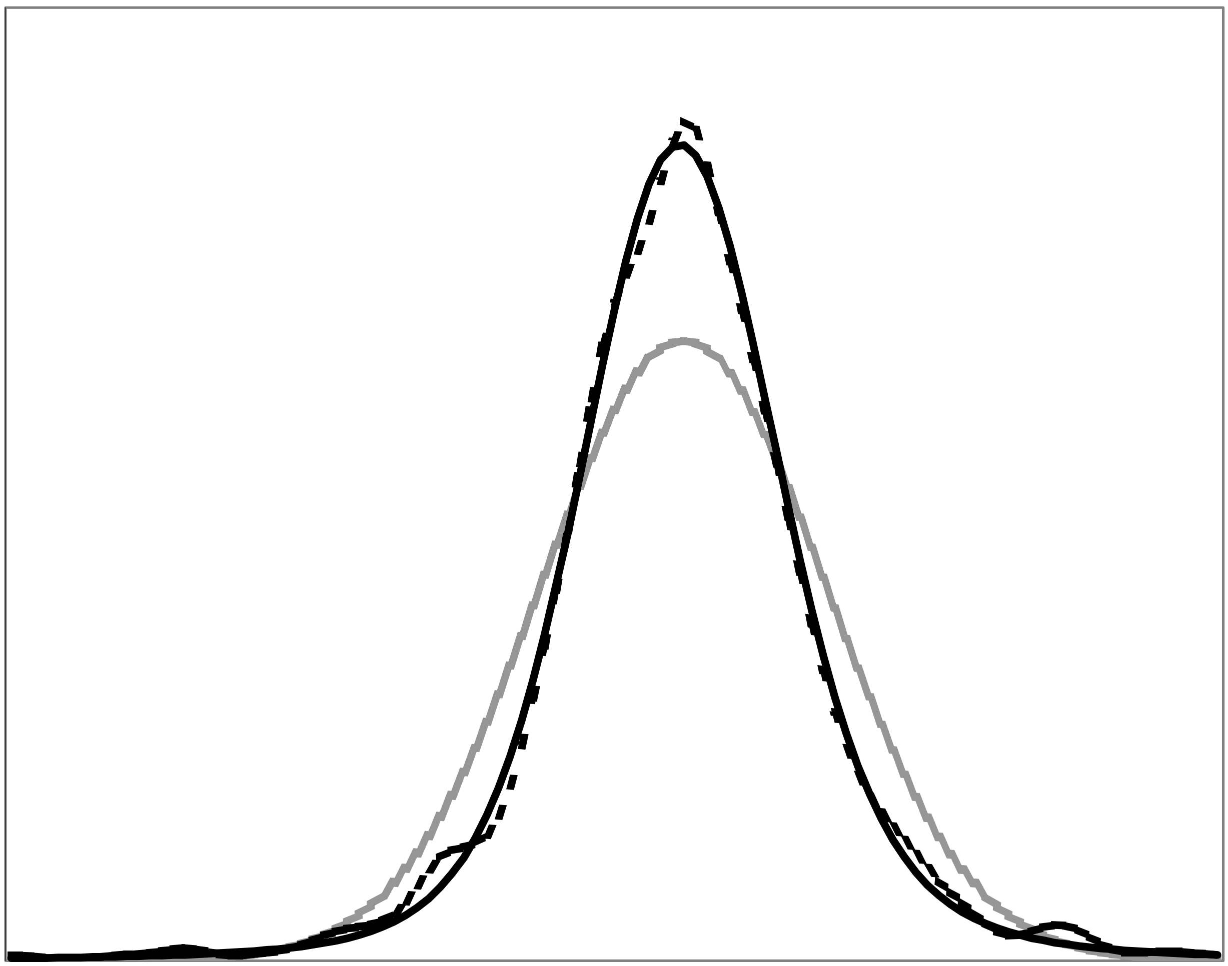


Figure 1

Construction of The Sierpinski Gasket*

* This figure is adapted from Frame (1996).



Note: The solid black line denotes the stable fitted density, the dotted line denotes the empirical smoothed data density, and the solid grey line denotes the Normal fitted density.

Figure 2

Kansas City Wheat
Comparison of Density Plots

References

Andrews, D. "Heteroskedasticity and Autocorrelation Consistent Covariance Matrix Estimation." *Econometrica* 59 (1991): 817-58.

Barkoulas, John, Walter C. Labys, and Joseph Onochie "Fractional Dynamics in International Commodity Prices." *The J. of Futures Markets* 17-2 (1997): 161-89.

Bollerslev, Tim, and Hans Ole Mikkelsen "Modeling and Pricing Long Memory in Stock Market Volatility." *J. of Econometrics* 73 (1996): 151-84.

Booth, G. Geoffrey and Yiuman Tse "Long Memory in Interest Rate Futures Markets: A Fractional Cointegration Analysis." *The J. of Futures Markets* 15-5 (1995): 573-84.

Breidt, F. Jay, Nuno Crato, and Pedro de Lima "The Detection and Estimation of Long Memory in Stochastic Volatility." *J. of Econometrics* 83 (1998): 325-48.

Brock, W. A. "Distinguishing Random and Deterministic Systems: Abridged Version." *J. of Econ. Theory* 40 (1986): 168-95.

Burton, M. "Some Illustrations of Chaos in Commodity Models." *J. of Agri. Econ.* 44 (1993): 38-49.

Campbell, John Y., Andrew W. Lo, and A. Craig MacKinlay *The Econometrics of Financial Markets*. Princeton, New Jersey: Princeton University Press, 1997.

Chambers, Marcus J. and Roy E. Bailey "A Theory of Commodity Price Fluctuations." *J. of Polit. Econ.* 104-5 (1996): 924-57.

Cheng, Pao L. and M. King Deets "Portfolio Returns and the Random Walk Theory." *The J. of Finance* 26-1 (1971): 11-30.

Cogley, Timothy "International Evidence on the Size of the Random Walk in Output." *The J. of Polit. Econ*. 98-3 (1990): 501-18.

Copeland, Thomas E. "A Model of Asset Trading under The Assumption of Sequential Information Arrival." *The J. of Finance* 31-4 (1976): 1149-68.

Corazza, Marco, A. G. Malliaris, and Carla Nardelli "Searching for Fractal Structure in Agricultural Futures Markets." *The J. of Futures Markets* 17-4 (1997): 433-73.

Cornew, R. W., D. E. Town, and L. D. Crowson "Stable Distributions, Futures Prices, and the Measurement of Trading Performance." *The J. of Futures Markets* 4-4 (1984): 531-57.

Decoster, Gregory P., Walter C. Labys, and Douglas W. Mitchell "Evidence of Chaos in Commodity Futures Prices." *The J. of Futures Markets* 12-3 (1992): 291-305.

DuMouchel, W. H. "On the asymptotic normality of the maximum-likelihood estimate when sampling from a stable distribution." *Annals of Statistics* 1 (1973): 948-57.

Falconer, K. *Fractal Geometry.* New York: Wiley, 1990.

Fama, Eugene "Mandelbrot and the Stable Paretian Hypothesis." *J. of Business* 36 (1963): 420-29.

Fama, Eugene "Portfolio Analysis in a Stable Paretian Market." *Manage. Science* 11 (1965a): 404-19.

Fama, Eugene "The Behavior of Stock Market Prices." *J. of Business* 38 (1965b): 34-105.

Fama, Eugene and Richard Roll "Parameter Estimates for Symmetric Stable Distributions." *J. of Ameri. Stat. Assoc.* 66-334 (1971): 331-38.

Fielitz, B. D. and J. P. Rozelle "Stable Distributions and the Mixtures of Distributions Hypothesis for Common Stock Returns." *J..of Ameri. Stat. Assoc.* 78 (1983): 28-36.

Frame, Michael "Fractals and Education: Helping Liberal Arts Students to See Science." *Fractal Horizons: The Future Use of Fractals,* pp 35-74, Edited by Clifford A. Pickover, New York: St. Martin's Press, 1996.

Fung, Hung-Gay and Wai-Chung Lo "Memory in Interest Rate Futures." *The J. of Futures Markets* 13-8 (1993): 865-72.

Geweke, John and Susan Porter-Hudak "The Estimation and Application of Long Memory Time Series Models." *The J. of Time Series Analysis* 4-4 (1983): 221-38.

Greene, Myron T. and Bruce D. Fielitz "Long-Term Dependence in Common Stock Returns." *J. of Financial Econ.* 4 (1977): 339-49.

Gribbin, Donald W., Randy W. Harris, and Hon-Shiang Lau "Futures Prices Are Not Stable-Paretian Distributed." *The J. of Futures Markets* 12-4 (1992): 475-.87.

Grossman, Sanford J. and Joseph E. Stiglitz "On the Impossibility of Informationally Efficient Markets." *Ameri. Econ. Review* 70 (1980): 393-408.

Hall, Joyce A., B. Wade Brorsen, and Scott H. Irwin. "The Distribution of Futures Prices: A Test of the Stable Paretian and Mixture of Normals Hypotheses." *J. of Finan. and Quant. Analysis* 21-1 (1989): 105-16.

Heiner, Ronald A. "The Origin of Predictable Behavior" *The Ameri. Econ. Review* 73-4 (1983): 560-95.

Helms, Billy P. and Terrence F. Martell. "An Examination of the Distribution of Futures Price Changes." *The J. of Futures Markets* 5-2 (1985): 259-72.

Holt, Donald R. and Edwin L. Crow "Tables and Graphs of the Stable Probability Density Functions." *J. of Research of the National Bureau of Standards - B. Mathematical Sciences* 77B-3&4 (1973): 143-97.

Hurst, H. E. "The Long-Term Dependence in Stock Returns." *Transactions of the Ameri. Society of Civil Engineers* 116 (1951).

Kaen, F. R. and R. E. Rosenman "Predictable Behavior in Financial Markets: Some Evidence in Support of Heiner's Hypothesis." *The Ameri. Econ. Review* 76-1 (1986): 212-20.

Kao, G. W. and Christopher K. Ma "Memories, Heteroscedasticity, and Price Limit in Currency Futures Markets." *The J. of Futures Markets* 12-6 (1992): 679-92.

Kwiatkowski, D., P. Phillips, P. Schmidt, and Y. Shin "Testing the Null of Stationarity Against the Alternative of a Unit Root," *J. of Econometrics* 54 (1992): 159-78.

Lo, A. W. "Long-Term Memory in Stock Market Prices." *Econometrica* 59-5 (1991): 1279-1313.

Mackey, Michael C. "Commodity Price Fluctuations: Price Dependent Delays and Nonlinearities as Explanatory Factors." *J. of Econ. Theory* 48 (1989): 497-509.

Mandelbrot, B. B. "The Variation of Certain Speculative Prices." *J. of Business* 36 (1963a): 394-419.

Mandelbrot, B. B. "New Methods in Statistical Economics." *J. of Polit. Econ.* 71-5 (1963b): 421-40.

Mandelbrot, B. B. "Statistical Methodology for Non-Periodic Cycles: From the Covariance to R/S Analysis." *Ann. of Econ. & Soc. Mea.* 1 (1972): 259-50.

Mandelbrot, B. B. *The Fractal Geometry of Nature*. New York: W. H. Freeman, 1982

Mandelbrot, B. B. and Taqqu, M. S. "Robust R/S Analysis of Long Run Serial Correlation." Lecture presented at the International Statistical Institute, 42nd Session, Manila, 1979.

Mandelbrot, B. B. and J. W. van Ness "Fractional Brownian Motions, Fractional Noises, and Applications." *SIAM Review* 10 (1968): 422-37.

Mandelbrot, B. B. and Wallis, J. R. "Robustness of the Rescaled Range R/S in the Measurement of Noncyclic Long Run Statistical Dependence." *Water Resource Research* 5 (1969).

McCulloch, J. Huston "Simple Consistent Estimator of Stable Distribution Parameters." *Commun. in Stat., Simul., and Comp.* 15-4 (1986): 1109-36.

McCulloch, J. Huston "Financial Applications of Stable Distributions." *Statistical Methods in Finance, Handbook of Statistics* Vol. 14, Maddala, G. S. and Roa, C. R. (eds.) North-Holland, NY, 1996.

Mercer, A. and C. S. Smith "A Random Walk in which the Steps Occur Randomly in Time." *Biometrika* 46-1 (1959): 30-5.

Newey, W., and K. West "A Simple Positive Definite, Heteroscedasticity and Autocorrelation Consistent Covariance Matrix." *Econometrica* 55 (1987): 703-05.

Nolan, J. P. "Numerical computation of stable densities and distribution functions." *Commun. Stat.: Stochastic Models* 13 (1997): 759-74.

Nolan, J. P. "Fitting data and assessing goodness of fit with stable distributions." *In Proceedings of the Conference on Applications of Heavy Tailed Distributions in Economics, Engineering, and Statistics.* American University, Washington, DC, June 3-5, 1999.

Nolan, J. P. and A. K. Panorska "Data Analysis for Heavy Tailed Multivariate Samples." *Commun. Stat.: Stochastic Models* 13(1997): 687-702.

Pagan, Adrian R. and Y. S. Hong "Non-parametric Estimation and the Risk Premium." In W. Barnett, J. Powell and G. Tauchen, eds., *Semiparametric and Nonparametric Methods in Econometrics and Statistics*. Cambridge, England: Cambridge University Press, 1990.

Pan, M., Y. A. Liu, and H. Bastin "An Examination of the Short-Term and Long-Term Behavior of Foreign Exchange Rates." *The Finan. Review* 31 (1996): 603-22.

Panton, D. "Cumulative distribution function values for symmetric standardized stable distributions." *Commun. in Stat., Simul., and Comp.* 21 (1992): 485-92.

Peters, E. E. *Fractal Market Analysis.* New York: John Wiley & Sons, 1994.

Peters, E. E. *Chaos and Order in the Capital Markets.* New York: John Wiley & Sons, 1996.

So, Jacky C. "The Sub-Gaussian Distribution of Currency Futures: Stable Paretian or Nonstationary?" *The Review of Econ. and Stat.* 69-1 (1987): 100-07.

Sunder, S. "Market for Information: Experimental Evidence." *Econometrica* 60(1992):667-95.

Williams, J. and Brian Wright. *Storage and Commodity Markets*. Cambridge, England: Cambridge University Press, 1991.

Yang, Seung-Ryong and B. Wade Brorsen. "Nonlinear Dynamics of Daily Cash Prices." *Amer. J. of Agri. Econ.* 74 (1992): 706-15.

Appendix

Consider sample returns X_1, X_2, ..., X_N and fix sub-time series of length T, where T is less than or equal to N. Determine all possible non-overlapping sub-time series $X_{i,T}$ and for every sub-time series, calculate the sample mean $X^m_{i,T}$. Calculate cumulative sums of deviations for each sub-time series: $X^s_{i,T} = \Sigma_j (X_{j,i,T} - X^m_{i,T})$, where $X_{j,i}$ denotes each observation of sub-time series $X_{i,T}$. Calculate the rescaled range statistic:

(A-1) $$(R/S)_{i,T} = (1/S_{i,T}) [\text{Max} (X^s_{i,T}) - \text{Min} (X^s_{i,T})],$$

where $S_{i,T}$ is the standard deviation of sub-time series $X_{i,T}$.

Between $(R/S)_{i,T}$ and T_i, there is a mathematical relationship such that

(A-2) $$(R/S)_{i,T} \sim (a * T_i)^H.$$

Based on equation (A-2), fit OLS to the following log equation with error $\varepsilon_{i,T}$ to estimate H:

(A-3) $$\ln (R/S)_{i,T} = \ln (a) + H \ln (T_i) + \varepsilon_{i,T}.$$

Equation (A-2) could be specified in other ways than the log-linear relationship and still be consistent with the asymptotic relationship underlying it. For example, nonlinear least squares can be used for the equation to estimate H instead of taking the log on both sides.

Since the maximum of the sum of all deviations, $X^s_{i,T}$, is always nonnegative and the minimum is always nonpositive, the rescaled range is always nonnegative. By using several different starting points, average values of R/S may be computed for various lags. This estimation of H makes no assumption about the shape of the underlying distribution. The intercept of the regression, ln (a), is just a constant that has no particular meaning in this specific case.

Lo (1991) modified the R/S statistic by using $R/S^b_{i,T}$ such that

(A-4) $$(R/S^b)_{i,T} = (1/S^b_{i,T}) [\text{Max} (X^s_{i,T}) - \text{Min} (X^s_{i,T})],$$

where $$S^b_{i,T} \equiv S_{i,T} + 2 \Sigma_k w_k(q) r^b_k, \quad \text{for } k = 1, 2, ..., q.$$

$w_k(q) \equiv 1 - k / (q+1), \quad q < T.$

$\hat{r}_k = (1/T) \Sigma_j (X_{j,i} - X^m_{i,T}) (X_{j-k,i} - X^m_{i,T}), \quad$ for $j = k+1, k+2, ..., T.$

$w_k(q)$ is the weight of lag q.

$\hat{r}_k$ is the usual covariance estimators of X.

The modified R/S statistic differs from the classical R/S statistic only in its denominator, adding the weights $w_k(q)$ and covariance estimators to the standard deviation. The weight is suggested by Newey and West (1987), and q is the truncation lag. By allowing q to increase with the number of observations N but at a slower rate than N, the modified standard deviation $S^b_{i,T}$ adjusts appropriately for general forms of short-term dependence. In the modified R/S, a troublesome job is choosing the truncation lag q. Andrews (1991) showed that when q becomes large relative to the sample size N, the finite-sample distribution of the estimator can be radically different from its asymptotic limit. However, the value chosen for q must not be too small, since the autocorrelation beyond lag q may be substantial and should be included in the weighted sum. The truncation lag thus must be chosen with some consideration of the data at hand. Andrews (1991) provided a data-dependent rule for choosing q. Andrews's (1991) data-dependent formula is $q \equiv int[(3 \cdot n / 2)^{1/3} \cdot (2 \cdot \rho / (1 - \rho^2))^{2/3}]$, where *int* denotes the greatest integer less than or equal to the value of $[\cdot]$, n is the number of observations, and ρ is the first-order autocorrelation coefficient of the series.

Endnotes

[1] See Mandelbrot (1963a); Fama (1965b); Cornew, Town, and Crowson (1984); Helms and Martell (1985); So (1987); Kao and Ma (1992); Booth and Tse (1995); Corrazza, Malliaris, and Nardelli (1997); and Barkoulas, Labys, and Onochie (1997).

[2] For example, see Fama (1965a); Fama and Roll (1971); Fielitz and Rozelle (1983); Hall, Brorsen, Irwin (1989); Gribbin, Harris, and Lau (1992); McCulloch (1996); Nolan and Panorska (1997); and Nolan (1999).

[3] For examples, see Grossman and Stiglitz (1980); DeCoster, Labys, and Mitchell (1992); Sunder (1992); Kao and Ma (1992); Fung and Lo (1993); Bollerslev and Mikkelsen (1996); Barkoulas, Labys, and Onochie (1997); and Breidt, Crato, and Lima (1998).

[4] The word *stable* is used because the shape is stable or unchanged under the sums of same type processes. That is, if two independent random variables with the same type of distribution are combined linearly, then the resulting distribution is also a random variable with the same type. There are several different names for the stable distribution: Pareto-Levy distribution, stable Paretian distribution, stable family distribution, fractal distribution, and stable distribution.

[5] The characteristic function is the Fourier transform of a random variable y, i.e., $E[\exp(i \cdot t \cdot y)]$ for t, where t is any real number and i denotes the square root of -1. The Fourier transform is used to define the stable distributions, since with certain exceptions the probability distribution of the stable processes cannot be specified directly.

[6] Since a stable distribution is invariant under addition, the distribution of sums of the stable distribution is also a stable distribution with the same values of α [Fama (1963)]. The estimated values of the α must be equal across the sums, provided that the underlying leptokurtic distribution is stable.

[7] Setting $\alpha = 2$, $\beta = 0$, $\delta = \mu$, and $\gamma = \sigma^2/2$ yields the characteristic function of the normal distribution. The EMH essentially says that α must equal 2, i.e., the Gaussian assumption. But the fractal market hypothesis does not give any restriction, i.e., α can range from 1 to 2. That is a significant difference between the two market hypotheses. Cauchy distribution is the case with $\alpha = 1$, $\beta = 0$, $\gamma \in (0, +\infty)$, and $\delta \in (-\infty, +\infty)$. Levy distribution is the case with $\alpha = 1/2$, $\beta = 1$, $\gamma \in (0, +\infty)$, and $\delta \in (-\infty, +\infty)$.